On sequential parameter estimation in stochastic differential equations involving fractional Brownian motion.

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Asymptotical Statistics of Stochastic Processes VIII (Statistique Asymptotique des Processus Stochastiques VIII)
Universite' du Maine, Le Mans, 21-24 March, 2011

Outline

Consider simple statistical models that involve fractional Brownian motion.
Why fBm? At least 4 properties:
–fractal structure (fractals in finances; Mandelbrot)
–dependent but stationary increments (mechanics, hydrodynamics; Kolmogorov, Monin, Yaglom)
–self-similarity (Hurst; devices in physics and radioelectronics)
–long memory (economics, tele-traffic)
Gaussian property–good for mathematicians. But–...many generalizations...

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition

The (two-sided, normalized) fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}\}$ on $(\Omega, \mathcal{F}, P)$, having the properties

(i) $B_0^H = 0$,

(ii) $EB_t^H = 0, t \in \mathbb{R}$,

(iii) $EB_t^H B_s^H = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), s, t \in \mathbb{R}$.

Remark

Since $E(B_t^H - B_s^H)^2 = |t-s|^{2H}$ and B^H is a Gaussian process, it has a continuous modification, according to the Kolmogorov theorem.

The trajectories of fBm are Hölder up to order H. The increments are positively correlated for $H > 1/2$ and negatively correlated for $H < 1/2$. The properties of correlation function and spectral density allow to say that for $H > 1/2$ fBm has long memory, and for $H < 1/2$ it has short memory.

Representation of fBm via the Wiener Process on a Finite Interval

Sometimes it is convenient to consider a one-sided fBm $B^H = \{B_t^H, t \geq 0\}$ and to represent it as a functional of the form $B_t^H = \varphi(W_s, 0 \leq s \leq t)$, of some Wiener process $W = \{W_t, t \geq 0\}$. For this purpose consider the kernel

$$m_H(t,s) = C_H\Big(\Big(\frac{t}{s}\Big)^{\alpha}(t-s)^{\alpha} - \alpha s^{-\alpha}\int_s^t u^{\alpha-1}(u-s)^{\alpha}du\Big),$$

where

$$C_H = \left(\frac{2H\Gamma(1-\alpha)}{\Gamma(1-2\alpha)\Gamma(\alpha+1)}\right)^{\frac{1}{2}},$$

and $\alpha = H - \frac{1}{2}, \quad H \in (0,1)$. It can be simplified to

$$m_H(t,s) = C_H s^{-\alpha}\int_s^t u^{\alpha}(u-s)^{\alpha-1}du$$

for $H > 1/2$.

Then one has the representation

$$B_t^H = \int_0^t m_H(t,s)dW_s. \tag{1}$$

The inverse representation holds: the process

$$M_t^H = \int_0^t l_H(t,s)dB_s^H$$

with the kernel $l_H(t,s)$, specified later, is Gaussian martingale (Molchan martingale).

Parameter Estimates in the Models Involving fBm

First we consider the "pure" fractional diffusion (nonlinear) model and establish strong consistency and asymptotic normality of the maximum likelihood drift parameter estimate.

The Girsanov Theorem for the Pure Fractional Diffusion Model and Likelihood Ratio for Drift Parameter

We assume that the fBm B^H with $H \in (1/2, 1)$ is defined on a probability space $(\Omega, \mathcal{F}, P)$ and denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by B^H. Consider a diffusion equation containing a stochastic differential driven by B^H:

$$dX_t = \theta a(t, X_t)dt + b(t, X_t)dB_t^H, \ X_{t=0} = X_0 \in \mathbb{R}, \tag{2}$$

$$\theta \in \mathbb{R}, \ 0 \leq t \leq T, \ T > 0.$$

Differential equation (2) can be rewritten in the integral form

$$X_t = X_0 + \theta \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s^H, \ \ t \in [0, T]. \qquad (3)$$

Here we use pathwise construction of the integral w.r.t. fBm.

