On parameter estimation for periodic diffusion processes II

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Model and Problems

Let us consider the following model of "signal in noise" type

$$\mathrm{d}X_t = S\left(\vartheta, t\right)\,\mathrm{d}t + b\left(X_t\right)\,\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T.$$

Here $S\left(\vartheta, t\right)$ is deterministic τ-periodic signal and $b\left(X_t\right) + \sigma\left(X_t\right)\dot{W}_t$ is "diffusion noise". Our problem is to estimat the parameter ϑ by continuous time observations $X^T = \left(X_t, 0 \leq t \leq T\right)$.

We suppose that the periodic process X_t has ergodic properties and we describe the properties of estimators in the cases of **phase** $S\left(\vartheta, t\right) = S\left(t - \vartheta\right)$ and **frequency** $S\left(\vartheta, t\right) = S\left(\vartheta t\right)$ modulations (or we estiùmate the shift and scale parameters). Here $S\left(t\right)$ is a τ-periodic function, which can be **smooth** or **discontinuous**.

Conditions

We suppose that the functions $b(x), \sigma(x) \in \mathcal{C}_b^3$ and that diffusion coefficient satisfies

$$0 < \kappa \leq \sigma(x)^2 \leq K.$$

The parameter $\vartheta \in (\alpha, \beta) = \Theta$; signal

$$m \leq S(\vartheta, t) \leq M, \qquad t \in [0, \tau], \quad \vartheta \in \Theta.$$

Let us put $S_- = \min(m, 0)$ and $S_+ = \max(M, 0)$ and introduce condition: *there existe constants $A > 0$ and $\varepsilon > 0$ such that for $|x| > A$ we have*

$$2x 1\!\!1_{\{x<A\}} S_- + 2x 1\!\!1_{\{x>A\}} S_+ + 2xb(x) + \sigma(x)^2 < -\varepsilon$$

Under these conditions the diffusion process has ergodic properties (Höpfner and Löcherbach, 2011), i.e., there exists an invariant (periodic) density function $f_\vartheta\left(t,x\right)$ such that for any absolutely integrable function $h\left(t,x\right)$ we have (with probability 1) the following limit

$$\frac{1}{T}\int_0^T h\left(t,X_t\right)\,\mathrm{d}t \longrightarrow \frac{1}{\tau}\int_{-\infty}^{\infty}\int_0^\tau h\left(t,x\right)\,f_\vartheta\left(t,x\right)\mathrm{d}t\;\mathrm{d}x.$$

It can be shown that this convergence is uniform in ϑ.

Note that the existence of the periodic diffusion process, solution of stochastic differential equation with periodic coefficients was proved by Khasminskii in his book (1969).

Note that if $\sigma(x) = \sigma$, then we can denote

$$Y_t = X_t - \int_0^t b(X_s)\, \mathrm{d}s$$

and the problem is reduced to the well-known *signal in white Gaussian noise* model

$$\mathrm{d}Y_t = S(\vartheta, t)\, \mathrm{d}t + \sigma \mathrm{d}W_t, \quad Y_0 = 0, \quad 0 \le t \le T.$$

Of course, the case

$$\mathrm{d}X_t = S(\vartheta, t, X_t)\, \mathrm{d}t + \sigma(t, X_t)\, \mathrm{d}W_t$$

wher $S(\vartheta, t, x)$ and $\sigma(t, x)$ are τ-periodic functions can be treated by a similar way and the choice of our particular model was motivated by biological applications and by simplicity of exposition.

Example. Let

$$\mathrm{d}X_t = S(t)\,\mathrm{d}t - \gamma X_t \mathrm{d}t + \sigma \mathrm{d}W_t, \qquad X_0,$$

where $S(t)$ is a bounded τ-periodic function. Then all mentioned above conditions are fulfilled and

$$f(t,x) = \sqrt{\frac{\gamma}{\pi\sigma^2}} \exp\left\{ -\frac{\gamma\,(x - m(t))^2}{\sigma^2} \right\}$$

where

$$m(t) = \int_0^\tau \frac{e^{-\gamma s}}{1 - e^{-\gamma\tau}}\, S(t-s)\,\mathrm{d}s$$

Smooth Signal. Shift parameter.

