ASYMPTOTIC STATISTICS PROBLEMS WITH NUISANCE PARAMETERS ARISING IN PROCESSING OF MULTIDIMENSIONAL GEOPHYSICAL TIME SERIES

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LAN expansion for Gaussian time series. Conditions A.

Assume that noise time series $\boldsymbol{n}_t,\ t\in\mathbb{Z},\ \boldsymbol{n}\in R^m$ is Gaussian regular multidimensional time series (GRMTS):

$$E\left\{\boldsymbol{n}_t\boldsymbol{n}^*_{t+\tau}\right\}\equiv \boldsymbol{C}_\tau \ \text{ for } t\in\mathbb{Z},\ \sum_{\tau=-\infty}^{\infty}\left\|C_\tau\right\|<\infty\ ,\ \psi=exp\left\{\frac{1}{2\pi}\int_0^{2\pi} log\left[det\,2\pi\boldsymbol{F}(\lambda)\right]d\lambda\right\}>0,$$

where $\boldsymbol{F}(\lambda)=\sum_{\tau=-\infty}^{+\infty}\boldsymbol{C}_\tau e^{i\lambda\tau}$, $\lambda\in[0,2\pi]$ is matrix power spectral density (MPSD) of $\boldsymbol{n}_t$.

Conditions A.

Multidimensional time series $\boldsymbol{x}_t=\boldsymbol{s}_t(\boldsymbol{\theta})+\boldsymbol{n}_{t,\boldsymbol{\theta}}\in R^m,\ t\in\mathbb{Z}$, ***where*** $\boldsymbol{s}_t(\boldsymbol{\theta})\in R^m$ ***is deterministic***, $\boldsymbol{n}_t$ ***- GRMTS with MPSD*** $\boldsymbol{F}(\lambda,\boldsymbol{\theta})$, $\boldsymbol{\theta}\in\Theta$, $\Theta\in R^q$ ***is a compact, satisfies conditions*** **A**, ***if***:

A1. ***There exist*** $\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})=\dfrac{\partial}{\partial\theta_k}\boldsymbol{s}_t(\boldsymbol{\theta}),\ t\in\mathbb{Z},\ k\in\overline{1,q},\ \ \boldsymbol{\theta}\in\Theta$, ***such that***:

A1.1 $\dfrac{1}{N}\sum_{t=1}^{N}\left|\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})\right|^2<c,\ \ N\in\mathbb{N},\ k\in\overline{1,q},\ \ \boldsymbol{\theta}\in\Theta$;

A1.2 $\max_{1<t<N}\left|\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})\right|^2<CN^\beta$ ***for some*** $\beta\in(0,1),\ \ k\in\overline{1,q},\ \boldsymbol{\theta}\in\Theta$;

A1.3) $\lim_{N\to\infty}\dfrac{1}{N}\sum_{t=1}^{N}\left|\boldsymbol{s}'_{k,t}\left(\boldsymbol{\theta}_N-\dfrac{\boldsymbol{r}_N}{N}\right)-\boldsymbol{s}'_{k,t}(\boldsymbol{\theta}_N)\right|^2=0$ ***for any*** $\boldsymbol{\theta}_N\in\Theta$, $\left|\boldsymbol{r}_N\right|<c$;

A2. ***MPSD*** $\boldsymbol{F}(\lambda;\boldsymbol{\theta})$ ***satisfies the following conditions***:

A2.1 $det\,\boldsymbol{F}(\lambda;\boldsymbol{\theta})>\mu>0$ ***for any*** $\lambda\in[0,2\pi],\ \boldsymbol{\theta}\in\Theta$;

A2.2 **Lipshits condition**: $\sup_{\boldsymbol{\theta}\in\Theta}\left\|\boldsymbol{F}(\lambda_1;\boldsymbol{\theta})-\boldsymbol{F}(\lambda_2;\boldsymbol{\theta})\right\|<c\left|\lambda_1-\lambda_2\right|,\ \boldsymbol{\theta}\in\Theta$;

Theorem 1.

Let MTS $\boldsymbol{x}_t = \boldsymbol{s}_t(\boldsymbol{\theta}) + \boldsymbol{n}_{t,\boldsymbol{\theta}}$, $t \in \mathbb{Z}$ ***satisfies Conditions*** **A**. ***Then distribution family of observations*** $P(\boldsymbol{X}_N, \boldsymbol{\theta})$, $\boldsymbol{X}_N = (\boldsymbol{x}_t,\, t \in 1,\dots,N)$ ***admits LAN expansion*** (***in L Le Cam definition***):

$$ln \frac{dP\left(\boldsymbol{X}_N, \boldsymbol{\theta} + \frac{\boldsymbol{\vartheta}}{\sqrt{N}}\right)}{dP(\boldsymbol{X}_N, \boldsymbol{\theta})} = exp\left\{\boldsymbol{\vartheta}^T \boldsymbol{\Delta}(\boldsymbol{X}_N; \boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{\vartheta}^T \boldsymbol{\Gamma}_N(\boldsymbol{\theta})\boldsymbol{\vartheta} + \alpha_{\boldsymbol{\theta}}(\boldsymbol{X}_N; \boldsymbol{\vartheta})\right\},$$

where $\boldsymbol{\Delta}(\boldsymbol{X}_N;\boldsymbol{\theta}) = \boldsymbol{\varphi}(\boldsymbol{X}_N;\boldsymbol{\theta}) + \boldsymbol{\psi}(\boldsymbol{X}_N;\boldsymbol{\theta})$; $\boldsymbol{\Gamma}_N(\boldsymbol{\theta}) = \boldsymbol{\Phi}_N(\boldsymbol{\theta}) + \boldsymbol{\Psi}_N(\boldsymbol{\theta})$;

$$\boldsymbol{\varphi}(\boldsymbol{X}_N;\boldsymbol{\theta}) = \left(\frac{2}{\sqrt{N}} Re \sum_{j=1}^{N} \breve{\boldsymbol{s}}'^{*}_{k,j} \boldsymbol{F}_j^{-1}\left(\breve{\boldsymbol{x}}_j - \breve{\boldsymbol{s}}_j\right);\ k \in \overline{1,q}\right); \quad \boldsymbol{\Phi}_N(\boldsymbol{\theta}) = \left[\frac{1}{N}\sum_{j=1}^{N} \breve{\boldsymbol{s}}'^{*}_{k,j} \boldsymbol{F}_j^{-1} \breve{\boldsymbol{s}}'_{l,j},\ k,l \in \overline{1,q}\right];$$

$$\boldsymbol{\psi}(\boldsymbol{X}_N;\boldsymbol{\theta}) = \left(\frac{2}{\sqrt{N}}\sum_{j=1}^{N}\left(\breve{\boldsymbol{x}}_j - \breve{\boldsymbol{s}}_j\right)^{*} \boldsymbol{F}_j^{-1}\boldsymbol{F}'_{k,j}\boldsymbol{F}^{-1}\left(\breve{\boldsymbol{x}}_j - \breve{\boldsymbol{s}}_j\right) - tr\boldsymbol{F}_j^{-1}\boldsymbol{F}'_{k,j};\ k \in \overline{1,q}\right);$$

$$\boldsymbol{\Psi}_N(\boldsymbol{\theta}) = \left[\frac{1}{N}\sum_{j=1}^{N} tr\boldsymbol{F}_j^{-1}\boldsymbol{F}'_{k,j}\boldsymbol{F}_j^{-1}\boldsymbol{F}'_{l,j};\ k,l \in \overline{1,q}\right]; \quad \sup_{\substack{\boldsymbol{\theta}\in\Theta \\ |\boldsymbol{\vartheta}|<c}} P\left\{|\, a_{\boldsymbol{\theta}}(\boldsymbol{X}_N;\boldsymbol{\vartheta})| > \boldsymbol{\varepsilon}\right\} \to 0 \ (N \to \infty);$$

$$\breve{\boldsymbol{s}}_j = \breve{\boldsymbol{s}}_j(\boldsymbol{\theta}),\ \breve{\boldsymbol{s}}'_{k,j} = \breve{\boldsymbol{s}}_j(\boldsymbol{\theta});\ \boldsymbol{F}'_{k,j} = \frac{\partial}{\partial\theta_k}\boldsymbol{F}(\lambda_j;\boldsymbol{\theta});\ \boldsymbol{F}_j = \boldsymbol{F}(\lambda_j;\boldsymbol{\theta}),\ \lambda_j = \frac{2\pi j}{N};$$

$\breve{\boldsymbol{x}}_j = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\boldsymbol{x}_t e^{i\lambda_j t}$, $j = 1,\dots,N$, ***is the discrete finite Fourier transform*** (***DFFT***) ***of*** $\boldsymbol{x}_t$, $t \in 1,\dots,N$;

$\breve{\boldsymbol{s}}_j(\boldsymbol{\theta})$, $\breve{\boldsymbol{s}}'_{k,j}(\boldsymbol{\theta})$ ***are DFFT*** *of* $\boldsymbol{s}_t(\boldsymbol{\theta})$, $\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})$.

Asymptotic probability distribution of "spectral observations" $\breve{\boldsymbol{x}}_j,\ j \in 1,\ldots,N$ – DFFT of "time observations" $\boldsymbol{x}_t,\ t \in 1,\ldots,N$

Theorem 2. Let $\breve{\boldsymbol{x}}_j,\ j = 1,\ldots,N$ ***are DFFT of observations*** $\boldsymbol{x}_t,\ t = 1,\ldots,N$ ***of GRMTS*** $\boldsymbol{x}_t,\ t \in \mathbb{Z}$,

with MPSD $\boldsymbol{F}(\lambda),\ n = n(N) \to \infty\ (N \to \infty)$ ***and*** $\lambda_{j_n} = \dfrac{2\pi j_n}{N} \to \lambda_0 \in [0, 2\pi],\ (N \to \infty)$.

***Then*:** 1) $p_{\breve{\boldsymbol{X}}_{j_n}}(\dot{\boldsymbol{x}}) \xrightarrow[N\to\infty]{D} \dfrac{1}{\sqrt{2\pi \det \boldsymbol{F}(\lambda_0)}} \exp\left\{-\dot{\boldsymbol{x}}^* \boldsymbol{F}^{-1}(\lambda_0)\dot{\boldsymbol{x}}\right\},\ \dot{\boldsymbol{x}} = \boldsymbol{y} + i\boldsymbol{z}$; 2) $\max\limits_{j \neq k} \left\| E\left\{\breve{\boldsymbol{X}}_j \breve{\boldsymbol{X}}_k^*\right\}\right\| = O\left(\frac{1}{N}\right)$.