In order to introduce the pathwise integrals w.r.t. fBm consider two nonrandom functions f and g, defined on some interval $[a, b] \subset \mathbb{R}^+$. Suppose also that the limits
$f(u+) := \lim_{\delta\downarrow 0} f(u+\delta)$ and $g(u-) := \lim_{\delta\downarrow 0} g(u-\delta),\ a \le u \le b$ exist.
Put
$f_{a+}(x) := (f(x) - f(a+))1_{(a,b)}(x),\ g_{b-}(x) := (g(b-) - g(x))1_{(a,b)}(x).$
Suppose that $f_{a+} \in I_{a+}^{\alpha}(L_p[a, b])),\ g_{b-} \in I_{b-}^{1-\alpha}(L_q[a, b]))$, for some $p \ge 1,\ q \ge 1, 1/p + 1/q \le 1,\ 0 \le \alpha \le 1.$

Consider fractional derivatives

$$(D^{\alpha}_{a+} f_{a+})(x) = \frac{1}{\Gamma(1-\alpha)}\Big(\frac{f_{a+}(s)}{(s-a)^{\alpha}}$$

$$+\alpha \int_{\alpha}^{s} \frac{f_{a+}(s) - f_{a+}(u)}{(s-u)^{1+\alpha}} du\Big) 1_{(a,b)}(x)$$

$$(D^{1-\alpha}_{b-} g_{b-})(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\Big(\frac{g_{b-}(s)}{(b-s)^{1-\alpha}}$$

$$+(1-\alpha) \int_{s}^{b} \frac{g_{b-}(s) - g_{b-}(u)}{(s-u)^{2-\alpha}} du\Big) 1_{(a,b)}(x).$$

It is known that $D^{\alpha}_{a+} f_{a+} \in L_p[a,b]$, $D^{1-\alpha}_{b-} g_{b-} \in L_q[a,b]$.

Definition

Under above assumptions, the generalized (fractional) Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) := e^{i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f_{a+})(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx$$

$$+f(a+)(g(b-) - g(a+)),$$

and for $\alpha p < 1$ it can be simplified to

$$\int_a^b f(x)dg(x) := e^{i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} g_{b-})(x)dx.$$

For any $1 - H < \alpha < 1$ there exists fractional derivative $D_{b-}^{1-\alpha} B_{b-}^H(x) \in L_\infty[a, b]$ for any $0 \le a < b$. Therefore, for $f \in I_{a+}^{\alpha}(L_1[a, b])$ we can define the integral w.r.t. fBm in the following way:

Definition

The integral with respect to fBm is defined as

$$\int_a^b f dB^H := \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} B_{b-}^H)(x)dx. \tag{4}$$

The evident estimate follows immediately from (4):

$$\left| \int_a^b f dB^H \right| \leq \sup_{a \leq x \leq b} |(D_{b-}^{1-\alpha} B_{b-}^H)(x)| \int_a^b |(D_{a+}^{\alpha} f)(x)| dx. \tag{5}$$

We suppose that the coefficients satisfy the following assumptions
(i) There exists such $K > 0$ that for any $s \in [0, T]$ and $x \in \mathbb{R}$

$$|a(s,x)| + |b(s,x)| \leq K(1 + |x|).$$

(ii) There exists such $L > 0$ that for any $t \in [0, T]$ and $x, y \in \mathbb{R}$

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq L|x - y|.$$

(iii) The function $b(t,x)$ is differentiable in x and there exist such constant $B>0$ and parameter $\beta\in(1-H,1)$ that for any $s,t\in[0,T]$ and $x\in\mathbb{R}$

$$|a(s,x)-a(t,x)|+|b(s,x)-b(t,x)|+|\partial_x b(s,x)-\partial_x b(t,x)|$$

$$\le B|s-t|^{\beta}.$$

(iv) Partial derivative $\partial_x b(t,x)$ is Hölder continuous in x: there exist such constant $D>0$ and parameter $\rho\in(3/2-H,1)$ that for any $t\in[0,T]$ and $x,y\in\mathbb{R}$

$$|\partial_x b(t,x)-\partial_x b(t,y)|\le D|x-y|^{\rho}.$$

As it was stated by Nualart and Rascanu, under the conditions (i)–(iv) the equation (3) has the unique solution $\{X_t, t\in[0,T]\}$, and this solution has a.s. the trajectories that are Hölder up to order H.