Let

$$\mathrm{d}X_t = S\left(t-\vartheta\right)\,\mathrm{d}t + b\left(X_t\right)\,\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0, \quad 0\leq t\leq T.$$

where τ-periodic function $S\left(t\right)\in\mathcal{C}^2$. Then the MLE $\hat\vartheta_T$ and bayesian estimator $\tilde\vartheta_T$ (with positive continuous prior density and quadratic loss function) are consistent, asymptotically normal

$$\sqrt{T}\left(\hat\vartheta_T-\vartheta\right)\Longrightarrow\mathcal{N}\left(0,\mathrm{I}\left(\vartheta\right)^{-1}\right), \quad \sqrt{T}\left(\tilde\vartheta_T-\vartheta\right)\Longrightarrow\mathcal{N}\left(0,\mathrm{I}\left(\vartheta\right)^{-1}\right)$$

and we have convergence of moments and both estimators are asymptotically efficient.

Here the Fisher information is

$$\mathrm{I}\left(\vartheta\right)=\frac{1}{\tau}\int_{0}^{\tau}\int_{-\infty}^{\infty}\frac{\dot S\left(t-\vartheta\right)^{2}}{\sigma\left(x\right)^{2}}\,f_{\vartheta}\left(t,x\right)\,\mathrm{d}x$$

and from convergence of moments we have

$$\mathbf{E}_{\vartheta}\left(\hat\vartheta_{T}-\vartheta\right)^{2}=\frac{\mathrm{I}\left(\vartheta\right)^{-1}}{T}\left(1+o\left(1\right)\right)$$

(see Höpfner and K., 2010)

Smooth Signal. Scale parameter.

Let

$$\mathrm{d}X_t = S\left(\vartheta t\right)\,\mathrm{d}t + b\left(X_t\right)\,\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T.$$

where τ-periodic function $S\left(t\right) \in \mathcal{C}^2$. Then X_t is $\frac{\vartheta}{\tau}$-periodic with $\frac{\vartheta}{\tau}$-periodic invariant density $f_\vartheta\left(t,x\right)$.

The MLE $\hat\vartheta_T$ and bayesian estimator $\tilde\vartheta_T$ are consistent, asymptotically normal

$$T^{3/2}\left(\hat\vartheta_T - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}\left(\vartheta\right)^{-1}\right), \quad T^{3/2}\left(\tilde\vartheta_T - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}\left(\vartheta\right)^{-1}\right)$$

and we have convergence of moments. The both estimators are asymptotically efficient.

Here the Fisher information is

$$\mathrm{I}\left(\vartheta\right)=\frac{\vartheta}{3\tau}\int_{0}^{\tau/\vartheta}\int_{-\infty}^{\infty}\frac{\dot S\left(\vartheta t\right)^2}{\sigma\left(x\right)^2}\,f_\vartheta\left(t,x\right)\,\mathrm{d}x$$

and from convergence of moments we have

$$\mathbf{E}_\vartheta\left(\hat\vartheta_T-\vartheta\right)^2=\frac{\mathrm{I}\left(\vartheta\right)^{-1}}{T^3}\left(1+o\left(1\right)\right)$$

(see Höpfner and K., 2011)

The proof is based on two general theorems by Ibragimov and Khasminskii (1981). Note that the normalized likelihood ratios

$$Z_T(u) = \exp\left\{\int_0^T \frac{S(\vartheta_u, t) - S(\vartheta, t)}{\sigma(X_t)^2} \mathrm{d}X_t - \frac{1}{2}\int_0^T \frac{S(\vartheta_u, t)^2 - S(\vartheta, t)^2}{\sigma(X_t)^2} \mathrm{d}t\right\}$$

where $\vartheta_u = \vartheta + \frac{u}{T^{1/2}}$ and $\vartheta_u = \vartheta + \frac{u}{T^{3/2}}$ in the shift and scale estimation problems respectively. In these smooth cases the corresponding family of measures is LAN, i.e.,

$$Z_T(u) = \exp\left\{u\Delta_T - \frac{u^2}{2}\mathrm{I}(\vartheta) + r_T\right\} \Longrightarrow Z(u) = \exp\left\{u\Delta - \frac{u^2}{2}\mathrm{I}(\vartheta)\right\}$$

where $\Delta \sim \mathcal{N}(0, \mathrm{I}(\vartheta))$

Discontinuous Signal. Shift parameter.

