“Hitsuda - Beneŝ” approach to exponential martingale revisited. New results

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1. Outline

1. Introduction

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Let $M = (M_t)_{t\in[0,T]}$ be a local martingale with càdlàg paths from Skorokhod space $\mathbb{D}$, that is,

$$M = M^c + M^d$$

M^c and M^d are continuous and purely discontinuous martingale parts (independent)

$\triangle M_t := M_t - M_{t-}$ is “jump process” of M

$\langle M^c \rangle_t$ is “predictable quadratic variation” of M^c.

Assume $\triangle M_t > -1, \ t > 0$.

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Then Doleans-Dade's equation

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_{s-} dM_s$$

obeys positive unique solution known as the Doleans-Dade stochastic exponential:

$$\mathfrak{z}_t = \exp\left(M_t - \frac{1}{2}\langle M^c\rangle_t\right)\prod_{s=0}^{t}(1+\triangle M_s)e^{-\triangle M_s}$$

having

$$\mathsf{E}\mathfrak{z}_t \le 1,$$

for any $t \ge 0$.

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Moreover (here $\mathcal{F}_t$ denotes filtration generated by M)

$$\big(\mathfrak{z}_t, \mathcal{F}_t\big)_{t\ge 0} \text{ is supermartingale with } \mathsf{E}\mathfrak{z}_t \le 1.$$

If for some $T > 0$ $\boxed{\mathsf{E}\mathfrak{z}_T = 1}$, then

$$\big(\mathfrak{z}_t, \mathcal{F}_t\big)_{t\in[0,T]} \text{ is martingale with } \mathsf{E}\mathfrak{z}_t \equiv 1.$$

The martingale property represents an important question:

- finance: risk-neutral probabilistic measure
- weak solution of Itô's equations
- absolutely continuity of diffusion processes distributions
- Bayes' formula
- statistic of random processes.

4. $M \equiv M^c$. Sufficient conditions of $\mathsf{E}\mathfrak{z}_T \equiv 1$

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If $M \equiv M^c$, that is,

$$\mathfrak{z}_t = \exp\Big(M^c_t - \frac{1}{2}\langle M^c\rangle_t\Big),$$

then $\mathsf{E}\mathfrak{z}_T \equiv 1$ under the following conditions:

$\langle M^c\rangle_T \le \text{const.}$	Girsanov, 1960
$\sup\limits_{t\in[0,T]} \mathsf{E}\exp\big(\tfrac{1}{2}M^c_T\big) < \infty$	Kazamaki, 1977
$\mathsf{E}\exp\big(\tfrac{1}{2}\langle M^c\rangle_T\big) < \infty$	Novikov, 1979

$$\left.\begin{aligned} &\lim_{\varepsilon\downarrow 0} \varepsilon \log \mathsf{E}\exp\big((1-\varepsilon)\tfrac{1}{2}\langle M^c\rangle_T\big) < \infty, \\ &\lim_{\varepsilon\downarrow 0} \varepsilon \log \sup_{t\in[0,T]} \mathsf{E}\exp\big((1-\varepsilon)\tfrac{1}{2}M^c_t\big) < \infty \end{aligned}\right\} \text{Krylov, 2009}$$

5. Beneŝ' conditions.

It is also known the Beneŝ approach different from them.

Let B_t denote Brownian motion and put $M_t^c = \int_0^t \sigma(B_s)dB_s$. Then

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s \sigma(X_s)dB_s.$$

Assume $\sigma(x)$ satisfies a linear growth condition (known as Beneŝ' condition)

$$\sigma^2(x) \le \text{const.}\big[1 + x^2\big].$$

Then

$$\mathrm{E}\mathfrak{z}_T = 1, \quad \forall\ T > 0.$$

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Original proof uses piece-wise approximation of $\sigma(x)$ what might be inconvenient in many applications.

New proof.