Now, let $T > 0$ be fixed. We are in a position to find the likelihood ratio $\frac{dP_\theta(t)}{dP_0(t)}$ for the probability measure $P_\theta(t)$ corresponding to our model and the probability measure $P_0(t)$ corresponding to the model with zero drift for $t \in [0, T]$. Suppose that the following assumption holds:

(v) $b(t, X_t) \neq 0, t \in [0, T]$ and $\frac{a(t,X_t)}{b(t,X_t)}$ is a.s. Lebesgue integrable on $[0, T]$.

Denote $\psi(t, x) = \frac{a(t,x)}{b(t,x)}$, $\varphi(t) := \psi(t, X_t)$. Also, let the kernel

$$l_H(t, s) = c_H s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H} I_{\{0<s<t\}}, \text{ with } c_H = \left(\frac{\Gamma(3-2H)}{2H\Gamma(\frac{3}{2}-H)^3\Gamma(H+\frac{1}{2})}\right)^{\frac{1}{2}},$$

the integral $J_t = \int_0^t l_H(t, s)\varphi(s)ds$. Let also $M_t^H = \int_0^t l_H(t, s)dB_s^H$ be Gaussian martingale with square bracket $\langle M\rangle_t^H = t^{2-2H}$ (Molchan martingale).

Introduce the processes

$$Y_t = \int_0^t b^{-1}(s, X_s) dX_s = \theta \int_0^t \varphi(s) ds + B_t^H$$

and

$$Z_t = \int_0^t l_H(t, s) dY_s = \theta J_t + M_t^H.$$

Note that the process $Z_t = \int_0^t l_H(t, s) b^{-1}(s, X_s) dX_s$, so it is the functional of the observable process X. The following result guaranties semimartingale property of Z_t.

Lemma 1. *Let $\psi(t,x)\in C^1(\mathbb{R}^+)\times C^2(\mathbb{R})$. Then for any $t>0$*

$$J'(t)=(2-2H)C_H\psi(0,x_0)t^{1-2H}+\int_0^t l_H(t,s)\left(\psi'_t(s,X_s)+\theta\psi'_x(s,X_s)a(s,X_s)\right)$$

$$-\left(H-\frac{1}{2}\right)c_H\int_0^t s^{-\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\int_0^s\left(\psi'_t(u,X_u)+\theta\psi'_x(u,X_u)a(u,X_u)\right)duds$$

$$+(2-2H)c_Ht^{1-2H}\int_0^t s^{2H-3}\int_0^s u^{\frac{3}{2}-H}(s-u)^{\frac{1}{2}-H}\psi'_x(u,X_u)b(u,X_u)dB_u^Hds$$

$$+c_Ht^{-1}\int_0^t u^{\frac{3}{2}-H}(t-u)^{\frac{1}{2}-H}\psi'_x(u,X_u)b(u,X_u)dB_u^H,$$

where $C_H=B(\frac{3}{2}-H,\frac{3}{2}-H)c_H=\left(\frac{\Gamma(\frac{3}{2}-H)}{2H\Gamma(H+\frac{1}{2})\Gamma(3-2H)}\right)^{\frac{1}{2}}$, and all the involved integrals exist a.s.

The same as Z_t, the processes J_t and J'_t are the functionals of X. It is more convenient to consider the process $\chi(t) = (2-2H)^{-1}J'(t)t^{2H-1}$, so that

$$Z_t = (2-2H)\theta\int_0^t \chi(s)s^{1-2H}ds + M_t^H = \theta\int_0^t \chi(s)d\langle M^H\rangle_s + M_t^H.$$

Suppose that the following conditions hold:

(vi) $EI_T := E\int_0^T \chi_s^2 d\langle M^H\rangle_s < \infty$ for any $T > 0$ and

(vii) $I_\infty := \int_0^\infty \chi_s^2 d\langle M^H\rangle_s = \infty$ a.s.

