

“Hitsuda - Beneš” approach to exponential martingale revisited. New results

P. Chigansky, F. Klebaner, R. Liptser

Hebrew Uni., Monash Uni., Tel Aviv Uni.

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1. Outline

1. Introduction

P. Chigansky,
F. Klebaner,
R. Liptser

Let $M = (M_t)_{t \in [0, T]}$ be a local martingale with càdlàg paths from Skorokhod space \mathbb{D} , that is,

$$M = M^c + M^d$$

M^c and M^d are continuous and purely discontinuous martingale parts (independent)

$\Delta M_t := M_t - M_{t-}$ is “jump process” of M

$\langle M^c \rangle_t$ is “predictable quadratic variation” of M^c .

Assume $\Delta M_t > -1$, $t > 0$.

2. Doleans-Dade equation

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Then Doleans-Dade's equation

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_{s-} dM_s$$

obeys positive unique solution known as the Doleans-Dade stochastic exponential:

$$\mathfrak{z}_t = \exp\left(M_t - \frac{1}{2}\langle M^c \rangle_t\right) \prod_{s=0}^t (1 + \Delta M_s) e^{-\Delta M_s}$$

having

$$\mathbb{E}\mathfrak{z}_t \leq 1,$$

for any $t \geq 0$.

3. $M \equiv M^c$

Moreover (here \mathcal{F}_t denotes filtration generated by M)

$(z_t, \mathcal{F}_t)_{t \geq 0}$ is supermartingale with $E z_t \leq 1$.

If for some $T > 0$ $E z_T = 1$, then

$(z_t, \mathcal{F}_t)_{t \in [0, T]}$ is martingale with $E z_t \equiv 1$.

The martingale property represents an important question:

- finance: risk-neutral probabilistic measure
- weak solution of Itô's equations
- absolute continuity of diffusion processes distributions
- Bayes' formula
- statistic of random processes.

4. $M \equiv M^c$. Sufficient conditions of $E\mathfrak{z}_T \equiv 1$

If $M \equiv M^c$, that is,

$$\mathfrak{z}_t = \exp \left(M_t^c - \frac{1}{2} \langle M^c \rangle_t \right),$$

then $E\mathfrak{z}_T \equiv 1$ under the following conditions:

$$\langle M^c \rangle_T \leq \text{const.}$$

Girsanov, 1960

$$\sup_{t \in [0, T]} E \exp \left(\frac{1}{2} M_t^c \right) < \infty$$

Kazamaki, 1977

$$E \exp \left(\frac{1}{2} \langle M^c \rangle_T \right) < \infty$$

Novikov, 1979

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log E \exp \left((1 - \varepsilon) \frac{1}{2} \langle M^c \rangle_T \right) < \infty,$$

$$\lim_{\varepsilon \downarrow 0} \varepsilon \log \sup_{t \in [0, T]} E \exp \left((1 - \varepsilon) \frac{1}{2} M_t^c \right) < \infty$$

Krylov, 2009

5. Beneš' conditions.

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It is also known the Beneš approach different from them.

Let B_t denote Brownian motion and put $M_t^c = \int_0^t \sigma(B_s) dB_s$.
Then

$$\hat{\delta}_t = 1 + \int_0^t \hat{\delta}_s \sigma(X_s) dB_s.$$

Assume $\sigma(x)$ satisfies a linear growth condition (known as Beneš' condition)

$$\sigma^2(x) \leq \text{const.} [1 + x^2].$$

Then

$$E\hat{\delta}_T = 1, \quad \forall T > 0.$$

6. New proof

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Original proof uses piece-wise approximation of $\sigma(x)$ what might be inconvenient in many applications.

New proof.

Choose a localization sequence stopping times

$$\tau_n = \inf \{ t : (\beta_t \vee \sigma^2(B_t)) \geq n \}, \quad n \geq 1$$

and notice that, in view of the Beneš condition,
 $(\beta_{s \wedge \tau_n} \vee \sigma^2(X_{s \wedge \tau_n})) \leq \text{const.}$ depending on n . Therefore

$$\mathbb{E} \beta_{T \wedge \tau_n} = 1 \quad \text{and} \quad \beta_{T \wedge \tau_n} = 1 + \int_0^T I_{\{s \leq \tau_n\}} \beta_s \sigma(B_s) dB_s,$$

7. New proof. Uniform integrability

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Since by Beneš' condition $(T \wedge \tau_n) \xrightarrow[n \rightarrow \infty]{} T$, then

$$E\mathfrak{z}_T = 1 \quad \text{holds}$$

provided that the family $\{\mathfrak{z}_{T \wedge \tau_n}\}_{n \rightarrow \infty}$ is uniformly integrable.

