Ergodicity for $T-$periodic time inhomogeneous Markov processes

Reinhard Höpfner and Eva Löcherbach

- $(\xi_t)_{t\geq 0}$ time inhomogeneous strong Markov process
- values in Polish state space E
- transition semigroup $(P_{s,t})_{0\leq s\leq t}$
- $T-$periodic :

$$P_{s,t}(x,dy) \;=\; P_{kT+s,kT+t}(x,dy) \text{ for all } k\geq 0.$$

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$$P_{s,t}(x, dy) \ = \ P_{kT+s,kT+t}(x, dy) \text{ for all } k \geq 0.$$

Then

$$X = (X_k)_{k \geq 0}, \ X_k := \xi_{kT}$$

is a time homogeneous Markov chain. Question : When does the following hold

(H) : the grid chain $X = (\xi_{kT})_{k \geq 0}$ is positive Harris recurrent with invariant probability measure μ

? ? ?

Once (H) holds : **strong laws of large numbers** (Höpfner and Kutoyants, 2010) for functionals of the process

$$A = (A_t)_{t \geq 0} \quad , \quad A_t \;=\; \int_0^t F(s, \xi_s)\, \Lambda_T(ds)$$

where : $F(T+s, x) = F(s, x)$ for all s, x,
Λ_T $\sigma-$finite T-periodic measure : $\Lambda_T(B) = \Lambda_T(B - kT)$.

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Then

$$\lim_{t \to \infty} \frac{1}{t} A_t \;\; = \;\; \frac{1}{T} \int_0^T \int_E \Lambda_T(ds) [\mu P_{0,s}](dy) F(s, y) \quad \text{a.s.}$$

(provided the limit integral exists).

Example

Consider in one dimension $E = \mathbb{R}$,

$$d\xi_t \;=\; (\, S(t) - \gamma\,\xi_t\,)\; dt \;+\; \sigma\, dW_t \quad , \quad t \geq 0\,,$$

where S *is a* $T-$*periodic function,* $\gamma > 0$.

Example

$$d\xi_t \;=\; (S(t) - \gamma\xi_t)\, dt \;+\; dZ_t,$$

Z : *one-dimensional Lévy process with Lévy triplet* $(0, 0, \nu)$ *s.t.*

$$\int_{\{|x|>1\}} |x|\nu(dx) \;<\; \infty \text{ and}$$

$$[\varepsilon^2 \ln \frac{1}{\varepsilon}]^{-1} \int (x^2 \wedge \varepsilon^2)\nu(dx) \;\to\; +\infty \quad \text{as } \varepsilon \to 0+ \;.$$

Plan of the talk

Want : Sufficient conditions implying

(H) : the grid chain $X = (\xi_{kT})_{k \geq 0}$ is positive Harris recurrent with invariant probability measure μ

1) Recall classical recurrence conditions for Markov chains : Meyn and Tweedie.
2) Apply them to diffusions via classical upper and lower bounds for transition densities : "heat kernel bounds".
3) Non classical lower bound results permit to get rid of too heavy non-degeneracy conditions.

Classical recurrence conditions for Markov chains { general frame

Let $(X_n)_n$ be a Markov chain, values in Polish state space, transition operator P.

- **Doeblin** : For all $x \in E$,

$$P(x, \cdot) \geq \alpha \nu(\cdot).$$

  Then at each step : with probability α one forgets the past :

$$\|P_n(x, \cdot) - \mu\|_{TV} \leq C(1-\alpha)^n :$$

  geometric ergodicity.

- In the case of discrete state space, being a **Doeblin chain** is equivalent to say : **Dobrushins** ergodicity coefficient $\beta(P)$ is positive, where

$$\beta(P) = \sum_y \inf_x P(x, y).$$

- **Harris :** replace the Doeblin condition by a **localized** Doeblin condition : There exists a "small" set C (compact, ...) such that

$$(M) \qquad P(x, \cdot) \geq \alpha\nu(\cdot) \text{ for all } x \in C.$$

$\Longrightarrow$ lower bounds for densities.

- Second step is then : Need to control **excursions out of** C and their length.
$\Longrightarrow$ Lyapunov functions.

- Instead of asking for lower bound (M) on C it would also be sufficient to compare pairwise

$$\inf_{x,y\in C} \int_E [P(x,\cdot) \wedge P(y,\cdot)]dz > 0$$

(Dobrushin coupling condition). Less restrictive then "lower bounds approach"...

