

On parameter estimation for periodic diffusion processes II

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Model and Problems

Let us consider the following model of “signal in noise” type

$$dX_t = S(\vartheta, t) dt + b(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

Here $S(\vartheta, t)$ is deterministic τ -periodic signal and $b(X_t) + \sigma(X_t) \dot{W}_t$ is “diffusion noise”. Our problem is to estimate the parameter ϑ by continuous time observations $X^T = (X_t, 0 \leq t \leq T)$.

We suppose that the periodic process X_t has ergodic properties and we describe the properties of estimators in the cases of **phase** $S(\vartheta, t) = S(t - \vartheta)$ and **frequency** $S(\vartheta, t) = S(\vartheta t)$ modulations (or we estimate the shift and scale parameters). Here $S(t)$ is a τ -periodic function, which can be **smooth** or **discontinuous**.

Conditions

We suppose that the functions $b(x), \sigma(x) \in \mathcal{C}_b^3$ and that diffusion coefficient satisfies

$$0 < \kappa \leq \sigma(x)^2 \leq K.$$

The parameter $\vartheta \in (\alpha, \beta) = \Theta$; signal

$$m \leq S(\vartheta, t) \leq M, \quad t \in [0, \tau], \quad \vartheta \in \Theta.$$

Let us put $S_- = \min(m, 0)$ and $S_+ = \max(M, 0)$ and introduce condition: *there exist constants $A > 0$ and $\varepsilon > 0$ such that for $|x| > A$ we have*

$$2x \mathbb{I}_{\{x < -A\}} S_- + 2x \mathbb{I}_{\{x > A\}} S_+ + 2xb(x) + \sigma(x)^2 < -\varepsilon$$

Under these conditions the diffusion process has ergodic properties (Höpfner and Löcherbach, 2011), i.e., there exists an invariant (periodic) density function $f_{\vartheta}(t, x)$ such that for any absolutely integrable function $h(t, x)$ we have (with probability 1) the following limit

$$\frac{1}{T} \int_0^T h(t, X_t) dt \longrightarrow \frac{1}{\tau} \int_{-\infty}^{\infty} \int_0^{\tau} h(t, x) f_{\vartheta}(t, x) dt dx.$$

It can be shown that this convergence is uniform in ϑ .

Note that the existence of the periodic diffusion process, solution of stochastic differential equation with periodic coefficients was proved by Khasminskii in his book (1969).

Note that if $\sigma(x) = \sigma$, then we can denote

$$Y_t = X_t - \int_0^t b(X_s) ds$$

and the problem is reduced to the well-known *signal in white Gaussian noise* model

$$dY_t = S(\vartheta, t) dt + \sigma dW_t, \quad Y_0 = 0, \quad 0 \leq t \leq T.$$

Of course, the case

$$dX_t = S(\vartheta, t, X_t) dt + \sigma(t, X_t) dW_t$$

wher $S(\vartheta, t, x)$ and $\sigma(t, x)$ are τ -periodic functions can be treated by a similar way and the choice of our particular model was motivated by biological applications and by simplicity of exposition.

Example. Let

$$dX_t = S(t) dt - \gamma X_t dt + \sigma dW_t, \quad X_0,$$

where $S(t)$ is a bounded τ -periodic function. Then all mentioned above conditions are fulfilled and

$$f(t, x) = \sqrt{\frac{\gamma}{\pi\sigma^2}} \exp\left\{-\frac{\gamma(x - m(t))^2}{\sigma^2}\right\}$$

where

$$m(t) = \int_0^\tau \frac{e^{-\gamma s}}{1 - e^{-\gamma\tau}} S(t - s) ds$$

Smooth Signal. Shift parameter.

Let

$$dX_t = S(t - \vartheta) dt + b(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

where τ -periodic function $S(t) \in \mathcal{C}^2$. Then the MLE $\hat{\vartheta}_T$ and bayesian estimator $\tilde{\vartheta}_T$ (with positive continuous prior density and quadratic loss function) are consistent, asymptotically normal

$$\sqrt{T} \left(\hat{\vartheta}_T - \vartheta \right) \implies \mathcal{N} \left(0, \mathbf{I}(\vartheta)^{-1} \right), \quad \sqrt{T} \left(\tilde{\vartheta}_T - \vartheta \right) \implies \mathcal{N} \left(0, \mathbf{I}(\vartheta)^{-1} \right)$$

and we have convergence of moments and both estimators are asymptotically efficient.

Here the Fisher information is

$$I(\vartheta) = \frac{1}{\tau} \int_0^\tau \int_{-\infty}^{\infty} \frac{\dot{S}(t - \vartheta)^2}{\sigma(x)^2} f_\vartheta(t, x) dx$$

and from convergence of moments we have

$$\mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta \right)^2 = \frac{I(\vartheta)^{-1}}{T} (1 + o(1))$$

(see Höpfner and K., 2010)

Smooth Signal. Scale parameter.