Choose a localization sequence stopping times

$$\tau_n = \inf\left\{t : (\mathfrak{z}_t \vee \sigma^2(B_t)) \geq n\right\}, \; n \geq 1$$

and notice that, in view of the Beneŝ condition, $\left(\mathfrak{z}_{s\wedge\tau_n} \vee \sigma^2(X_{s\wedge\tau_n})\right) \leq$ const. depending on n. Therefore

$$\mathrm{E}\mathfrak{z}_{T\wedge\tau_n} = 1 \quad \text{and} \quad \mathfrak{z}_{T\wedge\tau_n} = 1 + \int_0^T I_{\{s\leq\tau_n\}}\mathfrak{z}_s\sigma(B_s)dB_s,$$

7. New proof. Uniform integrability

Since by Beneŝ' condition $(T \wedge \tau_n) \xrightarrow[n\to\infty]{} T$, then

$$\mathrm{E}\mathfrak{z}_T = 1 \quad \text{holds}$$

provided that the family $\{\mathfrak{z}_{T\wedge\tau_n}\}_{n\to\infty}$ is uniformly integrable.

Following Hitsuda one may apply Vallée-Poussin's theorem in a form:

$$\sup_n \mathrm{E}\mathfrak{z}_{T\wedge\tau_n} \log(\mathfrak{z}_{T\wedge\tau_n}) < \infty.$$

The use of obvious upper bound

$$\log(\mathfrak{z}_{T\wedge\tau_n}) \le \int_0^T I_{\{s\le\tau_n\}}\sigma(B_s)dB_s$$

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jointly with the Beneŝ condition implies

$$\mathrm{E}\mathfrak{z}_{T\wedge\tau_n}\log(\mathfrak{z}_{T\wedge\tau_n}) \le \mathrm{E}\int_0^T \mathfrak{z}_{s\wedge\tau_n}\sigma^2(B_{s\wedge\tau_n})ds$$
$$\le \mathrm{const.E}\int_0^T \mathfrak{z}_{s\wedge\tau_n}[1+B^2_{s\wedge\tau_n}]ds.$$

Since $\mathrm{E}\mathfrak{z}_{T\wedge\tau_n}=1$, the random process $\mathfrak{z}_{t\wedge\tau_n}$ is the martingale, that is, $\mathfrak{z}_{s\wedge\tau_n}=\mathrm{E}\big(\mathfrak{z}_{T\wedge\tau_n}|\mathcal{F}_{s\wedge\tau_n}\big)$, the obtained upper bound can be transformed into

$$E\mathfrak{z}_{T\wedge\tau_n}\log(\mathfrak{z}_{T\wedge\tau_n}) \le \mathrm{const.E}\mathfrak{z}_{T\wedge\tau_n}\int_0^T [1+B^2_{s\wedge\tau_n}]ds.$$

9. New proof. Uniform integrability

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On the other hand, $\mathsf{E}\mathfrak{z}_{T\wedge\tau_n} = 1$ enables to introduce a new probability measure: Q^n with

$$d\mathsf{Q}^n = \mathfrak{z}_{T\wedge\tau_n} d\mathsf{P}.$$

Denote $\widetilde{\mathsf{E}}^n$ is the expectation symbol of Q^n. Then

$$\mathsf{E}\mathfrak{z}_{T\wedge\tau_n} \log(\mathfrak{z}_{T\wedge\tau_n}) \le \text{const.}\Big[1 + \widetilde{\mathsf{E}}^n \int_0^T B^2_{s\wedge\tau_n} ds\Big]$$

and

$$\sup_n \widetilde{\mathsf{E}}^n \int_0^T B^2_{s\wedge\tau_n} ds < \infty$$

is a sufficient condition of $\mathsf{E}\mathfrak{z}_T = 1$ to hold.

10. New proof. Uniform integrability

By classical Girsanov's theorem the random process $(B_{t\wedge\tau_n}, \mathrm{Q}^n)_{[0,T]}$ is defined as:

$$B_{t\wedge\tau_n} = \int_0^t I_{\{s\le\tau_n\}}\sigma(B_{s\wedge\tau_n})ds + \widetilde{B}^n_t,$$

where $\widetilde{B}^n_t$ is Q^n-Brownian motion stopped at the time τ_n, i.e., $\langle\widetilde{B}^n\rangle_t \equiv \langle B\rangle_{t\wedge\tau_n} \equiv (t\wedge\tau_n)$
Then, by the Itô formula

$$B^2_{t\wedge\tau_n} = 2\int_0^t I_{\{s\le\tau_n\}}B_{s\wedge\tau_n}\sigma(B_{s\wedge\tau_n})ds$$
$$+2\int_0^t I_{\{s\le\tau_n\}}B_{s\wedge\tau_n}d\widetilde{B}^n_s + \langle\widetilde{B}^n\rangle_t.$$