Meyn-Tweedie : classical result

Consider the $T-$grid chain $X = (X_k)_k$, $X_k = \xi_{kT}$, with transition operator $P_{0,T}(x, dy) = p_{0,T}(x, y)\Lambda(dy)$.

Theorem

Assume that there exists some norm-like function V and some compact $K \subset E$ such that

$$P_{0,T}V \;\leq\; V - \varepsilon \quad \text{on} \;\; K^c \; ,$$

and such that on $K \times K$,

$$\inf_{x,y \in K} p_{0,T}(x,y) \;>\; 0 \; .$$

We also need : $P_{0,T}V$ is bounded on K.
Then **(H)** *holds.*

Idea of proof :

• $V(\xi_{kT})1_{\{k<\tau_K\}}$ nonnegative supermartingale, hence convergent $\Longrightarrow$ the process comes back to K almost surely.

• Apply Nummelin-splitting on K.

• Excursions out of K have finite expected length : stopping theorem (or Dynkin formula) :

$$E_x(\tau_K) \leq 1 + \varepsilon^{-1} P_{0,T} V(x),$$

for all x. That is why we need $P_{0,T} V$ bounded !

Remark

Instead of our condition $P_{0,T}V(x) \leq V(x) - \varepsilon$ *there are refined conditions such as*

$$P_{0,T}V(x) \leq V(x) - \Phi \circ V(x) + b1_C(x).$$

Here, $\Phi : [1, \infty[\to \mathbb{R}_+$ *increasing, concave.*

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Here, $\Phi : [1, \infty[\to \mathbb{R}_+$ *increasing, concave.* Φ *determines then the speed of convergence, for instance if* $\Phi(u) = u^\alpha$*, then the rate of convergence is polynomial*

$$n^{\frac{\alpha}{1-\alpha}}$$

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Φ *linear leads to geometric ergodicity.*

Example

OU-process, driven by Lévy noise :

$$d\xi_t \; = \; (S(t) - \gamma \xi_t)\, dt \; + \; dZ_t.$$

1. $\int_{\{|x|>1\}} |x| \nu(dx) \; < \; \infty$ *implies that we can take* $V(x) = |x|$.
2. $[\varepsilon^2 \ln \frac{1}{\varepsilon}]^{-1} \int (x^2 \wedge \varepsilon^2) \nu(dx) \; \to \; +\infty$ *implies (Bodnarchuk and Kulik 2008) : there is a transition density of the process* ξ *which is* C_b^∞ *in* x, y.
3. *Note that this last condition does not imply that the law of the driving Lévy process is smooth, the law of* Z_t *can be* **singular** *!*

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3. *Note that this last condition does not imply that the law of the driving Lévy process is smooth, the law of* Z_t *can be* **singular** *! Nice remark : the invariant probability* μ *is given by*

$$\mu = \mathcal{L}(Z_\tau + c), \quad \tau \sim \exp(\gamma) \text{ independent of } Z.$$

Applications to SDE's

d-dimensional SDE with T-periodic drift

$$d\xi_t \;=\; b(t,\xi_t)\,dt \;+\; \sigma(\xi_t)\,dW_t,\; b(t+T,x) \;=\; b(t,x),$$

under **uniform ellipticity**

$$\zeta^\top a(x)\,\zeta \;\;\geq\;\; cst\cdot|\zeta|^2, a=\sigma\sigma^\top,$$

smooth and bounded coefficients, bounded first derivatives (with respect to space).

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Aronson (1967) : then we have heat kernel bounds

$$p_{0,s}(x,y)\geq\kappa_1\left(\frac{1}{2\pi\,s\,\gamma_1^2}\right)^{d/2}\exp\{-\frac{1}{2}\frac{|y-x|^2}{s\,\gamma_1^2}\}$$

(similar upper bounds), see also Kusuoka-Stroock (1985), Norris-Stroock (1991), . . .

As a consequence :

Theorem

Grant Aronson's conditions + the following drift condition

$$(D) \qquad 2\,x^{\top} b(s,x) \;+\; \mathrm{tr}(a(x)) \;<\; -\varepsilon \quad \text{on } [0,T]\times\{|x|>R\}\;.$$

for R sufficiently large.
Then **(H)** *holds.*

Proof : Take $V(x) = |x|^2$, apply Ito to $V(\xi_t) = |\xi_t|^2$ and use the heat kernel bounds.