Following Hitsuda one may apply Vallée-Poussin's theorem in a form:

$$\sup_n E\mathfrak{z}_{T \wedge \tau_n} \log(\mathfrak{z}_{T \wedge \tau_n}) < \infty.$$

The use of obvious upper bound

$$\log(\mathfrak{z}_{T \wedge \tau_n}) \leq \int_0^T I_{\{s \leq \tau_n\}} \sigma(B_s) dB_s$$

8. New proof. Uniform integrability

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jointly with the Beneš condition implies

$$\begin{aligned} E_{\mathfrak{z}_{T \wedge \tau_n}} \log(\mathfrak{z}_{T \wedge \tau_n}) &\leq E \int_0^T \mathfrak{z}_{s \wedge \tau_n} \sigma^2(B_{s \wedge \tau_n}) ds \\ &\leq \text{const.} E \int_0^T \mathfrak{z}_{s \wedge \tau_n} [1 + B_{s \wedge \tau_n}^2] ds. \end{aligned}$$

Since $E_{\mathfrak{z}_{T \wedge \tau_n}} = 1$, the random process $\mathfrak{z}_{t \wedge \tau_n}$ is the martingale, that is, $\mathfrak{z}_{s \wedge \tau_n} = E(\mathfrak{z}_{T \wedge \tau_n} | \mathcal{F}_{s \wedge \tau_n})$, the obtained upper bound can be transformed into

$$E_{\mathfrak{z}_{T \wedge \tau_n}} \log(\mathfrak{z}_{T \wedge \tau_n}) \leq \text{const.} E_{\mathfrak{z}_{T \wedge \tau_n}} \int_0^T [1 + B_{s \wedge \tau_n}^2] ds.$$

9. New proof. Uniform integrability

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On the other hand, $E_{\mathfrak{z}_{T \wedge \tau_n}} = 1$ enables to introduce a new probability measure: Q^n with

$$dQ^n = \mathfrak{z}_{T \wedge \tau_n} dP.$$

Denote \tilde{E}^n is the expectation symbol of Q^n . Then

$$E_{\mathfrak{z}_{T \wedge \tau_n}} \log(\mathfrak{z}_{T \wedge \tau_n}) \leq \text{const.} \left[1 + \tilde{E}^n \int_0^T B_{s \wedge \tau_n}^2 ds \right]$$

and

$$\sup_n \tilde{E}^n \int_0^T B_{s \wedge \tau_n}^2 ds < \infty$$

is a sufficient condition of $E_{\mathfrak{z}_T} = 1$ to hold.

10. New proof. Uniform integrability

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By classical Girsanov's theorem the random process $(B_{t \wedge \tau_n}, Q^n)_{[0, T]}$ is defined as:

$$B_{t \wedge \tau_n} = \int_0^t I_{\{s \leq \tau_n\}} \sigma(B_{s \wedge \tau_n}) ds + \tilde{B}_t^n,$$

where \tilde{B}_t^n is Q^n -Brownian motion stopped at the time τ_n , i.e.,

$$\langle \tilde{B}^n \rangle_t \equiv \langle B \rangle_{t \wedge \tau_n} \equiv (t \wedge \tau_n)$$

Then, by the Itô formula

$$\begin{aligned} B_{t \wedge \tau_n}^2 &= 2 \int_0^t I_{\{s \leq \tau_n\}} B_{s \wedge \tau_n} \sigma(B_{s \wedge \tau_n}) ds \\ &\quad + 2 \int_0^t I_{\{s \leq \tau_n\}} B_{s \wedge \tau_n} d\tilde{B}_s^n + \langle \tilde{B}^n \rangle_t. \end{aligned}$$