Let

$$\mathrm{d}X_t = S\left(t-\vartheta\right)\,\mathrm{d}t + b\left(X_t\right)\,\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0, \quad 0\leq t\leq T.$$

where τ-periodic function $S\left(t\right)$ is continuously differentiable on the intervals $[0,\tau_*)$ and $(\tau_*,\tau]$. At point τ_* it has a jump

$$S\left(\tau_*+\right)-S\left(\tau_*-\right)=R\neq 0.$$

The likelihood ratio $Z_T\left(u\right)$ with $\vartheta_u=\vartheta+\frac{u}{T}$ converges to the process

$$Z\left(u\right)=\exp\left\{\Gamma\,W\left(u\right)-\frac{\left|u\right|}{2}\,\Gamma^2\right\}$$

where $W\left(\cdot\right)$ is double-sided Wiener process and

$$\Gamma^2=R^2\int_{-\infty}^{\infty}\frac{f_\vartheta\left(\tau_*+\vartheta,x\right)}{\sigma\left(x\right)^2}\,\mathrm{d}x.$$

Let us put

$$Z_0(u) = \exp\left\{W(u) - \frac{|u|}{2}\right\}$$

and introduce two random variables $\hat{u}$ and $\tilde{u}$ by the relations

$$Z_0(\hat{u}) = \sup_u Z_0(u), \qquad \tilde{u} = \frac{\int u\, Z_0(u)\,\mathrm{d}u}{\int Z_0(u)\,\mathrm{d}u}$$

Then the MLE $\hat{\vartheta}_T$ and bayesian estimator $\tilde{\vartheta}_T$ are consistent, have the following limits

$$T\left(\hat{\vartheta}_T - \vartheta\right) \Longrightarrow \frac{\hat{u}}{\Gamma^2}, \quad T\left(\tilde{\vartheta}_T - \vartheta\right) \Longrightarrow \frac{\tilde{u}}{\Gamma^2}$$

and the convergence of moments take place.

Remind that in such situation we have the following lower bound on the risk of all estimators $\bar\vartheta_T$

$$\lim_{\delta\to 0}\lim_{T\to\infty}\sup_{|\vartheta-\vartheta_0|<\delta} T^2\mathbf{E}_\vartheta\left(\bar\vartheta_T-\vartheta\right)^2\geq\frac{\mathbf{E}\left(\tilde u\right)^2}{\Gamma^4}.$$

As usual in such problems the BE are asymptotically efficient:

$$\lim_{\delta\to 0}\lim_{T\to\infty}\sup_{|\vartheta-\vartheta_0|<\delta} T^2\mathbf{E}_\vartheta\left(\tilde\vartheta_T-\vartheta\right)^2=\frac{\mathbf{E}\left(\tilde u\right)^2}{\Gamma^4}.$$

(see Ibragimov and Khasminskii, 1981). Remind that the values

$$\mathbf{E}\left(\hat u\right)^2=26,\qquad \mathbf{E}\left(\tilde u\right)^2=16\,\zeta\left(3\right)\approx 19,6,$$

are known (Terent'ev 1968, Golubev 1979 and Rubin and Song 1995).

Therefore, for the asymptotically efficient estimator we have

$$\mathbf{E}_\vartheta \left(\tilde{\vartheta}_T - \vartheta\right)^2 = \frac{19,6}{\Gamma^2}\,\frac{1}{T^2}\left(1 + o\left(1\right)\right).$$

If the function $S\left(t\right)$ has jumps in k points $0 < \tau_1, \ldots, \tau_k < \tau$ and is continuously differentiable between these points, then the estimators have the same asymptotic properties, but

$$\Gamma^2 = \sum_{l=1}^{k} \int \frac{\left[S\left(\tau_l +\right) - S\left(\tau_l -\right)\right]^2}{\sigma\left(x\right)^2}\, f_\vartheta\left(\tau_l + \vartheta, x\right)\,\mathrm{d}x$$

The problem becames a bit more complicate if

$$\mathrm{d}X_t = \sum_{l=1}^{k} S_l\left(t - \vartheta_l\right)\mathrm{d}t + b\left(X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\mathrm{d}W_t$$

and $\vartheta = \left(\vartheta_1, \ldots, \vartheta_k\right)$ but can be done too.

Discontinuous Signal. Scale parameter.

