

# Ergodicity for $T$ -periodic time inhomogeneous Markov processes

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- 1 Introduction
- 2 Classical conditions
- 3 Non classical lower bounds

- $(\xi_t)_{t \geq 0}$  time inhomogeneous strong Markov process
- values in Polish state space  $E$
- transition semigroup  $(P_{s,t})_{0 \leq s \leq t}$
- $T$ -periodic :

$$P_{s,t}(x, dy) = P_{kT+s, kT+t}(x, dy) \text{ for all } k \geq 0.$$

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$$P_{s,t}(x, dy) = P_{kT+s, kT+t}(x, dy) \text{ for all } k \geq 0.$$

Then

$$X = (X_k)_{k \geq 0}, X_k := \xi_{kT}$$

is a time homogeneous Markov chain. Question : When does the following hold

**(H)** : the grid chain  $X = (\xi_{kT})_{k \geq 0}$  is positive Harris recurrent with invariant probability measure  $\mu$

???

Once (H) holds : **strong laws of large numbers** (Höpfner and Kutoyants, 2010) for functionals of the process

$$A = (A_t)_{t \geq 0} \quad , \quad A_t = \int_0^t F(s, \xi_s) \Lambda_T(ds)$$

where :  $F(T + s, x) = F(s, x)$  for all  $s, x$ ,

$\Lambda_T$   $\sigma$ -finite  $T$ -periodic measure :  $\Lambda_T(B) = \Lambda_T(B - kT)$ .

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Then

$$\lim_{t \rightarrow \infty} \frac{1}{t} A_t = \frac{1}{T} \int_0^T \int_E \Lambda_T(ds) [\mu P_{0,s}](dy) F(s, y) \quad \text{a.s.}$$

(provided the limit integral exists).

## Example

Consider in one dimension  $E = \mathbb{R}$ ,

$$d\xi_t = (S(t) - \gamma \xi_t) dt + \sigma dW_t \quad , \quad t \geq 0 ,$$

where  $S$  is a  $T$ -periodic function,  $\gamma > 0$ .

## Example

$$d\xi_t = (S(t) - \gamma \xi_t) dt + dZ_t,$$

$Z$  : one-dimensional Lévy process with Lévy triplet  $(0, 0, \nu)$  s.t.

$$\int_{\{|x|>1\}} |x| \nu(dx) < \infty \text{ and}$$

$$\left[ \varepsilon^2 \ln \frac{1}{\varepsilon} \right]^{-1} \int (x^2 \wedge \varepsilon^2) \nu(dx) \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow 0+ .$$

# Plan of the talk

Want : Sufficient conditions implying

**(H)** : the grid chain  $X = (\xi_{kT})_{k \geq 0}$  is positive Harris recurrent with invariant probability measure  $\mu$

- 1) Recall classical recurrence conditions for Markov chains : Meyn and Tweedie.
- 2) Apply them to diffusions via classical upper and lower bounds for transition densities : “heat kernel bounds”.
- 3) Non classical lower bound results permit to get rid of too heavy non-degeneracy conditions.

# Classical recurrence conditions for Markov chains – general frame

Let  $(X_n)_n$  be a Markov chain, values in Polish state space, transition operator  $P$ .

- **Doebelin** : For all  $x \in E$ ,

$$P(x, \cdot) \geq \alpha \nu(\cdot).$$

Then at each step : with probability  $\alpha$  one forgets the past :

$$\|P_n(x, \cdot) - \mu\|_{TV} \leq C(1 - \alpha)^n :$$

**geometric ergodicity.**

- In the case of discrete state space, being a **Doebelin chain** is equivalent to say : **Dobrushins** ergodicity coefficient  $\beta(P)$  is positive, where

$$\beta(P) = \sum_x \inf_y P(x, y).$$



- **Harris** : replace the Doeblin condition by a **localized** Doeblin condition : There exists a “small” set  $C$  (compact, ...) such that

$$(M) \quad P(x, \cdot) \geq \alpha \nu(\cdot) \text{ for all } x \in C.$$

$\implies$  lower bounds for densities.

- Second step is then : Need to control **excursions out of  $C$**  and their length.

$\implies$  Lyapunov functions.

- Instead of asking for lower bound  $(M)$  on  $C$  it would also be sufficient to compare pairwise

$$\inf_{x, y \in C} \int_E [P(x, \cdot) \wedge P(y, \cdot)] dz > 0$$

(Dobrushin coupling condition). Less restrictive than “lower bounds approach” ...