Corollary 1. $\frac{1}{\sqrt{N}} \ln p_{\breve{\boldsymbol{X}}_N}(\boldsymbol{\theta}) = l(\boldsymbol{X}_N, \boldsymbol{\theta}) + \beta(\boldsymbol{X}_N, \boldsymbol{\theta})$, ***where***

$$l(\boldsymbol{X}_N, \boldsymbol{\theta}) = -\frac{\sqrt{N}}{2}\ln(2\pi) - \frac{1}{\sqrt{N}}\sum_{j=1}^{N}\left[\frac{1}{2}\ln\det F_j(\boldsymbol{\theta}) + \left(\breve{\boldsymbol{x}}_j - \boldsymbol{s}_j(\boldsymbol{\theta})\right)^* \boldsymbol{F}_j^{-1}(\boldsymbol{\theta})\left(\breve{\boldsymbol{x}}_j - \boldsymbol{s}_j(\boldsymbol{\theta})\right)\right],$$

$$\sup_{\theta \in \Theta} \beta(\boldsymbol{X}_N, \boldsymbol{\theta}) \xrightarrow{P} 0,\ (N \to \infty).$$

Corollary 2. **a)** $\dfrac{\partial}{\partial \theta_k} l(\boldsymbol{X}_N, \boldsymbol{\theta}) = \Delta_k(\boldsymbol{X}_N; \boldsymbol{\theta}) = \varphi_k(\boldsymbol{X}_N; \boldsymbol{\theta}) + \psi_k(\boldsymbol{X}_N; \boldsymbol{\theta})$, ***that is the derivatives above are the components of asymptotically sufficient statistic;*** **b)** $E\left\{\dfrac{\partial l(\boldsymbol{X}_N, \boldsymbol{\theta})}{\partial \theta_k} \dfrac{\partial l(\boldsymbol{X}_N, \boldsymbol{\theta})}{\partial \theta_l}\right\} = \Phi_{k,l}(\boldsymbol{\theta}) + \Psi_{k,l}(\boldsymbol{\theta})$**,** ***that is the values above are the components of asymptotic Fisher matrix of observations***.

LAN expansion for Marcov multidimensional time series

$$\boldsymbol{x}_t=\boldsymbol{s}_t(\boldsymbol{\theta})+\xi_t(\boldsymbol{\theta}),\quad p(\boldsymbol{X}_N;\boldsymbol{\theta})=\bar{\boldsymbol{W}}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_1-\boldsymbol{s}_1(\boldsymbol{\theta})\right)\prod_{t=1}^{N}\boldsymbol{W}_{\boldsymbol{\theta}}\left(\boldsymbol{x}_t-\boldsymbol{s}_t(\boldsymbol{\theta})\middle|\boldsymbol{x}_{t-1}-\boldsymbol{s}_{t-1}(\boldsymbol{\theta})\right),\ \text{where } \boldsymbol{X}_N=\left(\boldsymbol{x}_t^T,\ t=1,\dots,N\right)^T,$$

$\bar{\boldsymbol{W}}_{\boldsymbol{\theta}}(\boldsymbol{z})$ is a one dimensional probability density, $\boldsymbol{W}_{\boldsymbol{\theta}}\left(\boldsymbol{z}_t\mid\boldsymbol{z}_{t-1}\right)$ is transition probability density of Marcov MTS $\boldsymbol{x}_t$.

Theorem 3. 1) ***Assume that transition probability of Marcov multidimensional time series is differentiable in mean-square sense for the all their arguments, that is there exists vectors:***

$$\boldsymbol{g}'(\boldsymbol{u}|\boldsymbol{\theta})=\left(\frac{\partial}{\partial z_{(k)l}}ln\boldsymbol{W}_{\boldsymbol{\theta}}\left(\boldsymbol{z}_1|\boldsymbol{z}_0\right),\ k\in\overline{1,m},\ l\in\overline{0,1}\right),\quad \boldsymbol{f}'(\boldsymbol{u}|\boldsymbol{\theta})=\left(\frac{\partial}{\partial\theta_r}ln\boldsymbol{W}_{\boldsymbol{\theta}}\left(\boldsymbol{z}_1|\boldsymbol{z}_0\right),\ r\in\overline{1,q}\right)$$

and matrices: $\boldsymbol{V}(\boldsymbol{\theta})=E_{\boldsymbol{\theta}}\left\{\boldsymbol{g}'(\boldsymbol{u}|\boldsymbol{\theta})\boldsymbol{g}'^T(\boldsymbol{u}|\boldsymbol{\theta})\right\},\quad \boldsymbol{U}(\boldsymbol{\theta})=E_{\boldsymbol{\theta}}\left\{\boldsymbol{f}'(\boldsymbol{u}|\boldsymbol{\theta})\boldsymbol{f}'^T(\boldsymbol{u}|\boldsymbol{\theta})\right\},\quad \boldsymbol{u}=\left(\boldsymbol{z}_1^T,\boldsymbol{z}_0^T\right)^T$

2) ***Let deterministic time series*** $\boldsymbol{s}_t(\boldsymbol{\theta})$ ***has partial derivative for parameters*** θ_k: $\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})=\dfrac{\partial}{\partial\theta_k}\boldsymbol{s}_t(\boldsymbol{\theta})$,

such that for any $\boldsymbol{\theta}\in\Theta$: $\dfrac{1}{N}\sum_{t=1}^{N}\left|\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})\right|^2<c,\quad \lim_{n\to\infty}\max_{t\in 1,N}\left|\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})\right|^2\left(\sum_{t=1}^{N}\left|\boldsymbol{s}'_{k,t}(\boldsymbol{\theta})\right|^2\right)^{-1}=0.$

Then probability distribution of X_N ***admits the LAN expansion (in sense of Le Cam) with asymptotically sufficient statistic*** $\Delta(\boldsymbol{x}_N,\boldsymbol{\theta})=\boldsymbol{\varphi}(\boldsymbol{x}_N,\boldsymbol{\theta})+\boldsymbol{\psi}(\boldsymbol{x}_N,\boldsymbol{\theta})$, ***where***

$$\boldsymbol{\varphi}(\boldsymbol{x}_N,\boldsymbol{\theta})=\left(\frac{1}{\sqrt{N}}\sum_{t=1}^{N}\boldsymbol{g}'^T\left(\boldsymbol{x}_{t-1}^t-\boldsymbol{s}_{t-1}^t(\boldsymbol{\theta})\middle|\boldsymbol{\theta}\right)\boldsymbol{s}'^t_{k,t-1}(\boldsymbol{\theta});k\in\overline{1,q}\right);\ \boldsymbol{\psi}(\boldsymbol{x}_N,\boldsymbol{\theta})=\left(\frac{1}{\sqrt{N}}\sum_{t=p+1}^{N}f'_k\left(\boldsymbol{x}_{t-p}^t-\boldsymbol{s}_{t-p}^t(\boldsymbol{\theta})\middle|\boldsymbol{\theta}\right);k\in\overline{1,q}\right);$$

and asymptotic Fisher matrix $\boldsymbol{\Gamma}_N(\boldsymbol{\theta})=\boldsymbol{\Phi}_N(\boldsymbol{\theta})+\boldsymbol{U}(\boldsymbol{\theta})+\boldsymbol{Q}_N(\boldsymbol{\theta})$;

$$\boldsymbol{\Phi}_N(\boldsymbol{\theta})=\left[\frac{1}{N}\sum_{t=1}^{N}\left(\boldsymbol{s}'^t_{k,t-1}(\boldsymbol{\theta})\right)^T\boldsymbol{V}(\boldsymbol{\theta})\boldsymbol{s}'^t_{l,t-1}(\boldsymbol{\theta});\ k,l\in\overline{1,q}\right];\ \boldsymbol{Q}_N(\boldsymbol{\theta})=\left[\frac{1}{N}\sum_{t=p+1}^{N}\boldsymbol{w}_k^T(\boldsymbol{\theta})\boldsymbol{s}'^t_{l,t-1}(\boldsymbol{\theta}),\ k,l\in\overline{1,q}\right];$$

where $\boldsymbol{w}_k(\boldsymbol{\theta})=E_{\boldsymbol{\theta}}\left\{\boldsymbol{g}'\left(\boldsymbol{x}_{t-1}^t\middle|\boldsymbol{\theta}\right)f'_k\left(\boldsymbol{x}_{t-1}^t\middle|\boldsymbol{\theta}\right)\right\};\quad \boldsymbol{x}_{t-1}^t=\left(\boldsymbol{x}_t^T,\boldsymbol{x}_{t-1}^T\right)^T,\ \boldsymbol{s}'^t_{k,t-1}=\left(\boldsymbol{s}'^T_{k,t},\boldsymbol{s}'^T_{k,t-1}\right)^T.$

Asymptotically optimal algorithms for hypotheses testing under LAN conditions

Consider contiguous hypotheses about distributions $P_{\boldsymbol{X}_N}\left(X_N,\boldsymbol{\theta}\right)$ of the observations $\boldsymbol{X}_N=(\boldsymbol{x}_t^T,t=1,\dots,N)^T$:

$$H_{N,0}: L\{\boldsymbol{X}_N\}=P_{\boldsymbol{X}_N}\left(\boldsymbol{\theta}\right);\quad H_{N,1}: L\{\boldsymbol{X}_N\}=P_{\boldsymbol{X}_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right),\ \left|\boldsymbol{\vartheta}\right|<c,\ \gamma_N=\frac{1}{\sqrt{N}}. \tag{1}$$

and class K_α^{AP} of tests $\varphi(X_N)$ for hypotheses (1) testing which have for all $\left|\boldsymbol{\vartheta}\right|<c$ asymptotic power function:

$$\lim_{N\to\infty}\beta_{\varphi_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right)=\lim_{N\to\infty}E_{\theta+\gamma_N\vartheta}\left\{\varphi\left(\boldsymbol{X}_N\right)\right\}\geq\lim_{N\to\infty}\beta_{\varphi_N}\left(\boldsymbol{\theta}\right)=\alpha,\ \gamma_N=\frac{1}{\sqrt{N}}.$$

***Theorem 4.** Assume that $P_{\boldsymbol{X}_N}\left(X_N,\theta\right)$ has the LAN property at point θ with asymptotically sufficient statistic $\Delta\left(\boldsymbol{X}_N;\boldsymbol{\theta}\right)$ and asymptotic Fisher matrix $\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)$. Then for any test $\varphi\left(\boldsymbol{X}_N\right)\in K_\alpha^{AP}$ the test $\psi\left(\boldsymbol{X}_N\right)=E_{\boldsymbol{\theta}}\left\{\varphi\left(\boldsymbol{X}_N\right)\middle|\Delta\left(\boldsymbol{X}_N;\boldsymbol{\theta}\right)\right\}=\psi\left(\Delta\left(\boldsymbol{X}_N;\boldsymbol{\theta}\right)\right)$ has the same asymptotic power function:*

$$\lim_{N\to\infty}\sup_{\left|\boldsymbol{\vartheta}\right|<c}\left|\beta_{\varphi_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right)-\beta_{\psi_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right)\right|=0.$$

The test $\varphi\left(\Delta\left(\boldsymbol{x}_N;\boldsymbol{\theta}\right)\right)\in K_\alpha^{AP}$ is Bayesian asymptotically optimal if there is no other test $\tilde{\varphi}\left(\Delta\left(\boldsymbol{x}_N;\boldsymbol{\theta}\right)\right)\in K_\alpha^{AP}$ which has:

$$\lim_{N\to\infty}E_{P(\vartheta)}\left\{\beta_{\tilde{\varphi}_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right)\right\}>\lim_{N\to\infty}E_{P(\vartheta)}\left\{\beta_{\varphi_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right)\right\},$$ $P\left(\boldsymbol{\vartheta}\right)$ is a'priory distribution of parameter $\boldsymbol{\vartheta}$.