Under the additional assumption

(viii) $E\exp\{\int_0^T \chi_s dM_s^H - \frac{1}{2}\int_0^T \chi_s^2 d\langle M^H\rangle_s\} = 1$ for any $T > 0$

the process Y_t is an fBm on $[0, T]$ w.r.t. the measure P_θ defined via the relation

$$\frac{dP_\theta(t)}{dP_0(t)} = \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\},\ t \in [0, T], \tag{6}$$

where

$$L_t = \int_0^t \chi_s dM_s^H,$$

and we can consider the maximum likelihood estimate

$$\hat{\theta}_T = \frac{\int_0^T \chi_s dZ_s}{\int_0^T \chi_s^2 d\langle M^H\rangle_s} = \theta + \frac{\int_0^T \chi_s dM_s^H}{\int_0^T \chi_s^2 d\langle M^H\rangle_s}. \tag{7}$$

Under condition (vi) the process $\int_0^t \chi_s dM_s^H, t > 0$ is a square integrable martingale and it is famous result of Liptser, Shiryaev that $\frac{X_t}{\langle X \rangle_t} \to 0$ a.s. if X_t is a square-integrable martingale and $\langle X \rangle_t \to \infty$ a.s. In other words,

$$\frac{\int_0^T \chi_s dM_s^H}{\int_0^T \chi_s^2 d\langle M^H \rangle_s} \to 0, \ T \to \infty,$$

with P_θ-probability 1, and $\hat{\theta}_T$ is strongly consistent as $T \to \infty$.

Example

Consider the linear model of the form

$$dX_t = \theta X_t dt + X_t dB_t^H.$$

In this case $\varphi_t = 1$, and

$$\widehat{\theta}_t = \theta + \frac{\widetilde{\alpha} \int_0^t s^{-\alpha} dW_s}{C(H) t^{1-2\alpha}}.$$

Since $\widetilde{\alpha} \int_0^t s^{-\alpha} dW_s$ is the square-integrable martingale with the angle bracket $t^{1-2\alpha} \to \infty$ when $t \to \infty$, then, according to above statement, $\frac{\widetilde{\alpha} \int_0^t s^{-\alpha} dW_s}{C(H) t^{1-2\alpha}} \to 0$, a.s. as $t \to \infty$. So, the estimate $\widehat{\theta}_t$ is consistent with probability 1.

Introduce one more example when these conditions hold. Consider generalized fractional Ornstein-Uhlenbeck model

$$dX_t = \theta(b - X_t)dt + \gamma dB_t^H, t \geq 0, \gamma > 0, b \in \mathbb{R}.$$

The solution of this equation is the Gaussian process and has a form

$$X_t = X_0 e^{-\theta t} + b\left(1 - e^{-\theta t}\right) + \gamma e^{-\theta t}\int_0^t e^{\theta s} dB_s^H. \tag{8}$$

In this case $a(s,x) = \theta(b - x)$, $b(s,x) = \gamma$, $\psi(s,x) = \theta\gamma^{-1}(b - x)$,

$$J_t = \theta\gamma^{-1}\int_0^t l_H(t,s)(b - X_s)ds = \theta\gamma^{-1} b C_H t^{2-2H} -$$

$$\theta\gamma^{-1}\int_0^t l_H(t,s)\Big(X_0 + (b - X_0)(1 - e^{-\theta s}) + \gamma e^{-\theta s}\int_0^s e^{\theta u} dB_u^H\Big)ds,$$

and

$$
\begin{gathered}
J_t' = \theta\gamma^{-1}(2-2H)C_H(b-X_0)t^{1-2H} \\
+\theta\gamma^{-1}c_H\Big(H-\frac{1}{2}\Big)\int_0^t (t-s)^{\frac{1}{2}-H}s^{-\frac{1}{2}-H}\Big((b-X_0) \\
\times(1-e^{-\theta s}) + \gamma e^{-\theta s}\int_0^s e^{\theta u}dB_u^H\Big)ds \\
-\theta\gamma^{-1}(b-X_0)\int_0^t l_H(t,s)e^{-\theta s}ds \\
-\theta^2\int_0^t l_H(t,s)e^{-\theta s}\int_0^s e^{\theta u}dB_u^H ds - \theta M_t^H.
\end{gathered}
$$