Remark

Any other Lyapunov function (other than $|x|^2$) could work - and would just lead to another drift condition (D).

• Uniform ellipticity or other non-degeneracy conditions (hypoellipticity) are far from being necessary for our purposes.

• Actually, it suffices to obtain lower bounds for transition densities **only on suitable** compacts.

• And this can be achieved even in the case of vanishing σ (somewhere "outside").

• Uniform ellipticity or other non-degeneracy conditions (hypoellipticity) are far from being necessary for our purposes.
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Here is our frame :
– **unbounded coe cients, but bounded derivatives** (uniformly in time). Existence of $d+2$ derivatives (wrt space).
– Write $\lambda_*(x)$ for the smallest and $\lambda^*(x)$ for the largest eigenvalue of $a(x)$.

Theorem

Under the above conditions, impose the drift condition

$$(D) \qquad 2\,x^{\top} b(s,x) \,+\, \mathrm{tr}(a(x)) \,<\, -\varepsilon \quad \text{on } [0,T]\times\{|x|>R\}\;.$$

Then there exists some r which depends in an explicit way on R, the dimension d, the dimension of the BM W and the coefficients such that : If

$$\lambda_*(x) > 0 \;\; \text{for all } x : |x| \le R + r,$$

then $\{x : |x| \le R + r\}$ is a "small" set and (H) holds.

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then $\{x : |x| \le R + r\}$ is a "small" set and (H) holds.

The message : We need non-degeneracy only on a (sufficiently big) compact which is given by the Lyapunov condition.

Proof : We only need a lower bound for the transition densities. For this we use the following theorem of Bally (2006), Lower bounds for locally elliptic Ito processes.

Theorem 24 of Bally (2006) Under the above conditions on the coefficients : If there is some compact and convex set $K \subset \mathbb{R}^d$ which satisfies

$$\lambda_*(x) > 0 \text{ for all } x \in K \, ,$$

then

$$\inf_{x,y \in K} p_{0,T}(x,y) > 0 \, . \tag{1}$$

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In order to prove this, he constructs a differentiable path from x to y inside K and gives a control of lower bounds for transitions along small balls covering these paths, using conditional Malliavin calculus (only with respect to the space variable).

Example

Lower bounds exist even in highly degenerated situations :

$$d\xi_t^1 \;=\; b^1(t,\xi_t)\,dt \;+\; \sigma(\xi_t)\,dW_t \quad , \quad d\xi_t^2 \;=\; b^2(t,\xi_t)\,dt \;.$$

Coefficients : 5 *times differentiable, not necessarily bounded, bounded derivatives of orders* $1,\ldots,5$, *non-degeneracy condition : (Bally- Kohatsu Higa 2010)*

$$|\sigma(x)| \;\geq\; c \;\;,\;\; |\frac{\partial b^2}{\partial x^1}(x)| \;\geq\; c \;.$$

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Drift condition :

$$2\sum_{i=1,2} x_i\, b^i(t,x) \ + \ \sigma^2(x) \ < \ -\varepsilon \quad \text{“outside some compact”}.$$

Remark

Veretennikov (1997) (On polynomial mixing bounds for SDE's) works :

- *under Aronson's conditions, but without uniform ellipticity.*
- *Instead of uniform ellipticity, he imposes :* $\lambda^*(x) > 0$ *for all* $x \in \mathbb{R}^d$.
- *Different Lyapunov condition :* $\lambda_- := \inf_{x \neq 0} \frac{x^\top a(x)\, x}{|x|^2}$, $\lambda_+ := \sup_{x \neq 0} \frac{x^\top a(x)\, x}{|x|^2}$, $\widetilde{\Lambda} := \sup_x \operatorname{tr}(a(x))$. *Suppose that*

$$x^\top b(x) \le -r \, , \; |x| > M \quad , \quad 3\lambda_+ < 2r - (\widetilde{\Lambda} - \lambda_-).$$

- *This implies in particular that*

$$2x^\top b(x) + \operatorname{tr}(a(x)) < -2\lambda_+ \quad on\ [0, T] \times \{|x| > M\} \, .$$

Example

Some literature :

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