Test $\psi\left(\Delta\left(\boldsymbol{x}_N;\boldsymbol{\theta}\right)\right)\in K_\alpha^{AP}$ is asymptotically admissible if there is no other test $\hat{\psi}\left(\Delta\left(\boldsymbol{x}_N;\boldsymbol{\theta}\right)\right)\in K_\alpha^{AP}$ which has:

$$\lim_{N\to\infty}\beta_{\hat{\psi}_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right)>\lim_{N\to\infty}\beta_{\psi_N}\left(\boldsymbol{\theta}+\gamma_N\boldsymbol{\vartheta}\right)\ \text{for all}\ \left|\boldsymbol{\vartheta}\right|<c.$$

Consider hypotheses for normal distribution density of random variable $\boldsymbol{y}\in R^m$:

$$\bar{H}_0: L\{\boldsymbol{y}\}=\boldsymbol{N}\left(\boldsymbol{0},\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)\right),\quad \bar{H}_1: L\{\boldsymbol{y}\}=\boldsymbol{N}\left(\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)\boldsymbol{\vartheta},\ \boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)\right),\ \boldsymbol{\vartheta}\neq 0. \tag{2}$$

The Bayesian optimal test for testing hypotheses $\bar{H}_0$, $\bar{H}_1$ has the form:

$$\tilde{\varphi}^{Б}\left(\boldsymbol{y}\right)=\begin{cases}1,\ \text{if}\ \rho\left(\boldsymbol{y}\right)\geq k_\alpha,\\ 0,\ \text{if}\ \rho\left(\boldsymbol{y}\right)<k_\alpha,\end{cases}\quad\text{where}\quad \rho\left(\boldsymbol{y}\right)=\int exp\left\{\boldsymbol{u}^T\boldsymbol{y}-\frac{1}{2}\boldsymbol{u}^T\boldsymbol{\Gamma}\left(\boldsymbol{\theta}\right)\boldsymbol{u}\right\}dP\left(\boldsymbol{u}\right),$$

Test $\boldsymbol{\psi}(\boldsymbol{y})$ for testing hypotheses H_0, H_1 is an admissible if there is no other test $\hat{\boldsymbol{\psi}}(\boldsymbol{y})$ for which:

$$\beta_{\hat{\psi}}(\boldsymbol{\vartheta}) \geq \beta_{\psi}(\boldsymbol{\vartheta}),\ \boldsymbol{\vartheta} \neq 0, \quad \beta_{\hat{\psi}}(0) < \alpha,\ \beta_{\psi}(0) < \alpha,$$

where $\beta_{\psi}(\boldsymbol{\vartheta}) = E_{\boldsymbol{\vartheta}}\{\psi(\boldsymbol{y})\}$ - power function of the test $\psi(\boldsymbol{y})$.

Theorem 5. Complete class of admissible tests for testing hypotheses $\bar{H}_0$, $\bar{H}_1$ coincides with the following class of the tests $\psi(\boldsymbol{y})$

$$\psi(\boldsymbol{y}) = \begin{cases} 1, \text{ if } \boldsymbol{y} \in \bar{S}, \\ 0, \text{ if } \boldsymbol{y} \in S. \end{cases}$$

where $S \in R^q$ is arbitrary closed convex set in R^q.

Theorem 6. Let a test $\psi(\boldsymbol{y})$, $\boldsymbol{y} \in R^q$ for testing hypotheses (2) ***is the optimal Bayesian test (or any admissible test). Then the test $\varphi(\boldsymbol{X}_N) = \psi(\boldsymbol{\Delta}(\boldsymbol{X}_N;\boldsymbol{\theta}))$ for testing hypotheses*** (1) ***is Bayesian asymptotically optimal (BAO) (or asymptotically admissible).***

If $dP(\boldsymbol{u}) = \boldsymbol{N}(\boldsymbol{b}, \boldsymbol{B})$ then BAO test has the form:

$$\varphi_N^{Б}(\boldsymbol{X}_N) = \begin{cases} 1, \text{ если } r(\boldsymbol{X}_N) \geq k_\alpha, \\ 0, \text{ если } r(\boldsymbol{X}_N) < k_\alpha, \end{cases} \quad r(\boldsymbol{X}_N) = \left[\boldsymbol{\Delta}^T \boldsymbol{\Gamma}_N^{-1} \boldsymbol{\Delta} - \left(\boldsymbol{\Gamma}_N^{-1} \boldsymbol{\Delta} - \boldsymbol{B}\right)^T \boldsymbol{Q}_N^{-1} \left(\boldsymbol{\Gamma}_N^{-1} \boldsymbol{\Delta} - \boldsymbol{B}\right)\right],\ \boldsymbol{\Delta} = \boldsymbol{\Delta}(\boldsymbol{X}_N),\ \boldsymbol{\Gamma}_N = \boldsymbol{\Gamma}_N(\boldsymbol{\theta}),$$

k_α is chosen to provide asymptotical level of significance for the test: $\lim\limits_{N\to\infty} \beta_{\varphi_N^{Б}}(\boldsymbol{\theta}) = \alpha$.

When $\|\boldsymbol{B}\| \gg \left\|\boldsymbol{\Gamma}_N^{-1}(\boldsymbol{\theta}_0)\right\|$ the statistic of BAO test is equal: $r(\boldsymbol{X}_N) \approx \boldsymbol{\Delta}^T(\boldsymbol{X}_N;\boldsymbol{\theta}) \boldsymbol{\Gamma}_N^{-1}(\boldsymbol{\theta}) \boldsymbol{\Delta}(\boldsymbol{X}_N;\boldsymbol{\theta})$.

This statistic is widely used for detection of geophysical signals based on multidimensional observations in conditions of strong natural or man-made random noise.

Asymptotical power of this test: $\lim\limits_{N\to\infty} E_{P(\boldsymbol{\vartheta})}\left\{\beta_{\varphi_N^{Б}}(\boldsymbol{\theta} + \gamma_N \boldsymbol{\vartheta})\right\}$, is defined by asymptotical distribution of AS statistic $\boldsymbol{\Delta}(\boldsymbol{X}_N;\boldsymbol{\theta})$ under contiguous alternative $H_{N,1}$ in (1). It is known that

$L\{\boldsymbol{\Delta}(\boldsymbol{X}_N;\boldsymbol{\theta}) \mid \boldsymbol{\theta} + \gamma_N \boldsymbol{\vartheta}\} \xrightarrow[N\to\infty]{} \boldsymbol{N}(\boldsymbol{\Gamma}(\boldsymbol{\theta})\boldsymbol{\vartheta},\ \boldsymbol{\Gamma}(\boldsymbol{\theta}))$. Hence the test statistic has in limit the non central χ^2-distribution with parameter $\boldsymbol{\vartheta}^T \boldsymbol{\Gamma} \boldsymbol{\vartheta}$ $L\{r(\boldsymbol{X}_N) \mid \boldsymbol{\theta} + \gamma_N \boldsymbol{\vartheta}\} \xrightarrow[N\to\infty]{} \chi^2_{\boldsymbol{\vartheta}^T \boldsymbol{\Gamma} \boldsymbol{\vartheta}}$.

AO tests for detection of signals generated by a point source in elastic Earth medium and distorted by coherent noise

Model of observations: $\boldsymbol{x}_t = \boldsymbol{u}_t + \boldsymbol{n}_t \in R^m$, $t = 1,\dots,N$, where $\boldsymbol{u}_t = \boldsymbol{h}_t * s_{t,\boldsymbol{\theta}}$, $t \in \mathbb{Z}$, source function $s_{t,\boldsymbol{\theta}} \in R^1$ is a regular Gaussian time series with PSD $g_{\boldsymbol{\theta}}(\lambda) = \sum_{k=1}^{q} \theta_k g_k(\lambda)$; $\boldsymbol{n}_t \in R^m$ is a regular Gaussian time series with MPSD $\boldsymbol{F}(\lambda)$.

Asymptotically sufficient statistics $\boldsymbol{\Delta}(\boldsymbol{X}_N)$ and asymptotic Fisher matrix $\boldsymbol{\Gamma}_N$ of the observations $\boldsymbol{X}_N = \left(\boldsymbol{x}_t, t \in \overline{1,N}\right)$ are:

$$\boldsymbol{\Delta}(\boldsymbol{X}_N) = \left(\frac{1}{2\sqrt{N}} \sum_{j=1}^{N} g_{k,j} \left| \breve{\boldsymbol{h}}_j^* \boldsymbol{F}_j^{-1} \breve{\boldsymbol{x}}_j \right|^2 - c_{k,N},\ k \in \overline{1,q} \right), \quad c_{k,N} = \frac{1}{2\sqrt{N}} \sum_{j=1}^{N} g_{k,j} \breve{\boldsymbol{h}}_j^* \boldsymbol{F}_j^{-1} \breve{\boldsymbol{h}}_j;$$

$$\boldsymbol{\Gamma}_N = \left[\frac{1}{4N} \sum_{j=1}^{N} g_{k,j} g_{l,j} tr\left[\boldsymbol{F}_j^{-1} \breve{\boldsymbol{h}}_j \breve{\boldsymbol{h}}_j^* \right]^2;\ k,l \in \overline{1,q} \right], \quad g_{l,j} = g_l\left(\frac{2\pi j}{N}\right); \ \breve{\boldsymbol{x}}_j, \breve{\boldsymbol{h}}_j,\ j = 1,\dots,N \text{ are DFFT of } \boldsymbol{x}_t, \boldsymbol{h}_t, t = 1,\dots,N.$$

<u>Pure coherent noise</u>: $\boldsymbol{\xi}_t = \sum_{l=1}^{s} \boldsymbol{q}_t^l * \varsigma_t^l$, $t \in \mathbb{Z}$, $s < m$, where ς_t^l are random noise source functions with PSD $\delta_l(\lambda)$.

$$\boldsymbol{F}_{COH}(\lambda) = \sum_{l=1}^{s} \boldsymbol{q}_l(\lambda) \boldsymbol{q}_l^*(\lambda) \delta_l(\lambda) = \boldsymbol{U}_s \boldsymbol{U}_s^*;\ \boldsymbol{U}_s = \left[\boldsymbol{q}_l(\lambda) \rho_l(\lambda), l \in \overline{1,s} \right] \text{ is } m \times s\text{-matrix, } s < m;\ \rho_l(\lambda) \rho_l^*(\lambda) = \delta(\lambda).$$

<u>For real noise</u>: $\boldsymbol{F}_\sigma(\lambda) = \sigma^2 \boldsymbol{F}_0(\lambda) + \boldsymbol{F}_{COH}(\lambda)$, where $\boldsymbol{F}_0(\lambda)$ - nonsingular matrix, σ^2 - small parameter.