Evidently, J_t' is the Gaussian process with mean and variance that are bounded on any bounded interval. Therefore, conditions (vi) and (viii) hold. As to condition (vii), for technical simplicity we establish it in the most simple case when $b = X_0 = 0$, $\gamma = 1$, so that

$$J_t' = \theta c_H\Big(H - \frac{1}{2}\Big)\int_0^t l_H(t,s)s^{-1}\Big(e^{-\theta s}\int_0^s e^{\theta u}dB_u^H\Big)ds$$
$$-\theta^2\int_0^t l_H(t,s)(e^{-\theta s}\int_0^s e^{\theta u}dB_u^H)ds - \theta M_t^H$$
$$= -\int_0^t l_H(t,s)f(s)\Big(e^{-\theta s}\int_0^s e^{\theta u}dB_u^H\Big)ds - \theta M_t^H,$$

where $f(s) = -\theta c_H\Big(H - \frac{1}{2}\Big)s^{-1} + \theta^2$.

Consider for any $a > 0$ the moment generation functions $\Theta_T(a) = E\exp\{-aI_T\} = E\exp\{-a\int_0^T (J_t')^2 t^{2H-1}dt\}$ and $\Theta_\infty(a) = E\exp\{-aI_\infty\} = E\exp\{-a\int_0^\infty (J_t')^2 t^{2H-1}dt\}$, so that $\Theta_\infty(a) = \lim_{T\to\infty}\Theta_T(a)$. Evidently,

$$\int_0^T (J_t')^2 t^{2H-1}dt \geq T^{-1}\Big(\int_0^T J_t' t^{H-\frac{1}{2}}dt\Big)^2,$$

whence

$$\Theta_T(a) \leq \Theta_T^1(a) := E \exp\Big\{ - \frac{a}{T}\Big(\int_0^T J_t' t^{H-\frac{1}{2}} dt \Big)^2 \Big\}.$$

The random variable $\int_0^T J_t' t^{H-\frac{1}{2}} dt$ is Gaussian with zero mean and variance σ_T^2, say. Therefore $\Theta_T^1(a) = \Big(1 + \frac{2a\sigma_T^2}{T}\Big)^{-\frac{1}{2}}$. After long and detailed computations we can obtain that

$$\sigma_T^2 \sim c(H)\theta^4 T^{5-2H},\ T \to \infty,$$

so, condition (vii) holds.

Now we consider the linear mixed Brownian–fractional-Brownian diffusion model represented by the stochastic differential equation of the form

$$dX_t = \theta X_t dt + \sigma_1 X_t dB_t + \sigma_2 X_t dB_t^H, \tag{9}$$

$X_{t=0} = X_0 \in \mathbb{R},\ 0 \le t \le T,\ T > 0,\ \{\theta, \sigma_1, \sigma_2\} \subset \mathbb{R},\ \sigma_1\sigma_2 < 0$, θ is a parameter that we need to estimate.

We suppose that the Wiener process B and the fBm B^H in (9) are connected via the relations (1). The integral form of equation (9) is

$$X_t = X_0 + \theta \int_0^t X_s ds + \sigma_1 \int_0^t X_s dB_s + \sigma_2 \int_0^t X_s dB_s^H, \ 0 \leq t \leq T. \quad (10)$$

The solution of equation (10) exists and is unique.

The Girsanov Theorem for the Mixed Fractional Diffusion Model

First we try to change the probability measure P_θ for the another measure P_0, $P_\theta(T) \sim P_0(T)$ in order to exclude the drift $\theta X_t dt$ from equations (9) and (10).