# Meyn-Tweedie : classical result

Consider the  $T$ -grid chain  $X = (X_k)_k$ ,  $X_k = \xi_{kT}$ , with transition operator  $P_{0,T}(x, dy) = p_{0,T}(x, y)\Lambda(dy)$ .

## Theorem

Assume that there exists some norm-like function  $V$  and some compact  $K \subset E$  such that

$$P_{0,T}V \leq V - \varepsilon \quad \text{on } K^c,$$

and such that on  $K \times K$ ,

$$\inf_{x,y \in K} p_{0,T}(x, y) > 0.$$

We also need :  $P_{0,T}V$  is bounded on  $K$ .

Then **(H)** holds.

**Idea of proof :**

- $V(\xi_{kT})1_{\{k < \tau_K\}}$  nonnegative supermartingale, hence convergent  
 $\implies$  the process comes back to  $K$  almost surely.
- Apply Nummelin-splitting on  $K$ .
- Excursions out of  $K$  have finite expected length : stopping theorem (or Dynkin formula) :

$$E_x(\tau_K) \leq 1 + \varepsilon^{-1} P_{0,T} V(x),$$

for all  $x$ . That is why we need  $P_{0,T} V$  bounded !

## Remark

*Instead of our condition  $P_{0,T}V(x) \leq V(x) - \varepsilon$  there are refined conditions such as*

$$P_{0,T}V(x) \leq V(x) - \Phi \circ V(x) + b1_C(x).$$

*Here,  $\Phi : [1, \infty[ \rightarrow \mathbb{R}_+$  increasing, concave.*

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$$n^{\frac{\alpha}{1-\alpha}}$$

*(Douc, Fort, Moulines, Soulier (2004)).*

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$\Phi$  linear leads to geometric ergodicity.

## Example

*OU-process, driven by Lévy noise :*

$$d\xi_t = (S(t) - \gamma\xi_t) dt + dZ_t.$$

- 1  $\int_{\{|x|>1\}} |x| \nu(dx) < \infty$  implies that we can take  $V(x) = |x|$ .
- 2  $[\varepsilon^2 \ln \frac{1}{\varepsilon}]^{-1} \int (x^2 \wedge \varepsilon^2) \nu(dx) \rightarrow +\infty$  implies (Bodnarchuk and Kulik 2008) : there is a transition density of the process  $\xi$  which is  $C_b^\infty$  in  $x, y$ .
- 3 Note that this last condition does not imply that the law of the driving Lévy process is smooth, the law of  $Z_t$  can be **singular !**

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- 3 Note that this last condition does not imply that the law of the driving Lévy process is smooth, the law of  $Z_t$  can be **singular** ! Nice remark : the invariant probability  $\mu$  is given by

$$\mu = \mathcal{L}(Z_\tau + c), \quad \tau \sim \exp(\gamma) \text{ independent of } Z.$$



## Applications to SDE's

$d$ -dimensional SDE with  $T$ -periodic drift

$$d\xi_t = b(t, \xi_t) dt + \sigma(\xi_t) dW_t, \quad b(t + T, x) = b(t, x),$$

under **uniform ellipticity**

$$\zeta^\top a(x) \zeta \geq cst \cdot |\zeta|^2, \quad a = \sigma \sigma^\top,$$

smooth and bounded coefficients, bounded first derivatives (with respect to space).

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smooth and bounded coefficients, bounded first derivatives (with respect to space).

Aronson (1967) : then we have heat kernel bounds

$$p_{0,s}(x, y) \geq \kappa_1 \left( \frac{1}{2\pi s \gamma_1^2} \right)^{d/2} \exp\left\{ -\frac{1}{2} \frac{|y-x|^2}{s \gamma_1^2} \right\}$$

(similar upper bounds), see also Kusuoka-Stroock (1985), Norris-Stroock (1991), ...

As a consequence :

### Theorem

*Grant Aronson's conditions + the following drift condition*

$$(D) \quad 2x^\top b(s, x) + \text{tr}(a(x)) < -\varepsilon \quad \text{on } [0, T] \times \{|x| > R\} .$$

*for  $R$  sufficiently large.*

*Then **(H)** holds.*

Proof : Take  $V(x) = |x|^2$ , apply Ito to  $V(\xi_t) = |\xi_t|^2$  and use the heat kernel bounds.

### Remark

*Any other Lyapunov function (other than  $|x|^2$ ) could work - and would just lead to another drift condition (D).*

- Uniform ellipticity or other non-degeneracy conditions (hypoellipticity) are far from being necessary for our purposes.
- Actually, it suffices to obtain lower bounds for transition densities **only on suitable** compacts.
- And this can be achieved even in the case of vanishing  $\sigma$  (somewhere “outside”).