<u>Expansion of</u> $\boldsymbol{F}_\sigma^{-1}(\lambda)$: $\boldsymbol{F}_\sigma^{-1} = \left[\sigma^{-2} \boldsymbol{F}_0^{-1} - \sigma^{-4} \boldsymbol{F}_0^{-1} \boldsymbol{U}_s \left(\boldsymbol{I}_s + \sigma^{-2} \boldsymbol{U}_s^* \boldsymbol{F}_0^{-1} \boldsymbol{U}_s \right)^{-1} \boldsymbol{U}_s^* \boldsymbol{F}_0^{-1} \right] = \sigma^{-2} \boldsymbol{B}_s + \boldsymbol{C}_s + \boldsymbol{O}(\sigma^2)$,

where $\boldsymbol{B}_s = \left[\boldsymbol{F}_0^{-1} - \boldsymbol{F}_0^{-1} \boldsymbol{U}_s \left(\boldsymbol{U}_s^* \boldsymbol{F}_0^{-1} \boldsymbol{U}_s \right)^{-1} \boldsymbol{U}_s^* \boldsymbol{F}_0^{-1} \right]$, $\boldsymbol{C}_s = \boldsymbol{F}_0^{-1} \boldsymbol{U}_s \left(\boldsymbol{U}_s^* \boldsymbol{F}_0^{-1} \boldsymbol{U}_s \right)^{-2} \boldsymbol{U}_s^* \boldsymbol{F}_0^{-1}$,

Consider $\mathbb{C}^m_{F_0^{-1}} = L_{Q_s} + L_{Q_s}^{\perp}$; $L_{Q_s} = L\{\boldsymbol{q}_1,\dots,\boldsymbol{q}_s\}$, $s < m$, $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^* \boldsymbol{F}_0^{-1} \boldsymbol{y}$. We have: $\boldsymbol{B}_s \boldsymbol{x}_Q = 0$, $\boldsymbol{x}^* \boldsymbol{B}_s \boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{Q}}^{\perp *} \boldsymbol{F}_0^{-1} \boldsymbol{x}_{\boldsymbol{Q}}^{\perp}$.

$$\boldsymbol{\Gamma}_N = \sigma^{-4} \left[\frac{1}{4N} \sum_{j=1}^{N} g_{k,j} g_{l,j} tr\left[\boldsymbol{B}_{s,j} \boldsymbol{h}_j \boldsymbol{h}_j^* \right]^2;\ k,l \in \overline{1,q} \right] + \boldsymbol{O}_\sigma(1).$$

<u>Hence, for any asymptotically admissible test</u> $\varphi\left(\boldsymbol{\Delta}(\boldsymbol{x}_N; \boldsymbol{\theta})\right) \in K_\alpha^{AP}$ we have: $\lim_{N\to\infty} \beta_{\varphi_N}(\boldsymbol{\theta} + \gamma_N \boldsymbol{\vartheta}) = b(\boldsymbol{\vartheta}) \to 1$ when $\sigma \to 0$.

Asymptotically efficient estimates of parameters in the Le Cam LAN conditions

Assume that distributions family $p_{\boldsymbol{X}_N}(\boldsymbol{\theta})$ of observations has the Le Cam LAN expansion for $\boldsymbol{\theta} \in \Theta \subset R^q$ with AS statistic $\boldsymbol{\Delta}(\boldsymbol{X}_N;\boldsymbol{\theta})$ and AFM $\boldsymbol{\Gamma}_N(\boldsymbol{\theta})$. Le Cam introduced asymptotically efficient estimate of parameter $\boldsymbol{\theta}_0$ in the form

$$\bar{\boldsymbol{\theta}}_N(\boldsymbol{X}_N) = \boldsymbol{A}\left(\boldsymbol{X}_N,\hat{\boldsymbol{\theta}}_N\right) = \hat{\boldsymbol{\theta}}_N + \gamma_N \boldsymbol{\Gamma}_N^{-1}\left(\hat{\boldsymbol{\theta}}_N\right)\boldsymbol{\Delta}\left(\boldsymbol{X}_N;\hat{\boldsymbol{\theta}}_N\right),\ \gamma_N = \frac{1}{\sqrt{N}},\ \hat{\boldsymbol{\theta}}_N(\boldsymbol{X}_N) \text{ is any } \sqrt{N}\text{ - consistent estimate of } \boldsymbol{\theta}_0 \in \Theta.$$

Regarding $\boldsymbol{A}(\boldsymbol{X}_N,\boldsymbol{\theta})$ as "improving" operator for estimate $\hat{\boldsymbol{\theta}}_N$, we can implement it iteratively and to analyze the estimate:

$$\tilde{\boldsymbol{\theta}}_N(n,\boldsymbol{X}_N,r) = \boldsymbol{A}^n\left(\boldsymbol{X}_N,\boldsymbol{\theta}^{(0)}\right),\text{ where } \boldsymbol{\theta}^{(0)} \in S_r(\boldsymbol{\theta}_0) = \left\{\boldsymbol{\theta} : |\boldsymbol{\theta}-\boldsymbol{\theta}_0| < r\right\},\ \boldsymbol{\theta}^{(0)} \text{ is some initial point of iterations.}$$

If estimate $\breve{\boldsymbol{\theta}}(\boldsymbol{x}_N,r) = \lim\limits_{n\to\infty} \boldsymbol{A}^n\left(\boldsymbol{X}_N,\boldsymbol{\theta}^{(0)}\right)$ exists, it (under some restrictions) is a root of equation $\boldsymbol{\Delta}(\boldsymbol{X}_N,\boldsymbol{\theta}) = 0$.

We apply a restrictions **B** on $\boldsymbol{\Delta}(\boldsymbol{X}_N,\boldsymbol{\theta})$, $\boldsymbol{\Gamma}_N(\boldsymbol{\theta})$ regarded as functions of $\boldsymbol{\theta} \in \Theta$ in $\mathbb{C}(\Theta)$ with the measure induced by $p_{\boldsymbol{X}_N}(\boldsymbol{\theta}_0)$:

B1.1) *Random function* $\boldsymbol{\Delta}(\boldsymbol{X}_N;\boldsymbol{\theta}) \in \mathbb{C}(\Theta)$ *has in* $\mathbb{C}(\Theta)$ *partial derivatives* $\boldsymbol{Q}(\boldsymbol{X}_N,\boldsymbol{\theta}) = \left[\Delta'_{k,l}(\boldsymbol{X}_N;\boldsymbol{\theta});\ k,l \in \overline{1,q}\right]$:

$$\boldsymbol{\Delta}(\boldsymbol{X}_N,\boldsymbol{\rho}) - \boldsymbol{\Delta}(\boldsymbol{X}_N,\boldsymbol{\theta}) = \left[\boldsymbol{Q}(\boldsymbol{X}_N,\boldsymbol{\theta}) + \boldsymbol{\Lambda}(\boldsymbol{X}_N,\boldsymbol{\rho},\boldsymbol{\theta})\right](\boldsymbol{\rho}-\boldsymbol{\theta}),\ \lim_{r\to 0} P_{\boldsymbol{\theta}_0}\left\{\sup_{|\boldsymbol{\rho}-\boldsymbol{\theta}|<r} \left\|\boldsymbol{\Lambda}(\boldsymbol{X}_N,\boldsymbol{\rho},\boldsymbol{\theta})\right\| > \varepsilon\right\} = 0\ .$$

B1.2) *Random functions* $\gamma_N \Delta_k(\boldsymbol{X}_N;\boldsymbol{\theta})$ *and* $\gamma_N \Delta'_{k,l}(\boldsymbol{x}_N;\boldsymbol{\theta})$ *converge in* $\mathbb{C}(\Theta)$ *to* $t_k(\boldsymbol{\theta},\boldsymbol{\theta}_0)$ *and* $t_{k,l}(\boldsymbol{\theta},\boldsymbol{\theta}_0)$;

B1.3) *Matrix* $\boldsymbol{T}(\boldsymbol{\theta};\boldsymbol{\theta}_0) = \left[t_{k,l}(\boldsymbol{\theta},\boldsymbol{\theta}_0) = \dfrac{\partial t_k(\boldsymbol{\theta},\boldsymbol{\theta}_0)}{\partial \theta_k},\ k,l \in \overline{1,q}\right]$ *has* $\inf\limits_{\boldsymbol{\theta}\in\Theta} \left|\det \boldsymbol{T}(\boldsymbol{\theta};\boldsymbol{\theta}_0)\right| > 0$;

B2.1) $\left\|\boldsymbol{\Gamma}_N(\boldsymbol{\theta}) - \boldsymbol{\Gamma}_N(\boldsymbol{\vartheta})\right\| < c|\boldsymbol{\theta}-\boldsymbol{\vartheta}|$ *for all* $\boldsymbol{\theta},\boldsymbol{\vartheta} \in \Theta$;

B2.2) *There exists an uniform for* $\boldsymbol{\theta} \in \Theta$ *limit*: $\boldsymbol{\Gamma}(\boldsymbol{\theta}) = \lim\limits_{N\to\infty} \boldsymbol{\Gamma}_N(\boldsymbol{\theta})$, *and* $\boldsymbol{\Gamma}(\boldsymbol{\theta}) = -\boldsymbol{T}(\boldsymbol{\theta}_0;\boldsymbol{\theta}_0)$; $\inf\limits_{\boldsymbol{\theta}\in\Theta} \det \boldsymbol{\Gamma}(\boldsymbol{\theta}) = \delta > 0$.

Theorem 7. Let conditions* B *are satisfied. Then there exists* $r>0$*, such that for any initial point* $\boldsymbol{\theta}^{(0)} \in S_r(\boldsymbol{\theta}_0)$ *the estimate* $\breve{\boldsymbol{\theta}}(\boldsymbol{X}_N,r) = \lim\limits_{n\to\infty} \boldsymbol{A}^n\left(\boldsymbol{X}_N,\boldsymbol{\theta}^{(0)}\right)$ *is an unique root of equation* $\Delta(\boldsymbol{X}_N,\boldsymbol{\theta}) = 0,\ \boldsymbol{\theta} \in S_r(\theta_0)$, *and

$$\sqrt{N}\left(\breve{\boldsymbol{\theta}}(\boldsymbol{X}_N,r) - \boldsymbol{\theta}_0\right) \xrightarrow[N\to\infty]{D} N\left(\boldsymbol{\theta}_0,\boldsymbol{\Gamma}^{-1}(\boldsymbol{\theta}_0)\right).$$

Class $K^{AN}(\delta)$ of asymptotically normal estimates based on random functions $\delta(X_N,\theta)$

Consider random function $\delta(X_N,\theta)$, in $\mathbb{C}(\Theta)$ with the measure induced by $p_{X_N}(\theta_0)$, and matrix functions $W_N(\theta)$.

Theorem 8.

Assume that: 1) ***Statistic*** $\delta(X_N,\theta_0)$ ***is asymptotically normal with parameters*** $(0,U(\theta_0))$ ***when*** $L\{X_N\}=P_{X_N}(\theta_0)$

2) $\delta(X_N,\theta)$ ***and*** $W_N(\theta)$ ***satisfy conditions*** **B** (*to be substituted instead* Δ *and* Γ).