11. New proof. Gronwall-Bellman inequality

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Then, $\tilde{\mathbb{E}}^n B_{t \wedge \tau_n}^2 = 2 \int_0^t \tilde{\mathbb{E}}^n I_{\{s \leq \tau_n\}} B_{s \wedge \tau_n} \sigma(B_{s \wedge \tau_n}) ds + \tilde{\mathbb{E}}^n(t \wedge \tau_n)$.
Denote

$$V_t^n := \tilde{\mathbb{E}}^n B_{t \wedge \tau_n}^2$$

and evaluate from above the right hand side of the equality
(r is a generic constant):

$$\left| 2 \tilde{\mathbb{E}}^n \int_0^t I_{\{s \leq \tau_n\}} B_{s \wedge \tau_n} \sigma(B_{s \wedge \tau_n}) ds \right|$$

$$\leq r \int_0^t \tilde{\mathbb{E}}^n \sqrt{[1 + B_{s \wedge \tau_n}^2] \sigma^2(B_{s \wedge \tau_n})} ds$$

$$\leq r \int_0^t \tilde{\mathbb{E}}^n [1 + B_{s \wedge \tau_n}^2] ds$$

$$\leq \tilde{\mathbb{E}}^n(t \wedge \tau_n) \leq T.$$

12. Gronwall-Bellman inequality

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Hence

$$V_t^n \leq r \left[1 + \int_0^t V_s^n ds \right],$$

that is, $\int_0^T V_s^n ds \leq e^{rT} - 1$

Thus

$$\sup_n \int_0^T \tilde{E}^n B_{S \wedge \tau_n}^2 ds \leq e^{rT} - 1.$$

13. General setting. Notations

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- $(B_t)_{t \in [0, T]}$ - Brownian motion;
- $\mu(dt, dz)$ - integer-valued random measure on $[0, T] \times \mathbb{R}_+$;
- $\nu(dt, dz) := dtK(dz)$ - Levý's measure of $\mu(dt, dz)$;
- $K(dz)$ - σ -finite measure on \mathbb{R} , $\int_{\mathbb{R}} z^2 K(dz) < \infty$.
- \mathbb{D} - Skorokhod space of cádlág function $(x_t)_{t \in [0, T]}$;

14. (X_t, \mathfrak{z}_t) -processes.

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$$\begin{aligned} X_t &= X_0 + \int_0^t a(s, X_{s-}) ds + \int_0^t b(s, X_{s-}) dB_s \\ &\quad + \int_0^t \int_{\mathbb{R}} h(s, X_{s-}, z) [\mu(ds, dz) - dsK(dz)] \\ \mathfrak{z}_t &= 1 + \int_0^t \mathfrak{z}_{s-} \left\{ \sigma(s, X_{s-}) dB_s \right. \\ &\quad \left. + \int_{\mathbb{R}} \varphi(s, X_{s-}, z) [\mu(ds, dz) - dsK(dz)] \right\}. \end{aligned}$$

Compare:

- Cheridito, Damir, Yor. (2005), Equivalent and absolutely continuous measure changes for jump-diffusion processes. *The Annals of Applied Probability* Vol. 15, No. 3, p. 1713 - 1732.

15. Beneš condition and the uniform integrability

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R. Liptser

Since

$$\begin{aligned} \mathfrak{z}_t &= 1 + \int_0^t \mathfrak{z}_{s-} \left\{ \sigma(s, X_{s-}) dB_s \right. \\ &\quad \left. + \int_{\mathbb{R}} \varphi(s, X_{s-}, z) [\mu(ds, dz) - dsK(dz)] \right\}, \end{aligned}$$

we have

$$\begin{aligned} M_t^c &= \int_0^t \sigma(s, X_{s-}) dB_s \\ M_t^d &= \int_0^t \int_{\mathbb{R}} \varphi(s, X_{s-}, z) [\mu(ds, dz) - dsK(dz)] \end{aligned}$$

with $\varphi(s, X_{s-}, z) \geq 0$ providing $M_t^d - M_{t-}^d \geq 0$.

16. Beneš condition and the uniform integrability

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Doleans-Dade's formula can be rewritten as

$$\mathfrak{z}_{t \wedge \tau_n} = \exp(\mathcal{M}_{t \wedge \tau_n} - \mathcal{A}_{t \wedge \tau_n}),$$

where $\mathcal{M}_{t \wedge \tau_n}$ is a square integrable martingale and $\mathcal{A}_{t \wedge \tau_n} \geq 0$:

$$\begin{aligned} \mathcal{M}_{T \wedge \tau_n} &= \int_0^T I_{\{s \leq \tau_n\}} \sigma(s, X_{s-}) dB_s \\ &+ \int_0^T \int_{\mathbb{R}} I_{\{s \leq \tau_n\}} \varphi(s, X_{s-}, z) [\mu(ds, dz) - dsK(dz)] \end{aligned}$$