11. New proof. Gronwall-Bellman inequality

Then, $\widetilde{\mathsf{E}}^n B^2_{t\wedge\tau_n} = 2\int_0^t \widetilde{\mathsf{E}}^n I_{\{s\le\tau_n\}} B_{s\wedge\tau_n}\sigma(B_{s\wedge\tau_n})ds + \widetilde{\mathsf{E}}^n(t\wedge\tau_n)$.
Denote

$$V_t^n := \widetilde{\mathsf{E}}^n B^2_{t\wedge\tau_n}$$

and evaluate from above the right hand side of the equality (r is a generic constant):

$$\begin{aligned}
&\Big|2\widetilde{\mathsf{E}}^n \int_0^t I_{\{s\le\tau_n\}} B_{s\wedge\tau_n}\sigma(B_{s\wedge\tau_n})ds\Big| \\
&\qquad \le \mathrm{r}\int_0^t \widetilde{\mathsf{E}}^n \sqrt{[1+B^2_{s\wedge\tau_n}]\sigma^2(B_{s\wedge\tau_n})}\,ds \\
&\qquad \le \mathrm{r}\int_0^t \widetilde{\mathsf{E}}^n[1+B^2_{s\wedge\tau_n}]ds
\end{aligned}$$

- $\widetilde{\mathsf{E}}^n(t\wedge\tau_n) \le T$.

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Hence

$$V_t^n \le \mathrm{r}\left[1 + \int_0^t V_s^n ds\right],$$

that is, $\int_0^T V_s^n ds \le e^{\mathrm{r}T} - 1$

Thus

$$\sup_n \int_0^T \widetilde{\mathrm{E}}^n B^2_{s\wedge\tau_n} ds \le e^{rT} - 1.$$

13. General setting. Notations

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- $(B_t)_{t\in[0,T]}$ - Brownian motion;
- $\mu(dt,dz)$ - integer-valued random measure on $[0,T]\times\mathbb{R}_+$;
- $\nu(dt,dz) := dtK(dz)$ - Levý's measure of $\mu(dt,dz)$;
- $K(dz)$ - σ-finite measure on $\mathbb{R}$, $\int\limits_{\mathbb{R}} z^2K(dz) < \infty$.
- $\mathbb{D}$ - Skorokhod space of cádlág function $(x_t)_{t\in[0,T]}$;

14. $(X_t, \mathfrak{z}_t)$-processes.

$$
\begin{aligned}
X_t &= X_0 + \int_0^t a(s, X_{s-})ds + \int_0^t b(s, X_{s-})dB_s \\
&\quad + \int_0^t \int_{\mathbb{R}} h(s, X_{s-}, z)[\mu(ds, dz) - dsK(dz)] \\
\mathfrak{z}_t &= 1 + \int_0^t \mathfrak{z}_{s-}\Big\{\sigma(s, X_{s-})dB_s \\
&\quad + \int_{\mathbb{R}} \varphi(s, X_{s-}, z)[\mu(ds, dz) - dsK(dz)]\Big\}.
\end{aligned}
$$

Compare:

- Cheridito, Damir, Yor. (2005), Equivalent and absolutely continuous measure changes for jump-diffusion processes. *The Annals of Applied Probability* Vol. 15, No. 3, p. 1713 - 1732.

15. Beneŝ condition and the uniform integrability

Since

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_{s-}\Big\{\sigma(s, X_{s-})dB_s + \int_{\mathbb{R}} \varphi(s, X_{s-}, z)[\mu(ds, dz) - dsK(dz)]\Big\},$$

we have

$$M_t^c = \int_0^t \sigma(s, X_{s-})dB_s$$

$$M_t^d = \int_0^t \int_{\mathbb{R}} \varphi(s, X_{s-}, z)[\mu(ds, dz) - dsK(dz)]$$

with $\varphi(s, X_{s-}, z) \geq 0$ providing $M_t^d - M_{t-}^d \geq 0$.