Then exists $r>0$ ***such that estimate***
$$\hat{\theta}_N(X_N,r)=\begin{cases} \text{root of equation}: \delta(X_N;\theta)=0,\ \theta\in S_r(\theta_0), \\ \text{when this exists and unique;} \\ \text{any } \theta\in S_r(\theta_0) \text{ in opposite case;}\end{cases}$$

is asymptotically normal: $\sqrt{N}\left(\hat{\theta}(X_N,r)-\theta_0\right)\xrightarrow[N\to\infty]{D} N(\theta_0,D(\theta_0))$, ***where*** $D(\theta_0)=W^{-1}(\theta_0)U(\theta_0)W^{-1}(\theta_0)$.

Note 1. If asymptotically normal estimate $\breve{\theta}(X_N)$ has the all moments starting form some N, then it is asymptotically minimax.

Note 2. Condition 1) of Theorem 8 implies the equation: $t_k(\theta_0,\theta_0)=0$. Validity of this equation determines a class $K^{AN}(P)=\{P_{X_N}(\theta_0)\}$ of distributions for which estimate $\hat{\theta}_N(X_N)$ can preserve its γ_N-consistency and asymptotic normality. Class $K^{AN}(P)$ is a class of "stability" of the estimate asymptotical properties when a real observations distribution deviates from supposed one.

Note 2. For Gaussian observations satisfying conditions **A** the estimate $\breve{\theta}(X_N,r)$ can be calculated as maximal point for $\theta\in S_r(\theta_0)$ of the asymptotic likelihood function for DFFT of observations:

$$\breve{\theta}(X_N,r)=\underset{\theta\in S_r(\theta_0)}{arg\,max}\left\{\sum_{j=1}^{N}\left[\frac{1}{2}\ln\det F_j(\theta)+\left(\breve{x}_j-s_j(\theta)\right)^{*}F_j^{-1}(\theta)\left(\breve{x}_j-s_j(\theta)\right)\right]\right\};$$

where $$\breve{x}_j=\frac{1}{\sqrt{N}}\sum_{j=1}^{N}x_t e^{i\lambda_j t},\ \lambda_j=\frac{2\pi j}{N},\quad j=1,\dots,N\ .$$

Geophysical estimation problems

On the basis of theorems 7 and 8 we derived computationally efficient estimates for number geophysical problems:

1. Estimation of MPSD $\boldsymbol{F}(\lambda)$ of Gaussian multidimensional observations $\boldsymbol{x}_t \in R^m$, $t=1,\dots,N$, by approximation $\boldsymbol{F}(\lambda)$ with MPSD of multidimensional autoregressive – moving averaged time series: $\boldsymbol{F}(\lambda)$ is approximated by the function

$$\boldsymbol{F}(\lambda,\boldsymbol{\theta}) = \boldsymbol{A}^{-1}(\lambda)\boldsymbol{D}(\lambda)\boldsymbol{A}^{-1}(\lambda);\ \ \boldsymbol{A}(\lambda) = -\sum_{k=0}^{p}\boldsymbol{A}_k e^{ik\lambda},\ \boldsymbol{A}_0 = -\boldsymbol{I};\ \ \boldsymbol{D}(\lambda) = \sum_{k=-q}^{q}\boldsymbol{D}_k e^{ik\lambda},$$

where $\boldsymbol{\theta} = \left\{A_{j,k}, B_{j,k}; j,k \in \overline{1,m}\right\}$.

2. Estimation parameters of Earth medium models based on multidimensional observations.
3. Estimation of coordinates of seismic sources based on observations of seismic arrays.

The common model of observations in problems 2, 3 is the following:

$$\boldsymbol{x}_t = \boldsymbol{G}_t(\boldsymbol{\vartheta}) * \boldsymbol{y}_t + \boldsymbol{\zeta}_t;\ \ \boldsymbol{x}_t \in R^m,\quad \boldsymbol{y}_t = \boldsymbol{s}_t(\boldsymbol{\rho}) + \boldsymbol{\mu}_{t,\boldsymbol{\rho}},\ \ \boldsymbol{y}_t \in R^r,\ \ t \in (1,\dots,N), \tag{1}$$

where $\boldsymbol{G}_t(\boldsymbol{\vartheta})$ is impulse response of a Earth medium modeled by a linear operator depending on parameters $\boldsymbol{\vartheta} \in R^r$, which are informative parameters of the estimation problems, $\boldsymbol{y}_t$ - sounding signal for medium investigation or seismic source signal depending from unknown nuisance parameters $\boldsymbol{\rho} \in R^p$.

In common case $\boldsymbol{y}_t$ has deterministic component $\boldsymbol{s}_t$ and random component $\boldsymbol{\mu}_{t,\boldsymbol{\rho}}$, regarded as Gaussian regular time series with MPSD $\boldsymbol{F}_\mu(\lambda,\boldsymbol{\rho})$ depending from parameters $\boldsymbol{\rho} \in R^p$, ζ_t is Gaussian regular time series modeling the noise .

The total parameters to be estimated in this case are $\boldsymbol{\theta} = \left(\boldsymbol{\vartheta}^T, \boldsymbol{\rho}^T\right)^T$. The observable multivariate time series $\boldsymbol{x}_t$ has a form: $\boldsymbol{x}_t = \boldsymbol{a}_t(\theta) + \boldsymbol{\eta}_t$, where $\boldsymbol{a}_t(\boldsymbol{\theta}) = \boldsymbol{G}_t(\boldsymbol{\vartheta}) * \boldsymbol{s}_t(\boldsymbol{\rho})$ is deterministic, $\boldsymbol{\eta}_t$ is a regular Gaussian time series with MPSD

$$\boldsymbol{F}(\lambda,\boldsymbol{\theta}) = \boldsymbol{F}_\zeta(\lambda) + \boldsymbol{G}(\lambda,\boldsymbol{\vartheta})\boldsymbol{F}_\mu(\lambda,\boldsymbol{\rho})\boldsymbol{G}^*(\lambda,\boldsymbol{\vartheta}), \tag{2}$$

If $\boldsymbol{a}_t(\boldsymbol{\theta})$ and $\boldsymbol{F}(\lambda,\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \Theta$ satisfy conditions **A**, then distribution of the observations $\boldsymbol{X}_N = \left(\boldsymbol{x}_t^T, t \in \overline{1,N}\right)$ has LAN property. If then AS statistic $\boldsymbol{\Delta}(\boldsymbol{X}_N,\boldsymbol{\theta})$ and AFM $\boldsymbol{\Gamma}_N(\boldsymbol{\theta})$ satisfy conditions **B** it is possible to construct AE estimate of the parameters $\boldsymbol{\theta} = \left(\boldsymbol{\vartheta}^T, \boldsymbol{\rho}^T\right)^T$ as the root of equation $\boldsymbol{\Delta}(\boldsymbol{X}_N,\boldsymbol{\theta}) = 0$.

Example of $N^{1/2+\nu}$-consistent estimate, $\nu < 1/2$

Consider the case when the output signal $\boldsymbol{x}_t$ of multidimensional linear system $\boldsymbol{H}_t(\boldsymbol{\theta})$ and its input signal $\boldsymbol{y}_t$ both are observing on a random noise background:

$$\boldsymbol{x}_t = \left(\boldsymbol{z}_t^T, \boldsymbol{y}_t^T\right)^T \text{, where } \boldsymbol{z}_t = \boldsymbol{H}_t(\boldsymbol{\vartheta}) * \boldsymbol{s}_t(\boldsymbol{\rho}) + \boldsymbol{\eta}_t,\ \boldsymbol{s}_t(\boldsymbol{\rho}) \text{ is deterministic; } \quad \boldsymbol{y}_t = \boldsymbol{s}_t(\boldsymbol{\rho}) + \boldsymbol{\xi}_t.$$

Then $\boldsymbol{x}_t = \boldsymbol{G}_t(\boldsymbol{\vartheta}) * \boldsymbol{s}_t(\boldsymbol{\rho}) + \boldsymbol{\zeta}_t$; where $\boldsymbol{G}_t(\boldsymbol{\vartheta}) = \left[\boldsymbol{H}_t^T(\boldsymbol{\vartheta}), \boldsymbol{I}\right]^T$, $\boldsymbol{\zeta}_t = \left(\boldsymbol{\eta}_t^T, \boldsymbol{\xi}_t^T\right)^T$, $t \in (1,\dots,N)$.

In this case AS statistic has the form:

$$\boldsymbol{\varDelta}(\boldsymbol{X}_N, \boldsymbol{\theta}) = \left(\frac{1}{\sqrt{N}} \operatorname{Re} \sum_{j=1}^{N} \left(\breve{\boldsymbol{x}}_j - \boldsymbol{u}_j(\boldsymbol{\theta})\right)^* \boldsymbol{F}_{\zeta,j}^{-1} \boldsymbol{u}'^{\alpha}_{k,j}(\boldsymbol{\theta});\ \alpha \in \{\vartheta, \rho\},\ k \in (1,\dots,q_\alpha) \right), \quad \boldsymbol{\theta} = \left(\boldsymbol{\vartheta}^T, \boldsymbol{\rho}^T\right)^T;$$

$$\boldsymbol{u}_j(\boldsymbol{\theta}) = \boldsymbol{H}(\lambda_j, \boldsymbol{\vartheta}) \boldsymbol{s}_j(\boldsymbol{\rho}),\ \boldsymbol{u}'^{\vartheta}_{k,j} = \frac{\partial}{\partial \vartheta_k} \boldsymbol{u}_j,\ \boldsymbol{u}'^{\rho}_{k,j} = \frac{\partial}{\partial \rho_k} \boldsymbol{u}_j,\ q_\vartheta = p,\ q_\rho = r.$$

In many practical case noise processes $\boldsymbol{\eta}_t$, $\boldsymbol{\xi}_t$ are high correlated because they have the structure:

$$\boldsymbol{\xi}_t = \boldsymbol{Q}_{1t} * \boldsymbol{\varepsilon}_t + \boldsymbol{v}_{1t},\ \boldsymbol{\eta}_t = \boldsymbol{Q}_{2t} * \boldsymbol{\varepsilon}_t + \boldsymbol{v}_{2t},$$

where $\boldsymbol{v}_{1t}$, $\boldsymbol{v}_{2t}$, $\boldsymbol{\varepsilon}_t$ are mutually non-correlated.