Let

$$\mathrm{d}X_t = S\left(\vartheta t\right)\,\mathrm{d}t + b\left(X_t\right)\,\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq T.$$

where 1-periodic $(\tau = 1)$ function $S\left(t\right)$ is continuously differentiable on the intervals $[0, \tau_*)$ and $(\tau_*, \tau]$. As before, at point τ_* it has a jump

$$S\left(\tau_*+\right) - S\left(\tau_*-\right) = R \neq 0.$$

The likelihood ratio $Z_T\left(u\right)$ with $\vartheta_u = \vartheta + \frac{u}{T^2}$ converges to the same process $Z\left(u\right)$ with the parameter

$$\Gamma^2 = R^2 \int_{-\infty}^{\infty} \frac{f_\vartheta\left(\frac{\tau_*}{\vartheta}, x\right)}{2\sigma\left(x\right)^2}\,\mathrm{d}x.$$

The estimators $\hat{\vartheta}_T, \tilde{\vartheta}_T$ are consistent, the limit distributions they have with a different normalization

$$T^2\left(\hat{\vartheta}_T-\vartheta\right) \Longrightarrow \frac{\hat{u}}{\Gamma^2}, \quad T^2\left(\tilde{\vartheta}_T-\vartheta\right) \Longrightarrow \frac{\tilde{u}}{\Gamma^2}$$

and the convergence of moments provides

$$\mathbf{E}_\vartheta\left(\hat{\vartheta}_T-\vartheta\right)^2=\frac{26}{\Gamma^4}\,\frac{1}{T^4}\left(1+o\left(1\right)\right)$$

Therefore we have four problems with for different rates

$$\text{smooth} \qquad \mathbf{E}_\vartheta\left(\hat\vartheta_T-\vartheta\right)^2\sim\frac{c}{T} \qquad \mathbf{E}_\vartheta\left(\hat\vartheta_T-\vartheta\right)^2\sim\frac{c}{T^3}$$

$$\text{discontinuous} \qquad \mathbf{E}_\vartheta\left(\hat\vartheta_T-\vartheta\right)^2\sim\frac{c}{T^2} \qquad \mathbf{E}_\vartheta\left(\hat\vartheta_T-\vartheta\right)^2\sim\frac{c}{T^4}.$$

It is natural to ask: how far can we go in the rate of convergence? What is the best choice of the signal?

The similar statement for *signal in white Gaussian noise* problem

$$\mathrm{d}X_t = S\left(\vartheta,t\right)\mathrm{d}t + \mathrm{d}W_t, \quad X_0 = 0, \quad 0 \le t \le T$$

was considered by M. Burnashev (1985). It was shown that for signals satisfying

$$\frac{1}{T}\int_0^T S\left(\vartheta,t\right)^2 \mathrm{d}t \le L$$

the best choice yields $(T \to \infty)$

$$\inf_{S,\bar\vartheta_T}\ \sup_{\vartheta\in[0,1]} \mathbf{E}_\vartheta \left(\hat\vartheta_T - \vartheta\right)^2 = \exp\left\{-\frac{L}{6}T\left(1+o\left(1\right)\right)\right\}.$$

Therefore the rate can be even exponential.

The similar result was obtained for inhomogeneous Poisson process model with intensity function

$$0 \leq \lambda\left(\vartheta, t\right) \leq L.$$

The best choice of intensity function and estimator provides the relation

$$\inf_{\lambda, \bar\vartheta_T} \sup_{\vartheta \in [0,1]} \mathbf{E}_\vartheta \left(\hat\vartheta_T - \vartheta\right)^2 = \exp\left\{-\frac{L}{6} T\left(1 + o\left(1\right)\right)\right\}$$

see Burnashev and K. (2001).

For periodic diffusion processes this question is open.

Another interesting statement related to the set Θ was considered by Ibragimov and Khasminskii (1974). First it was shown that if the observed process is

$$\mathrm{d}X_t = A \sin(\vartheta t)\, \mathrm{d}t + \sigma \mathrm{d}W_t, \quad 0 \le t \le T$$

and $\Theta = (\alpha, \infty)$, then the consistent estimation is impossible. Moreover, if $\Theta = (\alpha, \beta_T)$ and $\beta_T \to \infty$ then we have two situations. The first one is : if there exists $\varepsilon > 0$ such that

$$\beta_T < \exp\left\{(L - \varepsilon)\, T\right\}, \qquad L = \frac{A^2}{4\sigma^2}$$

then the MLE $\hat{\vartheta}_T$ is consistent. The second case: if

$$\beta_T > \exp\left\{(L + \varepsilon)\, T\right\},$$

then the consistent estimation of ϑ is impossible.