17. Beneš condition and the uniform integrability

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Then

$$\begin{aligned} \mathbf{E}_{\mathfrak{z}_{T \wedge \tau_n}} \log (\mathfrak{z}_{T \wedge \tau_n}) &\leq \mathbf{E}_{\mathfrak{z}_{T \wedge \tau_n}} \mathcal{M}_{T \wedge \tau_n} \tilde{\mathbf{E}}^n \\ &= \tilde{\mathbf{E}}^n \int_0^T I_{\{s \leq \tau_n\}} \left[\sigma^2(s, X_{s-}) + \int_{\mathbb{R}} \varphi^2(s, X_{s-}, z) K(dz) \right] ds, \end{aligned}$$

where $\tilde{\mathbf{E}}^n$ is the expectation symbol of \mathbf{Q}^n the new probability measure with $\frac{d\mathbf{Q}^n}{d\mathbf{P}} = \mathfrak{z}_{T \wedge \tau_n}$.

Then a choice of the Beneš condition

$$\sigma^2(s, y) + \int_{\mathbb{R}} \varphi^2(s, y, z) K(dz) \leq r [1 + y^2], \quad s \leq T, \quad y \in \mathbb{R}$$

18. Beneš condition and the uniform integrability

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makes sense if we assume to check

$$\sup_n E \mathfrak{z}_{T \wedge \tau_n} \log (\mathfrak{z}_{T \wedge \tau_n}) \leq \sup_n \tilde{E}^n \int_0^T X_{S \wedge \tau_n}^2 ds < \infty$$

with the help of Gronwall-Bellman's inequality

$$V_t^n \leq r \left[1 + \int_0^t V_s^n ds \right], \quad V_s^n \equiv X_{S \wedge \tau_n}^2.$$

To this end, we have to assume

- $X_0^2 \leq \text{const.}$
- $\mathcal{L}(s, y) = 2ya(s, y) + b^2(s, y) + \int_{\mathbb{R}} h^2(s, y, z)K(dz) \leq r[1 + y^2] \quad s \leq T$
 $\mathfrak{L}(s, y) = \mathcal{L}(s, y) + 2yb(s, y)\sigma(s, y)$
+ $2y \int_{\mathbb{R}} h(s, y, z)\varphi(s, y, z)K(dz)$
+ $\int_{\mathbb{R}} h^2(s, y, z)^2\varphi(s, y, z)K(dz) \leq r[1 + y^2], \quad s \leq T.$

19. Example 1.

P. Chigansky,
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$$X_t = \int_0^t \int_{\mathbb{R}} z [\mu(ds, dz) - K(dz)ds]$$

$$\mathfrak{z}_t = 1 + \int_0^t \int_{\mathbb{R}} \mathfrak{z}_{s-} \underbrace{|zX_{s-}|}_{\geq 0} [\mu(ds, dz) - K(dz)ds].$$

- $X_0^2 = 0$
- $\mathcal{L}(s, y) = \int_{\mathbb{R}} z^2 K(dz) < \infty$
- $\mathcal{L}(s, y) = 2y \int_{\mathbb{R}} z^2 K(dz) + y^2 \int_{\mathbb{R}} |z|^3 K(dz)$
- Beneš condition: $\int_{\mathbb{R}} \varphi(s, y, z) K(dz) = y^2 \int_{\mathbb{R}} z^2 K(dz)$

Thus, $E\mathfrak{z}_T = 1$ provided that $\int_{\mathbb{R}} |z|^3 K(dz) < \infty$.

20. Example 2. Cox model

P. Chigansky,
F. Klebaner,
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$$X_t = 1 + \int_0^t X_s ds + \int_0^t \sqrt{X_s^+} dB_s, \quad \mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s \sqrt{X_s^+} dB_s.$$

The process X_t obeys the unit nonnegative solution absorbed at zero at time $\vartheta > 0$. Then

$$E \mathfrak{z}_{T \wedge \vartheta} = 1$$

Since

- $X_0^2 = 1$
- $\mathcal{L}(s, y) \leq 2y^2 + |y|$
- $\mathfrak{L}(s, y) \leq 4y^2$
- Beneš condition $\sigma^2(s, y) \leq |y| \leq 1 + y^2$.