If $\boldsymbol{v}_{1t}$, $\boldsymbol{v}_{2t}$ are absent, $\det \boldsymbol{F}_\zeta(\lambda) \equiv 0$ and it is impossible to use equation $\boldsymbol{\varDelta}(\boldsymbol{X}_N, \boldsymbol{\theta}) = 0$ to derive the AE estimate $\breve{\boldsymbol{\theta}}(\boldsymbol{X}_N)$

Let us replace the inverse matrix $\boldsymbol{F}_\zeta^{-1}(\lambda)$ by pseudo–inverse matrix $\boldsymbol{B}_\zeta(\lambda)$ and consider nonlinear equation:

$$\boldsymbol{\delta}(\boldsymbol{X}_N, \boldsymbol{\theta}) = \left(\frac{1}{\sqrt{N}} \operatorname{Re} \sum_{j=1}^{N} \left(\breve{\boldsymbol{x}}_j - \boldsymbol{u}_j(\boldsymbol{\theta})\right)^* \boldsymbol{B}_{\zeta,j} \boldsymbol{u}'^{\alpha}_{k,j}(\boldsymbol{\theta});\ \alpha \in \{\vartheta, \rho\},\ k \in (1,\dots,q_\alpha) \right) = 0, \quad \boldsymbol{\theta} \in S_r(\boldsymbol{\theta}_0), \tag{1}$$

Random function $\boldsymbol{\delta}(\boldsymbol{X}_N, \boldsymbol{\theta})$ satisfies restrictions of the Theorem 8.

Theorem 9. Estimate $\bar{\boldsymbol{\theta}}(\boldsymbol{X}_N)$ ***- root of equation* (1), *is*** $N^{(1/2+\nu)}$***-consistent*:**

$$P_{N,\boldsymbol{\theta}_0}\left\{N^{(1/2+\nu)} \left|\bar{\boldsymbol{\theta}}(\boldsymbol{X}_N) - \boldsymbol{\theta}_0\right| > c\right\} \to 0\ (N \to \infty), \textbf{\textit{ where }} \nu = (1-\beta)/2;$$

$$\beta \in (0,1) \textbf{\textit{ satisfies the relation: }} \max_{1<t<N} \left|\boldsymbol{u}'_{k,t}(\boldsymbol{\theta})\right|^2 < CN^\beta,\ \boldsymbol{u}_t(\boldsymbol{\theta}) = \boldsymbol{H}(\boldsymbol{\vartheta}) * \boldsymbol{s}_t(\boldsymbol{\rho}).$$

Example of estimation problem with increasing proportionally N number of nuisance parameters

The model of observations: $\boldsymbol{x}_t = \boldsymbol{h}_t(\boldsymbol{r}) * s_t + \boldsymbol{\zeta}_t \in R^m$, $s_t \in R^1$ is a seismic source signal, $\boldsymbol{r}$ are coordinates of the source, $\boldsymbol{h}_t(\boldsymbol{r})$ is impulse response of Earth medium for the source signal, $\boldsymbol{\zeta}_t$ is GRTS with MPSD $\boldsymbol{F}(\lambda)$. If $s_t = \mu_{t,\boldsymbol{\rho}}$ is a GRTS with PSD $f_\mu(\lambda,\boldsymbol{\rho})$ the AE estimate of parameter $\boldsymbol{\theta} = (\boldsymbol{r},\boldsymbol{\rho})$ can be calculated as (accordingly Note 2 for Theorem 7):

$$\breve{\boldsymbol{\theta}}(\boldsymbol{X}_N, r) = \underset{\boldsymbol{\theta}\in S_r(\boldsymbol{\theta}_0)}{arg\,max}\left\{\sum_{j=1}^{N}\left[\frac{1}{2} ln\,det\,\boldsymbol{F}_j(\boldsymbol{\theta}) + \breve{\boldsymbol{x}}_j^{\,*}\boldsymbol{F}_j^{-1}(\boldsymbol{\theta})\breve{\boldsymbol{x}}_j\right]\right\} \quad \text{where } \boldsymbol{F}_j(\boldsymbol{\theta}) = \boldsymbol{F}_\zeta(\lambda_j) + f_\mu(\lambda_j,\boldsymbol{\rho})\boldsymbol{h}(\lambda_j,\boldsymbol{r})\boldsymbol{h}^*(\lambda_j,\boldsymbol{r}).$$

or:
$$\breve{\boldsymbol{\theta}}(\boldsymbol{X}_N) = \underset{\boldsymbol{\theta}\in S_r(\boldsymbol{\theta}_0)}{arg\,max}\sum_{j=1}^{N}\left(\frac{1}{2} ln\,det\left[\boldsymbol{F}_{\zeta,j} + f_{\mu,j}(\boldsymbol{\rho})\boldsymbol{h}_j(\boldsymbol{r})\boldsymbol{h}_j^*(\boldsymbol{r})\right] + \frac{\left|\boldsymbol{h}_j^*(\boldsymbol{r})\boldsymbol{F}_{\zeta,j}^{-1}\breve{\boldsymbol{x}}_j\right|^2}{f_{\mu,j}^{-1}(\boldsymbol{\rho}) + \boldsymbol{h}_j^*(\boldsymbol{r})\boldsymbol{F}_{\zeta,j}^{-1}\boldsymbol{h}_j(\boldsymbol{r})}\right).$$

If there is not a'priory information about source signal s_t it can be regarded as some completely unknown time series. In this case all $s_1,\ldots,s_N$ are nuisance parameters. We can try to estimate of informative parameters $\boldsymbol{r}$ as following:

$$\breve{\boldsymbol{r}}(\boldsymbol{X}_N, r) = \underset{\substack{\boldsymbol{r}\in S_r(\boldsymbol{r}_0)\\ \breve{s}_j,\, j\in\overline{1,N}}}{arg\,max}\left\{\sum_{j=1}^{N}\left[\frac{1}{2} ln\,det\,\boldsymbol{F}_{\zeta,j} + \left(\breve{\boldsymbol{x}}_j - \boldsymbol{h}_j(\boldsymbol{r})s_j\right)^*\boldsymbol{F}_{\zeta,j}^{-1}\left(\breve{\boldsymbol{x}}_j - \boldsymbol{h}_j(\boldsymbol{r})s_j\right)\right]\right\},$$

$$\breve{\boldsymbol{r}}(\boldsymbol{X}_N, r) = \underset{\boldsymbol{r}\in S_r(\boldsymbol{r}_0)}{arg\,max}\sum_{j=1}^{N}\frac{\left|\boldsymbol{h}_j(\boldsymbol{r})^*\boldsymbol{F}_{\zeta,j}^{-1}\breve{\boldsymbol{x}}_j\right|^2}{\boldsymbol{h}_j(\boldsymbol{r})^*\boldsymbol{F}_{\zeta,j}^{-1}\boldsymbol{h}_j(\boldsymbol{r})}. \qquad (1)$$

Theorem 10. Let **1)** $\boldsymbol{a}_t(\boldsymbol{r}) = \boldsymbol{h}_t(\boldsymbol{r}) * s_t$ ***satisfy conditions*** **A1** ***and exists limit*** $T(\lambda) = \lim_{N\to\infty}\sum_{\lambda_j<\lambda}\frac{1}{N}\left|\sum_{t=1}^{N}s_t e^{i\frac{\lambda_j t}{N}}\right|^2 \in C[0,2\pi]$ **,**
2) $det\,\boldsymbol{F}_\zeta(\lambda) > 0$ ***and*** $\left\|\boldsymbol{F}(\lambda_1) - \boldsymbol{F}(\lambda_2)\right\| < c|\lambda_1 - \lambda_2|$. ***Then there exists*** $r > 0$ ***such that estimate*** (1) ***is*** $\sqrt{N}$ ***-consistent and asymptotically normal with parameters*** $(\theta_0,\boldsymbol{\Phi})$ ***where*** $\boldsymbol{\Phi} = \boldsymbol{\Psi}^{-1}(\boldsymbol{r}) + \boldsymbol{\Psi}^{-1}(\boldsymbol{r})\boldsymbol{\Sigma}(\boldsymbol{r})\boldsymbol{\Psi}^{-1}(\boldsymbol{r})$,

$$\boldsymbol{\Psi}(\boldsymbol{r})=\left[\frac{1}{2\pi}\int_0^{\pi} \boldsymbol{d}_k'^*(\lambda,\boldsymbol{r})\left(\boldsymbol{I}-\boldsymbol{K}(\lambda,\boldsymbol{r})\right)\boldsymbol{d}_l'(\lambda,\boldsymbol{r})\,dT(\lambda);\ k,l\in 1,3\right];\quad \boldsymbol{\Sigma}(\boldsymbol{r})=\left[\frac{1}{2\pi}\int_0^{\pi} tr\left\lfloor \boldsymbol{K}_k'(\lambda,\boldsymbol{r})\boldsymbol{K}_l'(\lambda,\boldsymbol{r})\right\rfloor d\lambda;\ k,l\in 1,3\right];$$

$$\boldsymbol{d}(\lambda,\boldsymbol{r})=\boldsymbol{F}_\zeta^{-1/2}(\lambda)\boldsymbol{h}(\lambda,\boldsymbol{r});\quad \boldsymbol{K}(\lambda,\boldsymbol{r})=\frac{\boldsymbol{d}(\lambda,\boldsymbol{r})\boldsymbol{d}^*(\lambda,\boldsymbol{r})}{|\boldsymbol{d}(\lambda,\boldsymbol{r})|^2},\quad \boldsymbol{d}_k'(\lambda,\boldsymbol{r})=\frac{\partial \boldsymbol{d}(\lambda,\boldsymbol{r})}{\partial r_k},\quad \boldsymbol{K}_k'(\lambda,\boldsymbol{r})=\frac{\partial \boldsymbol{K}(\lambda,\boldsymbol{r})}{\partial r_k}$$

When $\boldsymbol{F}_\zeta(\lambda)=\boldsymbol{I}$ we got well known in seismology algorithm called Emission Seismic Tomography (EST) derived using some heuristic considerations:

$$\breve{\boldsymbol{r}}(\boldsymbol{X}_N,r)=\underset{r\in S_r(r_0)}{arg\,max}\sum_{j=1}^{N}\frac{\left|\boldsymbol{h}_j(\boldsymbol{r})^*\breve{\boldsymbol{x}}_j\right|^2}{\boldsymbol{h}_j(\boldsymbol{r})^*\boldsymbol{h}_j(\boldsymbol{r})}.$$

Linear functional relationship (LFR):

$$\boldsymbol{y}_t=\boldsymbol{s}_t+\boldsymbol{\xi}_{1,t}\in R^p,\ \boldsymbol{z}_t=\boldsymbol{B}\boldsymbol{s}_t+\boldsymbol{\xi}_{2,t}\in R^q,\ t\in\overline{1,N}\ ,\ \boldsymbol{B}\text{ is a } p\times q \text{ matrix, } p<q$$

LFR can be rewritten as: $\boldsymbol{x}_t=\boldsymbol{G}\boldsymbol{s}_t+\boldsymbol{\zeta}_t,\ \boldsymbol{x}_t=\left(\boldsymbol{y}_t^T,\boldsymbol{z}_t^T\right)^T,\ \boldsymbol{G}=\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{B}\end{bmatrix},\ \boldsymbol{\zeta}_t=\left(\xi_{1,t}^T,\xi_{1,t}^T\right)^T$ are independent vectors with $L\{\boldsymbol{\zeta}_t\}=N(0,\boldsymbol{F}),\ \boldsymbol{s}_t\in R^p$ is an unknown sequence of no random vectors. The problem is to estimate $\boldsymbol{B}$ from a samples $\boldsymbol{x}_t,\ t\in\overline{1,N}$ assuming $\boldsymbol{F}$ known. M. Nusbaum (1984) have found the low bound for estimates risks in LFR problem and proved that it is reached by maximum likelihood estimate of matrix $\boldsymbol{B}$.