Let

$$dX_t = S(\vartheta t) dt + b(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

where τ -periodic function $S(t) \in \mathcal{C}^2$. Then X_t is $\frac{\vartheta}{\tau}$ -periodic with $\frac{\vartheta}{\tau}$ -periodic invariant density $f_{\vartheta}(t, x)$.

The MLE $\hat{\vartheta}_T$ and bayesian estimator $\tilde{\vartheta}_T$ are consistent, asymptotically normal

$$T^{3/2} \left(\hat{\vartheta}_T - \vartheta \right) \implies \mathcal{N} \left(0, \mathbf{I}(\vartheta)^{-1} \right), \quad T^{3/2} \left(\tilde{\vartheta}_T - \vartheta \right) \implies \mathcal{N} \left(0, \mathbf{I}(\vartheta)^{-1} \right)$$

and we have convergence of moments. The both estimators are asymptotically efficient.

Here the Fisher information is

$$I(\vartheta) = \frac{\vartheta}{3\tau} \int_0^{\tau/\vartheta} \int_{-\infty}^{\infty} \frac{\dot{S}(\vartheta t)^2}{\sigma(x)^2} f_{\vartheta}(t, x) dx$$

and from convergence of moments we have

$$\mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 = \frac{I(\vartheta)^{-1}}{T^3} (1 + o(1))$$

(see Höpfner and K., 2011)

The proof is based on two general theorems by Ibragimov and Khasminskii (1981). Note that the normalized likelihood ratios

$$Z_T(u) = \exp \left\{ \int_0^T \frac{S(\vartheta_u, t) - S(\vartheta, t)}{\sigma(X_t)^2} dX_t - \frac{1}{2} \int_0^T \frac{S(\vartheta_u, t)^2 - S(\vartheta, t)^2}{\sigma(X_t)^2} dt \right\}$$

where $\vartheta_u = \vartheta + \frac{u}{T^{1/2}}$ and $\vartheta_u = \vartheta + \frac{u}{T^{3/2}}$ in the shift and scale estimation problems respectively. In these smooth cases the corresponding family of measures is LAN, i.e.,

$$Z_T(u) = \exp \left\{ u\Delta_T - \frac{u^2}{2} I(\vartheta) + r_T \right\} \implies Z(u) = \exp \left\{ u\Delta - \frac{u^2}{2} I(\vartheta) \right\}$$

where $\Delta \sim \mathcal{N}(0, I(\vartheta))$

Discontinuous Signal. Shift parameter.

Let

$$dX_t = S(t - \vartheta) dt + b(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

where τ -periodic function $S(t)$ is continuously differentiable on the intervals $[0, \tau_*)$ and $(\tau_*, \tau]$. At point τ_* it has a jump

$$S(\tau_*+) - S(\tau_*-) = R \neq 0.$$

The likelihood ratio $Z_T(u)$ with $\vartheta_u = \vartheta + \frac{u}{T}$ converges to the process

$$Z(u) = \exp \left\{ \Gamma W(u) - \frac{|u|}{2} \Gamma^2 \right\}$$

where $W(\cdot)$ is double-sided Wiener process and

$$\Gamma^2 = R^2 \int_{-\infty}^{\infty} \frac{f_{\vartheta}(\tau_* + \vartheta, x)}{\sigma(x)^2} dx.$$

Let us put

$$Z_0(u) = \exp \left\{ W(u) - \frac{|u|}{2} \right\}$$

and introduce two random variables \hat{u} and \tilde{u} by the relations

$$Z_0(\hat{u}) = \sup_u Z_0(u), \quad \tilde{u} = \frac{\int u Z_0(u) du}{\int Z_0(u) du}$$

Then the MLE $\hat{\vartheta}_T$ and bayesian estimator $\tilde{\vartheta}_T$ are consistent, have the following limits

$$T \left(\hat{\vartheta}_T - \vartheta \right) \Longrightarrow \frac{\hat{u}}{\Gamma^2}, \quad T \left(\tilde{\vartheta}_T - \vartheta \right) \Longrightarrow \frac{\tilde{u}}{\Gamma^2}$$

and the convergence of moments take place.

Remind that in such situation we have the following lower bound on the risk of all estimators $\bar{\vartheta}_T$

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \sup_{|\vartheta - \vartheta_0| < \delta} T^2 \mathbf{E}_{\vartheta} \left(\bar{\vartheta}_T - \vartheta \right)^2 \geq \frac{\mathbf{E}(\tilde{u})^2}{\Gamma^4}.$$

As usual in such problems the BE are asymptotically efficient:

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \sup_{|\vartheta - \vartheta_0| < \delta} T^2 \mathbf{E}_{\vartheta} \left(\tilde{\vartheta}_T - \vartheta \right)^2 = \frac{\mathbf{E}(\tilde{u})^2}{\Gamma^4}.$$

(see Ibragimov and Khasminskii, 1981). Remind that the values

$$\mathbf{E}(\hat{u})^2 = 26, \quad \mathbf{E}(\tilde{u})^2 = 16 \zeta(3) \approx 19,6,$$

are known (Terent'ev 1968, Golubev 1979 and Rubin and Song 1995).