21. Vector case. Diffusion process

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F. Klebaner,
R. Liptser

$$X_t = X_0 + \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dB_s$$
$$\delta t = 1 + \int_0^t \delta s \sigma^*(s, X_s) dB_s$$

B_t Brownian motion with independent entries, $*$ is the transposition symbol.

Then $E \delta t = 1$, if

$$\|X\|_0^2 \leq r$$

$$\mathcal{L}(s, x_s) = 2x_s^* a_s(x) + \text{trace}(b_s^*(x_s) b_s(x_s)) \leq r[1 + \|x\|_s^2]$$

$$\|\sigma(s, x_s)\|^2 \leq r[1 + \|x\|_s^2]$$

$$\mathfrak{L}_s(x) = 2x_s^* [a_s(x_s) + b_s(x_s) \sigma_s(x_s)] \\ + \text{trace}(b_s^*(x_s) b_s(x_s)) \leq r[1 + \|x\|_s^2].$$

22. Example 3. SDE with cubic stabilizing drift

P. Chigansky,
F. Klebaner,
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$$X_t = 1 - \int_0^t X_s^3 ds + \int_0^t X_s dB_s \quad \text{and} \quad \beta_t = 1 + \int_0^t \beta_s X_s dB_s.$$

Hence

- $X_0 = 0$
- $\mathcal{L}(s, x_s) = -2x_s^4 + x_s^2 \leq r[1 + x_s^2]$
- Beneš' condition holds since $\sigma_s^2(x_s) = x_s^2 \leq r[1 + x_s^2]$
- $\mathfrak{L}(s, x_s) \leq -2x_s^4 + x_s^2 + 2|x_s^3| \leq r[1 + x_s^2]$.

Then $E\beta_T = 1$.

23. Brownian bridge

P. Chigansky,
F. Klebaner,
R. Liptser

Gaussian random process $(X_t)_{t \in [0,1]}$ with

$$EX_t \equiv 0 \quad \text{and} \quad EX_t X_{t'} = (t' \wedge t)[1 - (t' \vee t)]$$

is known as Brownian Bridge and can be simulated as the unique solution of Itô' equation

$$X_t = - \int_0^t \frac{X_s}{1-s} ds + B_t, \quad t \in [0, 1), \quad \lim_{t \uparrow 1} X_t = 0.$$

Let $\partial_t = 1 + \int_0^t \partial_s X_s dB_s$.

$E\partial_T = 1$ since $X_0 = 0$, $\mathcal{L}_s(x_s) = -\frac{2x^2}{1-s} + 1 \leq 1$, $s \leq 1$.

$\mathcal{L}(s, x_s) = -\frac{2x^2}{1-s} + 1 + 2x^2 \leq r[1 + x_s^2]$, $s \leq 1$.

Beneš' condition holds since $\sigma_s^2(x_s) = x_s^2 \leq r[1 + x_s^2]$, $s \leq 1$

24. Prolongation of Mijitovic and Urusov

P. Chigansky,
F. Klebaner,
R. Liptser

It is known that for $\alpha \in (-1, 1]$ and

$$X_t = 1 + \int_0^t |X_s|^\alpha ds + B_t, \quad \hat{X}_t = 1 + \int_0^t \hat{\beta}_s X_s dB_s,$$

$$E \hat{\beta}_T = 1.$$

Consider the case $\alpha = -1$, that is,

$$X_t = 1 + \int_0^t \frac{ds}{X_s} + B_t$$

It is well known that $X_t \stackrel{law}{=} X'_t$, where

$$X'_t = \sqrt{(W'_t)^2 + (W''_t)^2 + (W'''_t + 1)^2}$$

and W'_t, W''_t, W'''_t are independent Wiener processes is the

25. Continuation

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Hence X_t is positive diffusion.