If we rewrite the last geophysical estimation problem in the "frequency domain" assuming that N is sufficiently large, we get the following asymptotic variant of LFR model:

$\breve{\boldsymbol{x}}_j=\boldsymbol{h}(\lambda_j,\boldsymbol{r})*\breve{s}_j+\boldsymbol{\zeta}_j\in R^m,\ j\in\overline{1,N},\ \boldsymbol{\zeta}_j$ are asymptotically independent vectors with $L\{\zeta_j\}=N(0,\boldsymbol{F}(\lambda_j)),\ \boldsymbol{\lambda}_j=\dfrac{2\boldsymbol{\pi}\,j}{N}$

This variant is more complicated then classical LFR and there is no published results similar to the results of M. Nussbaum.

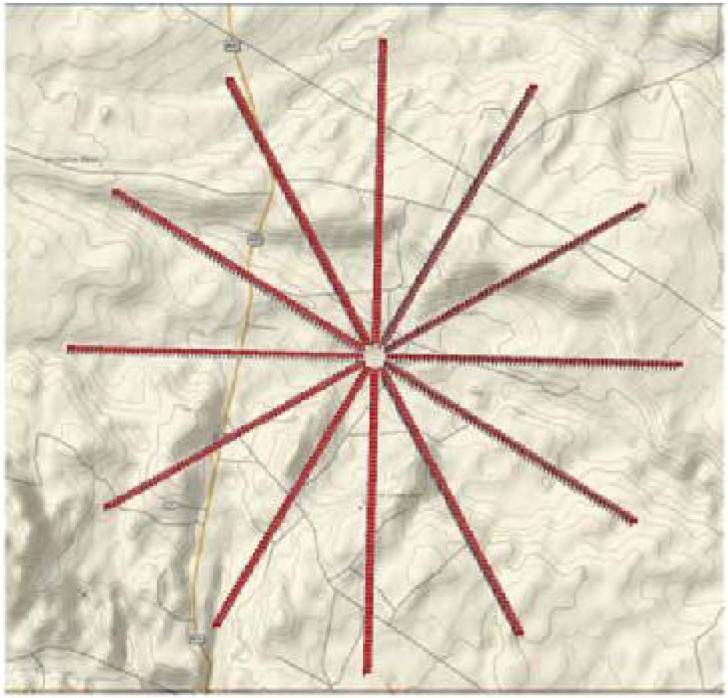

Figure 1: Typical layout for surface array monitor of a vertical well frac. Arms are 3300 m. Station spacing is 35 m.