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Doleans-Dade's formula can be rewritten as

$$\mathfrak{z}_{t\wedge\tau_n} = \exp\left(\mathcal{M}_{t\wedge\tau_n} - \mathcal{A}_{t\wedge\tau_n}\right),$$

where $\mathcal{M}_{t\wedge\tau_n}$ is a square integrable martingale and $\mathcal{A}_{t\wedge\tau_n} \geq 0$:

$$\begin{aligned}\mathcal{M}_{T\wedge\tau_n} &= \int_0^T I_{\{s\leq\tau_n\}}\sigma(s, X_{s-})dB_s \\ &+ \int_0^T \int_{\mathbb{R}} I_{\{s\leq\tau_n\}}\varphi(s, X_{s-}, z)[\mu(ds, ds) - dsK(dz)]\end{aligned}$$

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Then

$$\mathrm{E}\mathfrak{z}_{T\wedge\tau_n}\log\left(\mathfrak{z}_{T\wedge\tau_n}\right)\le \mathrm{E}\mathfrak{z}_{T\wedge\tau_n}\mathcal{M}_{T\wedge\tau_n}\widetilde{\mathrm{E}}^n$$
$$=\widetilde{\mathrm{E}}^n\int_0^T I_{\{s\le\tau_n\}}\Big[\sigma^2(s,X_{s-})+\int_{\mathbb{R}}\varphi^2(s,X_{s-},z)K(dz)\Big]ds,$$

where $\widetilde{\mathrm{E}}^n$ is the expectation symbol of Q^n the new probability measure with $\frac{d\mathrm{Q}^n}{d\mathrm{P}}=\mathfrak{z}_{T\wedge\tau_n}$.
Then a choice of the Beneŝ condition

$$\sigma^2(s,y)+\int_{\mathbb{R}}\varphi^2(s,y,z)K(dz)\le \mathrm{r}\Big[1+y^2\Big],\ s\le T,\ y\in\mathbb{R}$$

18. Beneŝ condition and the uniform integrability

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makes sense if we assume to check

$$\sup_n \mathrm{E}\mathfrak{z}_{T\wedge\tau_n} \log\left(\mathfrak{z}_{T\wedge\tau_n}\right) \le \sup_n \widetilde{\mathrm{E}}^n \int_0^T X^2_{s\wedge\tau_n} ds < \infty$$

with the help of Gronwall-Bellman's inequality

$$V^n_t = \le \mathrm{r}\left[1 + \int_0^t V^n_s ds\right], \quad V^n_s \equiv X^2_{s\wedge\tau_n}.$$

To this end, we have to assume

- $X^2_0 \le$ const.
- $\mathcal{L}(s,y) = 2ya(s,y) + b^2(s,y) + \int_{\mathbb{R}} h^2(s,y,z)K(dz) \le \mathrm{r}[1+y^2]$ $s \le T$
- $\mathfrak{L}(s,y) = \mathcal{L}(s,y) + 2yb(s,y)\sigma(s,y)$
  $+2y\int_{\mathbb{R}} h(s,y,z)\varphi(s,y,z)K(dz)$
  $+\int_{\mathbb{R}} h^2(s,y,z)^2\varphi(s,y,z)K(dz) \le \mathrm{r}[1+y^2],\ s \le T.$

19. Example 1.

$$X_t = \int_0^t \int_{\mathbb{R}} z[\mu(ds,dz) - K(dz)ds]$$

$$\mathfrak{z}_t = 1 + \int_0^t \int_{\mathbb{R}} \mathfrak{z}_{s-} \underbrace{|zX_{s-}|}_{\geq 0}[\mu(ds,dz) - K(dz)ds].$$

- $X_0^2 = 0$
- $\mathcal{L}(s,y) = \int_{\mathbb{R}} z^2 K(dz) < \infty$
- $\mathfrak{L}(s,y) = 2y \int_{\mathbb{R}} z^2 K(dz) + y^2 \int_{\mathbb{R}} |z|^3 K(dz)$
- Beneŝ condition: $\int_{\mathbb{R}} \varphi(s,y,z)K(dz) = y^2 \int_{\mathbb{R}} z^2 K(dz)$

Thus, $\mathrm{E}\mathfrak{z}_T = 1$ provided that $\int_{\mathbb{R}} |z|^3 K(dz) < \infty$.

20. Example 2. Cox model

$$X_t = 1 + \int_0^t X_s ds + \int_0^t \sqrt{X_s^+} dB_s, \quad \mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s \sqrt{X_s^+} dB_s.$$

The process X_t obeys the unit nonnegative solution absorbed at zero at time $\vartheta > 0$. Then

$$\mathsf{E}\mathfrak{z}_{T\wedge\vartheta} = 1$$

Since

- $X_0^2 = 1$
- $\mathcal{L}(s,y) \le 2y^2 + |y|$
- $\mathfrak{L}(s,y) \le 4y^2$
- Beneŝ condition $\sigma^2(s,y) \le |y| \le 1 + y^2$.