Then, $E_{\mathfrak{z}_T} = 1$ since

- $X_0 = 1$
- $\mathcal{L}_s(x_s) = 2s \leq 2T, s \leq T$
- Beneš' condition holds: $\sigma_s^2(x_s) = x_s^2 \leq r[1 + x_s^2], s \leq T$
- $\mathcal{L}_s(x_s) = 2s + 2x_s^2 \leq r[1 + x_s^2], s \leq T.$

26. Past-dependent process

P. Chigansky,
F. Klebaner,
R. Liptser

$$\begin{aligned} X_t &= X_0 + \int_0^t a_s(X_{[0,s]}) ds + \int_0^t b_s([0,s]) dB_s \\ &\quad + \int_0^t \int_{\mathbb{R}} h_s(X_{[0,s]}, z) [\mu(ds, dz) - dsK(dz)], \\ \delta t &= 1 + \int_0^t \delta s - \left(\sigma_s(X_{[0,s]}) dB_s \right. \\ &\quad \left. + \int_{\mathbb{R}} \varphi_s(X_{[0,s]}, z) [\mu(ds, dz) - dsK(dz)] \right). \end{aligned}$$

27. Second Beneš and other conditions

P. Chigansky,
F. Klebaner,
R. Liptser

$$\underbrace{\sigma_s^2(x_{[0,s]}) + \int_{\mathbb{R}} \varphi_s(x_{[0,s]}, z) K(dz)}_{\text{Beneš condition: it is natural to compare } u(x_{[0,s]}) \text{ with } [1 + \sup_{s' \leq s} x_{s'-}^2]}, \quad s \leq T$$

Beneš condition: it is natural to compare $u(x_{[0,s]})$ with $[1 + \sup_{s' \leq s} x_{s'-}^2]$

Other conditions:

$$\mathcal{L}_s(x) := a_s^2(x) + b_s^2(x) + \int_{\mathbb{R}} h_s^2(x, z) K(dz)$$

$$\mathfrak{L}_s(x) := a_s^2(x) + b_s^2(x) + \int_{\mathbb{R}} h_s^2(x, z) K(dz) + b_s^2(x) \sigma_s^2(x)$$

$$+ \int_{\mathbb{R}} h_s^2(x, z) K(dz) \int_{\mathbb{R}} \varphi_s^2(x, z) K(dz)$$

$$+ \int_{\mathbb{R}} h_s^2(x, z) \varphi_s(x, z) K(dz).$$

29. Past-depended case. Result.

P. Chigansky,
F. Klebaner,
R. Liptser

Assume

$$\left. \begin{array}{l} \sigma_s^2(x_{[0,s]}) + \int_{\mathbb{R}} \varphi_s(x_{[0,s]}, z) K(dz) \\ \mathcal{L}_s(x_{[0,s]}) \\ \mathcal{L}_s(x_{[0,s]}) \end{array} \right\} \leq r \left[1 + \sup_{s' \leq s} x_{s'-}^2 \right].$$

Then $E_{\mathfrak{z}_T} = 1$.

Example. Suppose

$$\mathfrak{z}_t = 1 + \int_0^t \mathfrak{z}_s \sigma_s(B_{[0,s]}) dB_s.$$

Then $E_{\mathfrak{z}_T} = 1$ if $\sigma_s^2(x_{[0,s]}) \leq [1 + \sup_{s' \leq s} x_{s'-}^2]$, $s \leq T$

30. Example of $E\beta_T < 1$

P. Chigansky,
F. Klebaner,
R. Liptser

Choose the Bessel process

$$X_t = 1 + \int_0^t \frac{1}{X_s} ds + B_t$$

and $\beta_t \equiv \frac{1}{X_t}$. Then, by Itô's formula

$$\beta_t = 1 - \int_0^t \beta_s \frac{1}{X_s} dB_s.$$

Since $\sigma_s(x_s) = -\frac{1}{x_s} I_{\{x_s > 0\}}$, the Beneš condition fails:

$$\sigma_s^2(x_s) = \frac{1}{x_s^2} I_{\{x_s > 0\}} \not\leq r[1 + x_s^2]$$

So $E\beta_T = 1$ is questionable ?

31. The proof of $E_{\mathfrak{z}_T} < 1$ for any $T > 0$

P. Chigansky,
F. Klebaner,
R. Liptser

We show that $E_{\mathfrak{z}_T} < 1$ for any $T > 0$.

Assume opposite, that is, $E_{\mathfrak{z}_T} = 1$ for some $T > 0$. Since X_t is the Bessel process, $X_t > 0, \forall t \geq 0$, a.s.

Since

$$\mathfrak{z}_T = 1$$

a probability measure $Q \ll P$ with the density

$$\frac{dQ}{dP} = \mathfrak{z}_T$$

exists. Moreover

$$X_t = 1 + B_t, \quad \text{w.r.t. } Q$$

that is, X_t is Q-Gaussian process and so its positiveness a.s. fails. So the contradiction holds.