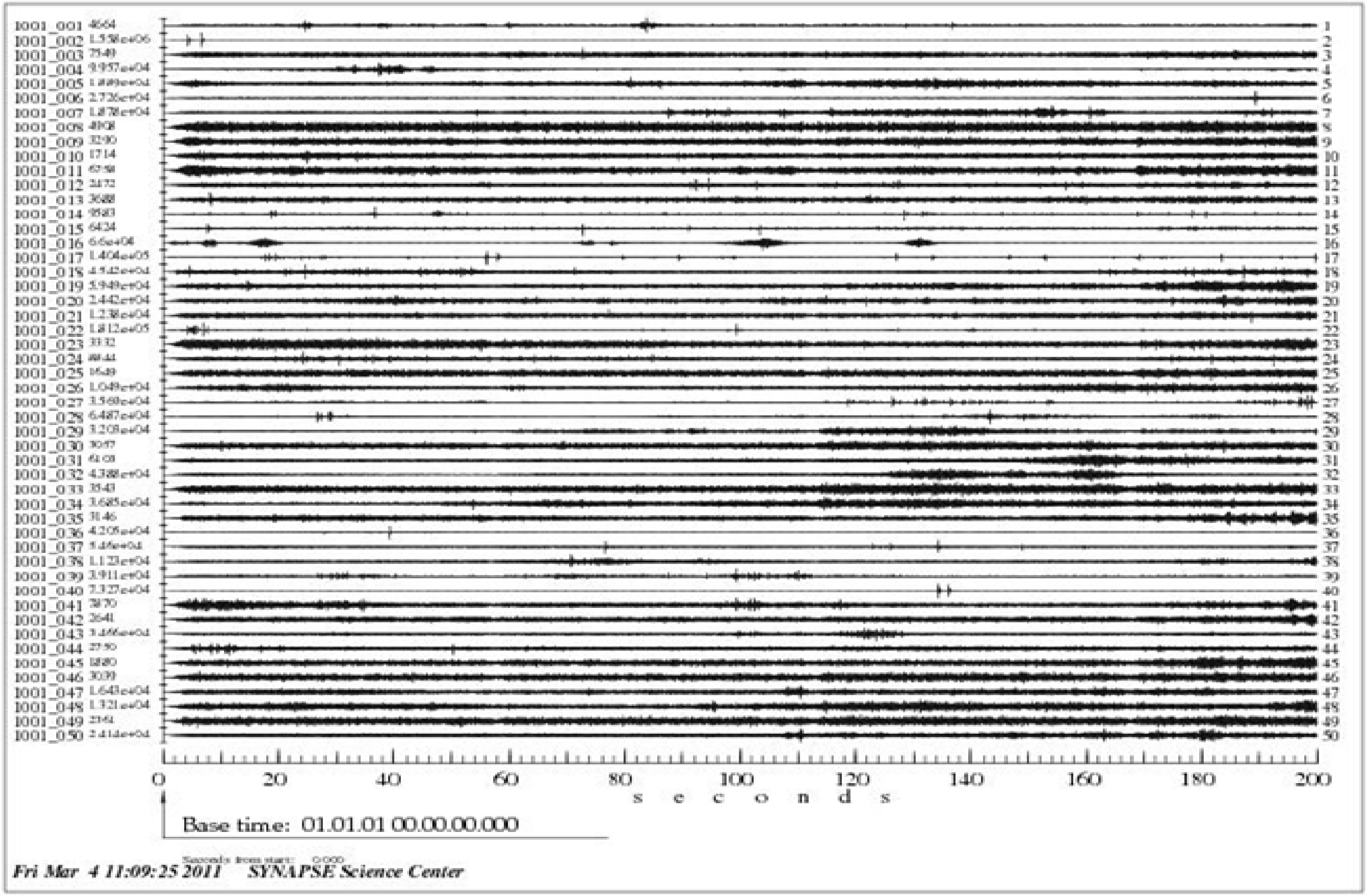

Figure 2: Example of typical mutichannel noise at hydrofracture site

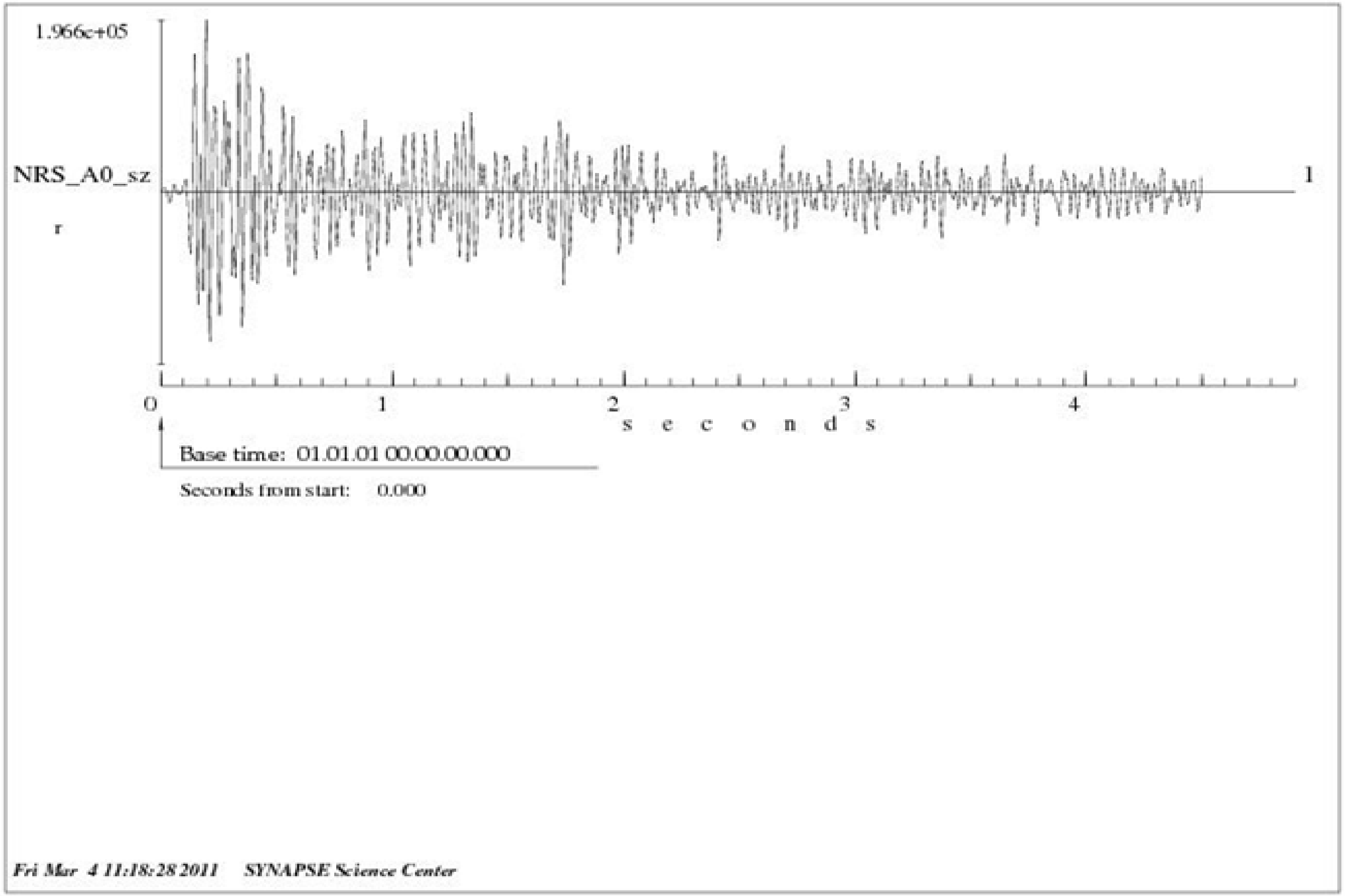

Figure 3: Example of typical seismic signal

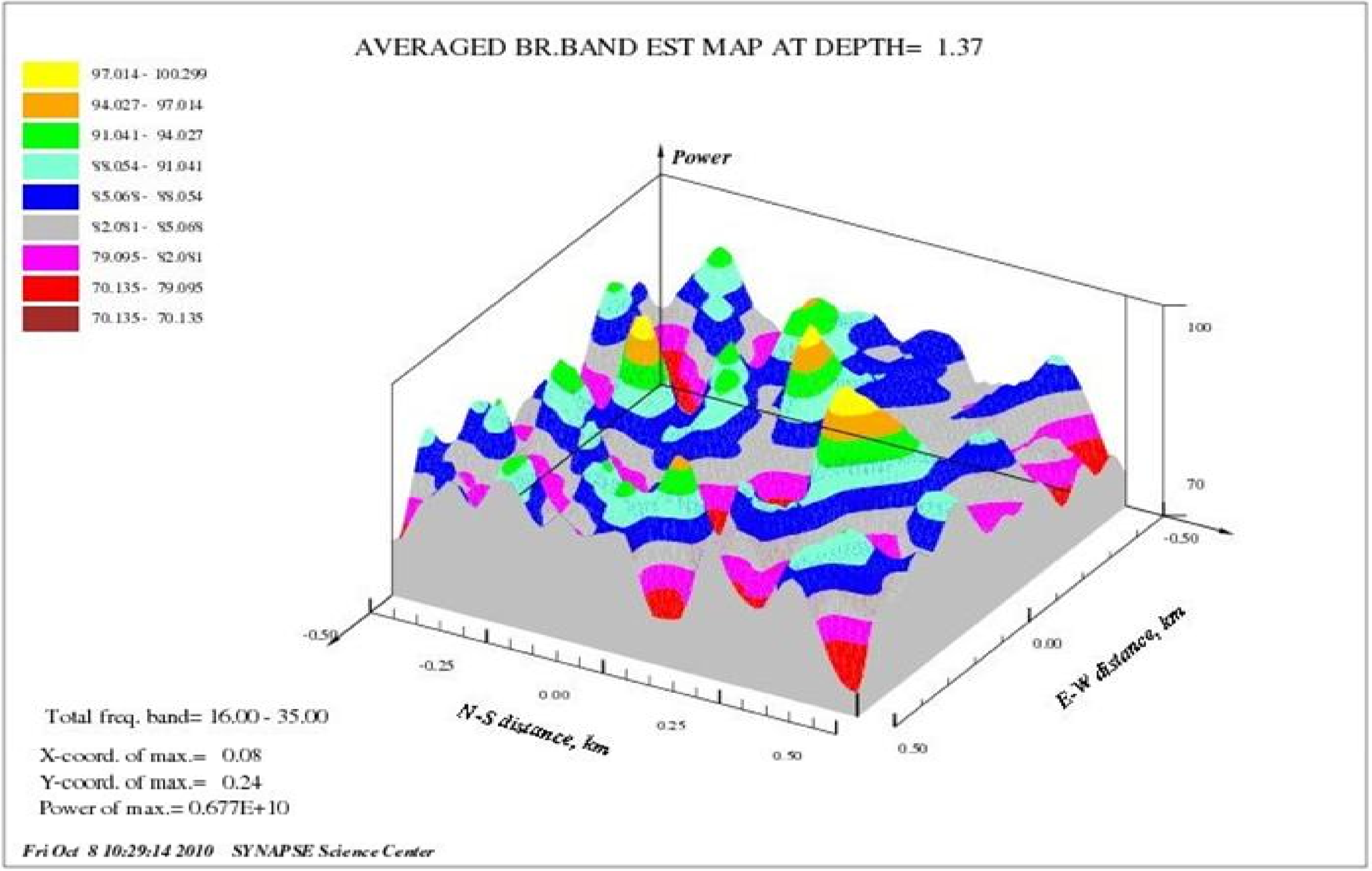

Figure 4. Traditional Emission Tomography functional map for location of micro seismic source. No source is detected.

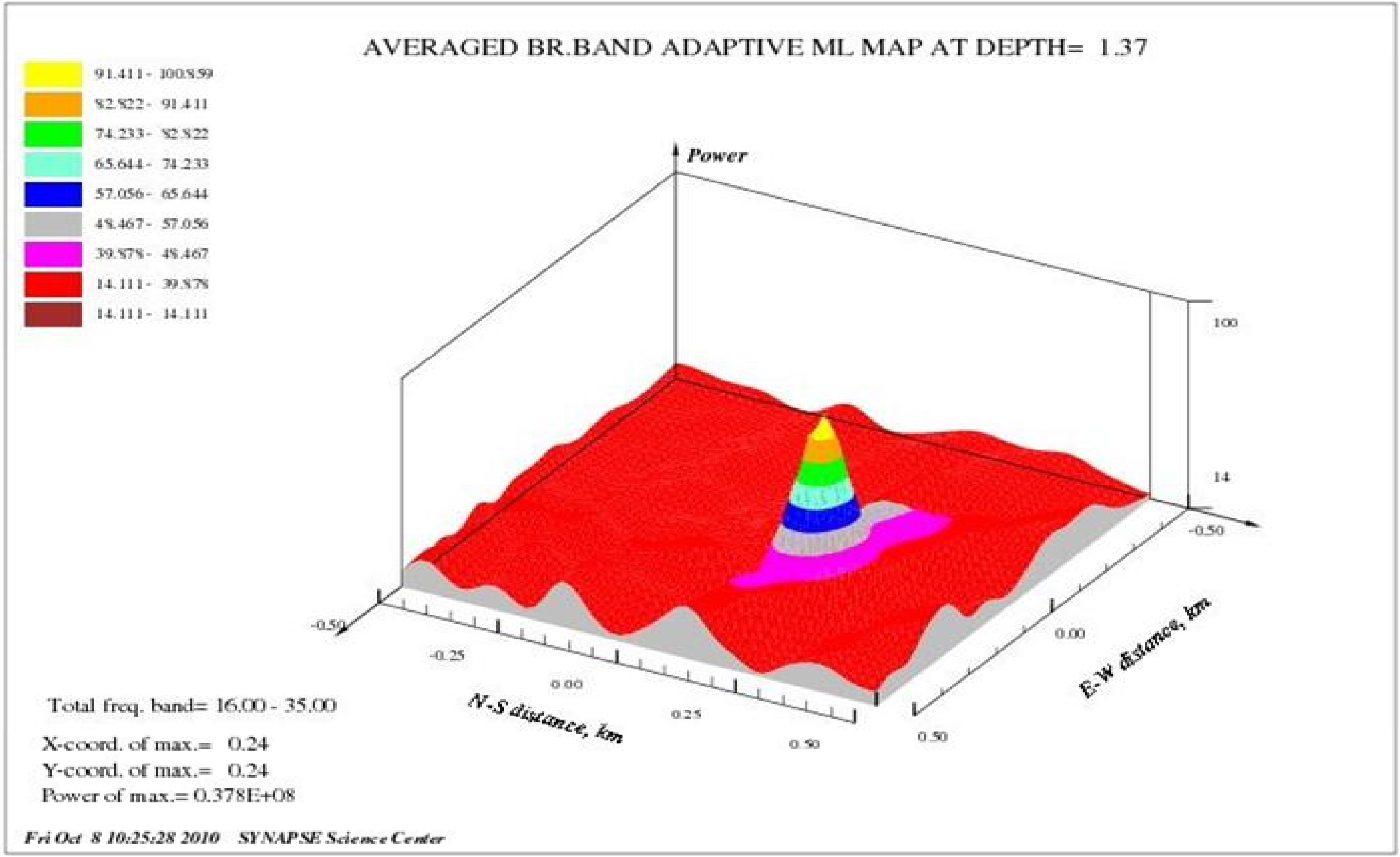

Figure 5: Statistic Maximum Likelihood functional map for location of micro seismic source. The source is successfully detected and accurately located.

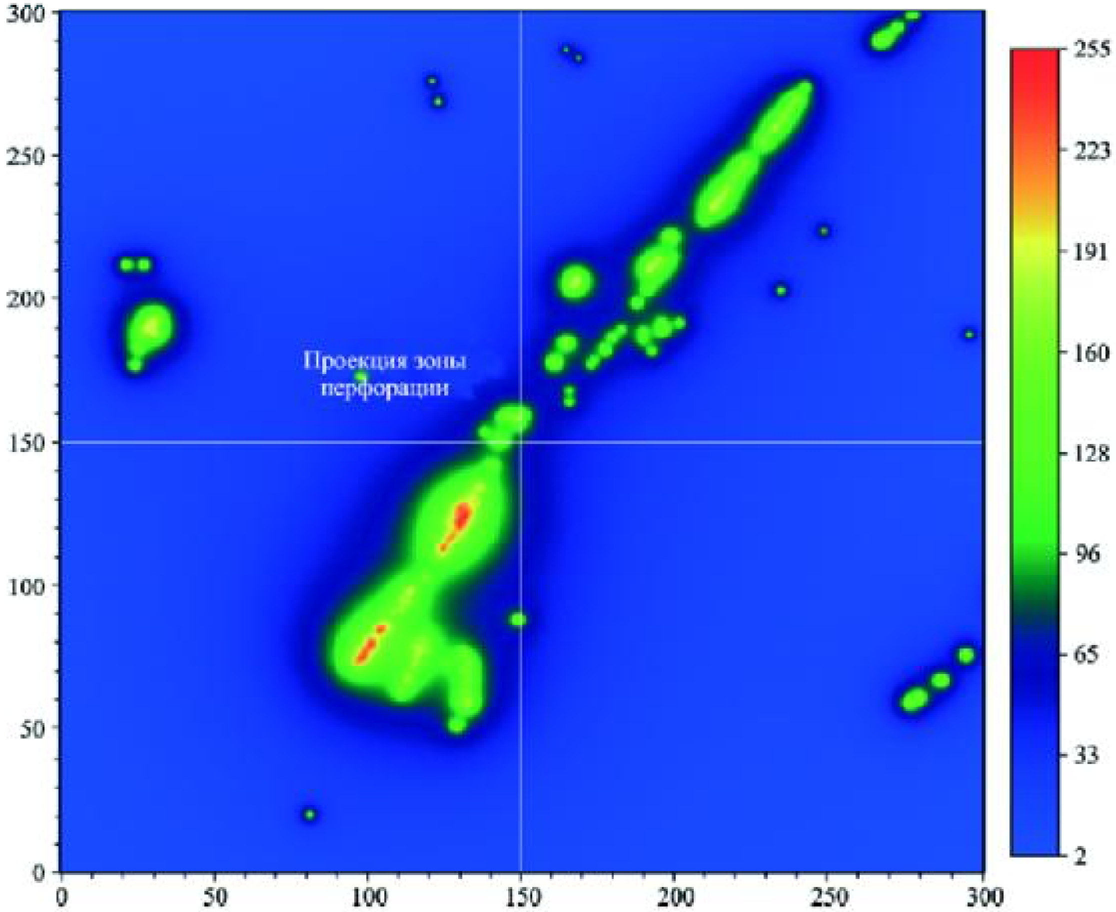

Рис. 1.6. Сейсмическая эмиссия ГРП скважины 5538 Мало-Балыкского месторождения.

Figure 6: Micro seismic emission map during hydrofracture: the sequence of micriseismic events occured during hydrofracture were located using data from surface sesmic array.