21. Vector case. Diffusion process

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$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s$$
$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s \sigma^*(s, X_s)dB_s$$

B_t Brownian motion with independent entries, * is the transposition symbol.
Then $\mathrm{E}\mathfrak{z}_T = 1$, if

$$\begin{aligned}
&\|X\|_0^2 \le \mathtt{r}\\
&\mathcal{L}(s, x_s) = 2x_s^* a_s(x) + \mathrm{trace}(b_s^*(x_s)b_s(x_s) \le \mathtt{r}[1 + \|x\|_s^2]\\
&\|\sigma(s, x_s)\|^2 \le \mathtt{r}[1 + \|x\|_s^2]\\
&\mathfrak{L}_s(x) = 2x_s^*[a_s(x_s) + b_s(x_s)\sigma_s(x_s)]\\
&\qquad\qquad\qquad + \mathrm{trace}(b_s^*(x_s)b_s(x_s) \le \mathtt{r}[1 + \|x\|_s^2].
\end{aligned}$$

22. Example 3. SDE with cubic stabilizing drift

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$$X_t = 1 - \int_0^t X_s^3 ds + \int_0^t X_s dB_s \quad \text{and} \quad \mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s X_s dB_s.$$

Hence

- $X_0 = 0$
- $\mathcal{L}(s, x_s) = -2x_s^4 + x_s^2 \le \mathrm{r}[1 + x_s^2]$
- Beneŝ' condition holds since $\sigma_s^2(x_s) = x_s^2 \le \mathrm{r}[1 + x_s^2]$
- $\mathfrak{L}(s, x_s) \le -2x_s^4 + x_s^2 + 2|x_s^3| \le \mathrm{r}[1 + x_s^2]$.

Then $\mathrm{E}\mathfrak{z}_T = 1$.

23. Brownian bridge

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Gaussian random process $(X_t)_{t\in[0,1]}$ with

$$\mathrm{E}X_t \equiv 0 \quad \text{and} \quad EX_tX_{t'} = (t' \wedge t)[1-(t' \vee t)]$$

is known as Brownian Bridge and can be simulated as the unique solution of Itô' equation

$$X_t = -\int_0^t \frac{X_s}{1-s}ds + B_t, \quad t\in[0,1), \quad \lim_{t\uparrow 1} X_t = 0.$$

Let $\qquad \mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s X_s dB_s.$

$\boxed{\mathrm{E}\mathfrak{z}_T = 1}$ since $X_0 = 0$, $\mathcal{L}_s(x_s) = -\frac{2x^2}{1-s} + 1 \le 1, \ s \le 1.$

$\mathfrak{L}(s, x_s) = -\frac{2x^2}{1-s} + 1 + 2x^2 \le \mathrm{r}[1+x_s^2], \ s \le 1.$

Beneŝ' condition holds since $\sigma_s^2(x_s) = x_s^2 \le \mathrm{r}[1+x_s^2], \ s \le 1$

24. Prolongation of Mijitovic and Urusov

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It is known that for $\alpha \in (-1, 1]$ and

$$X_t = 1 + \int_0^t |X_s|^\alpha ds + B_t, \quad \mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_{s-} X_s dB_s,$$

$$\mathsf{E}\mathfrak{z}_T = 1.$$

Consider the case $\alpha = -1$, that is,

$$X_t = 1 + \int_0^t \frac{ds}{X_s} + B_t$$

It is well known that $X_t \overset{law}{=} X'_t$, where

$$X'_t = \sqrt{(W'_t)^2 + (W''_t)^2 + (W'''_t + 1)^2}$$

and W'_t, W''_t, W'''_t are independent Wiener processes is the

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Hence X_t is positive diffusion.
Then, $\mathsf{E}\mathfrak{z}_T = 1$ since

- $X_0 = 1$
- $\mathcal{L}_s(x_s) = 2s \le 2T, \ s \le T$
- Beneŝ' condition holds: $\sigma_s^2(x_s) = x_s^2 \le \mathrm{r}[1 + x_s^2], \ s \le T$
- $\mathfrak{L}_s(x_s) = 2s + 2x_s^2 \le \mathrm{r}[1 + x_s^2], \ s \le T.$