- Uniform ellipticity or other non-degeneracy conditions (hypoellipticity) are far from being necessary for our purposes.
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- And this can be achieved even in the case of vanishing  $\sigma$  (somewhere “outside”).

Here is our frame :

- **unbounded coefficients, but bounded derivatives** (uniformly in time). Existence of  $d + 2$  derivatives (wrt space).
- Write  $\lambda_*(x)$  for the smallest and  $\lambda^*(x)$  for the largest eigenvalue of  $a(x)$ .

## Theorem

*Under the above conditions, impose the drift condition*

$$(D) \quad 2x^\top b(s, x) + \text{tr}(a(x)) < -\varepsilon \quad \text{on } [0, T] \times \{|x| > R\}.$$

*Then there exists some  $r$  which depends in an explicit way on  $R$ , the dimension  $d$ , the dimension of the BM  $W$  and the coefficients such that : If*

$$\lambda_*(x) > 0 \quad \text{for all } x : |x| \leq R + r,$$

*then  $\{x : |x| \leq R + r\}$  is a “small” set and (H) holds.*

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**The message :** We need non-degeneracy only on a (sufficiently big) compact which is given by the Lyapunov condition.

Proof : We only need a lower bound for the transition densities. For this we use the following theorem of Bally (2006), Lower bounds for locally elliptic Ito processes.

**Theorem 24 of Bally (2006)** Under the above conditions on the coefficients : If there is some compact and convex set  $K \subset \mathbb{R}^d$  which satisfies

$$\lambda_*(x) > 0 \text{ for all } x \in K ,$$

then

$$(1) \quad \inf_{x,y \in K} p_{0,T}(x,y) > 0 .$$



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In order to prove this, he constructs a differentiable path from  $x$  to  $y$  inside  $K$  and gives a control of lower bounds for transitions along small balls covering these paths, using conditional Malliavin calculus (only with respect to the space variable).

## Example

*Lower bounds exist even in highly degenerated situations :*

$$d\xi_t^1 = b^1(t, \xi_t) dt + \sigma(\xi_t) dW_t \quad , \quad d\xi_t^2 = b^2(t, \xi_t) dt .$$

*Coefficients : 5 times differentiable, not necessarily bounded,  
 bounded derivatives of orders 1, ..., 5, non-degeneracy condition :  
 (Bally- Kohatsu Higa 2010)*

$$|\sigma(x)| \geq c \quad , \quad \left| \frac{\partial b^2}{\partial x^1}(x) \right| \geq c .$$

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$$|\sigma(x)| \geq c \quad , \quad \left| \frac{\partial b^2}{\partial x^1}(x) \right| \geq c .$$

*Drift condition :*

$$2 \sum_{i=1,2} x_i b^i(t, x) + \sigma^2(x) < -\varepsilon \quad \text{“outside some compact”} .$$

## Remark

Veretennikov (1997) (*On polynomial mixing bounds for SDE's*) works :

- under Aronson's conditions, but without uniform ellipticity.
- Instead of uniform ellipticity, he imposes :  $\lambda^*(x) > 0$  for all  $x \in \mathbb{R}^d$ .

- Different Lyapunov condition :  $\lambda_- := \inf_{x \neq 0} \frac{x^\top a(x) x}{|x|^2}$ ,  
 $\lambda_+ := \sup_{x \neq 0} \frac{x^\top a(x) x}{|x|^2}$ ,  $\tilde{\Lambda} := \sup_x \text{tr}(a(x))$ . Suppose that

$$x^\top b(x) \leq -r, \quad |x| > M, \quad 3\lambda_+ < 2r - (\tilde{\Lambda} - \lambda_-).$$

- This implies in particular that

$$2x^\top b(x) + \text{tr}(a(x)) < -2\lambda_+ \quad \text{on } [0, T] \times \{|x| > M\}.$$

## Example

**Pearson type diffusion** ( $d = 1$ )

$$dX_t = \theta (S(t) - X_t) dt + \sigma(X_t) \sqrt{c_0 + (X_t - c_1)^2} dW_t,$$

$\theta > 0$ ,  $c_0 > 0$ ,  $\sigma(\cdot) > 0$  bounded (might converge to zero as  $|x| \rightarrow \infty$ ), first three derivatives of  $\sigma$  of order  $O(|x|^{-1})$  for  $|x| \rightarrow \infty$ .

(D) is implied by

$$2\theta > \sup_x \sigma^2(x),$$

and hence in this case (H) holds.

## Some literature :

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