Therefore, for the asymptotically efficient estimator we have

$$\mathbf{E}_{\vartheta} \left(\tilde{\vartheta}_T - \vartheta \right)^2 = \frac{19,6}{\Gamma^2} \frac{1}{T^2} (1 + o(1)).$$

If the function $S(t)$ has jumps in k points $0 < \tau_1, \dots, \tau_k < \tau$ and is continuously differentiable between these points, then the estimators have the same asymptotic properties, but

$$\Gamma^2 = \sum_{l=1}^k \int \frac{[S(\tau_l+) - S(\tau_l-)]^2}{\sigma(x)^2} f_{\vartheta}(\tau_l + \vartheta, x) dx$$

The problem becomes a bit more complicated if

$$dX_t = \sum_{l=1}^k S_l(t - \vartheta_l) dt + b(X_t) dt + \sigma(X_t) dW_t$$

and $\vartheta = (\vartheta_1, \dots, \vartheta_k)$ but can be done too.

Discontinuous Signal. Scale parameter.

Let

$$dX_t = S(\vartheta t) dt + b(X_t) dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T.$$

where 1-periodic ($\tau = 1$) function $S(t)$ is continuously differentiable on the intervals $[0, \tau_*)$ and $(\tau_*, \tau]$. As before, at point τ_* it has a jump

$$S(\tau_*+) - S(\tau_*-) = R \neq 0.$$

The likelihood ratio $Z_T(u)$ with $\vartheta_u = \vartheta + \frac{u}{T^2}$ converges to the same process $Z(u)$ with the parameter

$$\Gamma^2 = R^2 \int_{-\infty}^{\infty} \frac{f_{\vartheta}(\frac{\tau_*}{\vartheta}, x)}{2\sigma(x)^2} dx.$$

The estimators $\hat{\vartheta}_T, \tilde{\vartheta}_T$ are consistent, the limit distributions they have with a different normalization

$$T^2 \left(\hat{\vartheta}_T - \vartheta \right) \Longrightarrow \frac{\hat{u}}{\Gamma^2}, \quad T^2 \left(\tilde{\vartheta}_T - \vartheta \right) \Longrightarrow \frac{\tilde{u}}{\Gamma^2}$$

and the convergence of moments provides

$$\mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 = \frac{26}{\Gamma^4} \frac{1}{T^4} (1 + o(1))$$

Therefore we have four problems with for different rates

$$\begin{array}{ll} \text{smooth} & \mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 \sim \frac{c}{T} & \mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 \sim \frac{c}{T^3} \\ \text{discontinuous} & \mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 \sim \frac{c}{T^2} & \mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 \sim \frac{c}{T^4}. \end{array}$$

It is natural to ask: how far can we go in the rate of convergence?

What is the best choice of the signal?

The similar statement for *signal in white Gaussian noise* problem

$$dX_t = S(\vartheta, t) dt + dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T$$

was considered by M. Burnashev (1985). It was shown that for signals satisfying

$$\frac{1}{T} \int_0^T S(\vartheta, t)^2 dt \leq L$$

the best choice yields ($T \rightarrow \infty$)

$$\inf_{S, \hat{\vartheta}_T} \sup_{\vartheta \in [0,1]} \mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 = \exp \left\{ -\frac{L}{6} T (1 + o(1)) \right\}.$$

Therefore the rate can be even exponential.

The similar result was obtained for inhomogeneous Poisson process model with intensity function

$$0 \leq \lambda(\vartheta, t) \leq L.$$

The best choice of intensity function and estimator provides the relation

$$\inf_{\lambda, \hat{\vartheta}_T} \sup_{\vartheta \in [0,1]} \mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 = \exp \left\{ -\frac{L}{6} T (1 + o(1)) \right\}$$

see Burnashev and K. (2001).

For periodic diffusion processes this question is open.

Another interesting statement related to the set Θ was considered by Ibragimov and Khasminskii (1974). First it was shown that if the observed process is

$$dX_t = A \sin(\vartheta t) dt + \sigma dW_t, \quad 0 \leq t \leq T$$

and $\Theta = (\alpha, \infty)$, then the consistent estimation is impossible.

Moreover, if $\Theta = (\alpha, \beta_T)$ and $\beta_T \rightarrow \infty$ then we have two situations.

The first one is : if there exists $\varepsilon > 0$ such that

$$\beta_T < \exp\{(L - \varepsilon)T\}, \quad L = \frac{A^2}{4\sigma^2}$$

then the MLE $\hat{\vartheta}_T$ is consistent. The second case: if

$$\beta_T > \exp\{(L + \varepsilon)T\},$$

then the consistent estimation of ϑ is impossible.