26. Past-dependent process

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$$X_t = X_0 + \int_0^t a_s(X_{[0,s)})ds + \int_0^t b_s(_{[0,s)})dB_s + \int_0^t \int_{\mathbb{R}} h_s(X_{[0,s)}, z)[\mu(ds, dz) - dsK(dz)],$$

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_{s-}\Big(\sigma_s(X_{[0,s)})dB_s + \int_{\mathbb{R}} \varphi_s(X_{[0,s)}, z)[\mu(ds, dz) - dsK(dz)]\Big).$$

27. Second Beneŝ and other conditions

$$\underbrace{\sigma_s^2(x_{[0,s)}) + \int_{\mathbb{R}} \varphi_s(x_{[0,s)}, z)K(dz) \le \mathrm{r}\Big[1 + \sup_{s' \le s} x_{s'-}^2\Big], \quad s \le T}_{\text{Beneŝ condition: it is natural to compare } u(x_{[0,s)}) \text{ with } [1+\sup_{s'\le s} x_{s'-}^2]}$$

Other conditions:

$$\mathcal{L}_s(x) := a_s^2(x) + b_s^2(x) + \int_{\mathbb{R}} h_s^2(x,z)K(dz)$$

$$\begin{aligned}\mathfrak{L}_s(x) := {} & a_s^2(x) + b_s^2(x) + \int_{\mathbb{R}} h_s^2(x,z)K(dz) + b_s^2(x)\sigma_s^2(x) \\ & + \int_{\mathbb{R}} h_s^2(x,z)K(dz) \int_{\mathbb{R}} \varphi_s^2(x,z)K(dz) \\ & + \int_{\mathbb{R}} h_s^2(x,z)\varphi_s(x,z)K(dz).\end{aligned}$$

29. Past-depended case. Result.

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Assume

$$\left.\begin{array}{l}\sigma_s^2(x_{[0,s)}) + \int_{\mathbb{R}} \varphi_s(x_{[0,s)}, z)K(dz) \\ \mathcal{L}_s(x_{[0,s)}) \\ \mathfrak{L}_s(x_{[0,s)})\end{array}\right\} \le \mathrm{r}\Big[1 + \sup_{s' \le s} x_{s'-}^2\Big].$$

Then $\mathrm{E}\mathfrak{z}_T = 1$.

Example. Suppose

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s \sigma_s(B_{[0,s)}) dB_s.$$

Then $\mathrm{E}\mathfrak{z}_T = 1$ if $\sigma_s^2(x_{[0,s)}) \le \big[1 + \sup_{s' \le s} x_{s'-}^2\big]$, $s \le T$

30. Example of $\mathsf{E}\mathfrak{z}_T < 1$

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Choose the Bessel process

$$X_t = 1 + \int_0^t \frac{1}{X_s} ds + B_t$$

and $\mathfrak{z}_t \equiv \frac{1}{X_t}$. Then, by Itô's formula

$$\mathfrak{z}_t = 1 - \int_0^t \mathfrak{z}_s \frac{1}{X_s} dB_s.$$

Since $\sigma_s(x_s) = -\frac{1}{x_s} I_{\{x_s>0\}}$, the Beneŝ condition fails:

$$\sigma_s^2(x_s) = \frac{1}{x_s^2} I_{\{x_s>0\}} \not\leq \mathrm{r}[1 + x_s^2]$$

So $\mathsf{E}\mathfrak{z}_T = 1$ is questionable ?

31. The proof of $\mathsf{E}\mathfrak{z}_T < 1$ for any $T > 0$

We show that $\mathsf{E}\mathfrak{z}_T < 1$ for any $T > 0$.
Assume opposite, that is, $\mathsf{E}\mathfrak{z}_T = 1$ for some $T > 0$. Since X_t is the Bessel process, $X_t > 0$, $\forall t \ge 0$, a.s.
Since

$$\mathfrak{z}_T = 1$$

a probability measure $\mathsf{Q} \ll \mathsf{P}$ with the density

$$\frac{d\mathsf{Q}}{d\mathsf{P}} = \mathfrak{z}_T$$

exists. Moreover

$$X_t = 1 + B_t, \qquad \text{w.r.t. } \mathsf{Q}$$

that is, X_t is Q-Gaussian process and so its positiveness a.s. fails. So the contradiction holds.