We introduce probability measures $P_{0,i}, i = 1, 2$ and $P_{\theta,i}, i = 1, 2$ as follows. The probability measures $P_{0,1}(t)$ and $P_{\theta,1}(t)$ are determined by the following condition:

$$\frac{dP_{\theta,1}(t)}{dP_{0,1}(t)} = \exp\left\{\int_0^t \psi_s \, dB_s^{(1)} - \frac{1}{2}\int_0^t \psi_s^2 \, ds\right\}$$

for a nonrandom function ψ_s such that $\int_0^t \psi_s^2 \, ds < \infty$ and

$$E \exp\left\{\int_0^t \psi_s \, dB_s^{(1)} - \frac{1}{2}\int_0^t \psi_s^2 \, ds\right\} = 1.$$

Here the process $B_t^{(1)}$ is composed according to the Girsanov theorem,

$$B_t^{(1)} := B_t + \int_0^t \psi_s \, ds, \tag{11}$$

and $B_t^{(1)}$ is a standard Wiener process with respect to the probability measure $P_{0,1}(t)$.

The probability measures $P_{0,2}(t)$ and $P_{\theta,2}(t)$ satisfy the relation

$$\frac{dP_{\theta,2}(t)}{dP_{0,2}(t)} = \exp\left\{ \int_0^t s^{\alpha}\delta_s \, dB_s^{(2)} - \frac{1}{2}\int_0^t s^{2\alpha}\delta_s^2 \, ds \right\},$$

where δ_s satisfies the relation $\int_0^t l_H(t,s)|\delta_s| \, ds < \infty$, $t \in [0,T]$ and admits the following integral representation:

$$\int_0^t l_H(t,s)\varphi_s \, ds = \widetilde{\alpha} \int_0^t \delta_s \, ds, \tag{12}$$

the Wiener process $B_t^{(2)}$ is defined from the equation

$$\int_0^t l_H(t,s) \, dB_s^{H,2} = \widetilde{\alpha} \int_0^t s^{-\alpha} \, dB_s^{(2)},$$

and the process

$$B_t^{H,2} := B_t^H + \int_0^t \varphi_s \, ds \tag{13}$$

is a fractional Brownian motion on $[0, T]$ with respect to the measure $P_{0,2}(t)$.

So, the total drift coefficient equals

$$\sigma_1 \int_0^t \psi_s\, ds + \sigma_2 \int_0^t \varphi_s\, ds = \theta t,$$

and if we suppose that the functions ψ and φ are continuous, we obtain that

$$\sigma_1 \psi_t + \sigma_2 \varphi_t = \theta. \tag{14}$$

Obviously, from (11)–(13) and since the likelihood ratios $\frac{dP_{\theta,i}(t)}{dP_{0,i}(t)}$ must coincide, we obtain that

$$\widehat{M}_t^H = M_t^H + \widetilde{\alpha} \int_0^t s^{-\alpha} \psi_s ds$$

and

$$\widehat{M}_t^H = M_t^H + \widetilde{\alpha} \int_0^t \delta_s ds,$$

whence $t^{\alpha} \delta_t = \psi_t,\ t \in [0, T]$.

Moreover,

$$\int_0^t l_H(t,s)\varphi_s ds = \widetilde{\alpha} \int_0^t s^{-\alpha}\psi_s ds.$$

Multiplying by $(t-s)^{\alpha-1}$ and integrating, we obtain

$$C_H^{(5)} \int_0^t (t-s)^{\alpha-1} \int_0^s u^{-\alpha}(s-u)^{-\alpha}\varphi_u du\, ds$$

$$= \widetilde{\alpha} \int_0^t (t-s)^{\alpha-1} \int_0^s \delta_u du\, ds, \tag{15}$$

and the Fubini theorem applied to both sides of (15) gives

$$C(H,2) \int_0^t u^{-\alpha}\varphi_u du = \frac{\widetilde{\alpha}}{\alpha} \int_0^t (t-u)^{\alpha}\delta_u du,$$

whence

$$\varphi_t = \frac{1}{C(H,3)} t^{\alpha} \int_0^t (t-u)^{\alpha-1} u^{-\alpha}\psi_u du. \tag{16}$$

Here $C(H,2) = C_H^{(5)} B(\alpha, 1-\alpha), C(H,3) = \frac{\widetilde{\alpha}}{C(H,2)}$.
Substituting (16) into (14), we obtain a Volterra equation of the second kind, with weak singularity, of the form

$$\sigma_1 \psi_t + \frac{\sigma_2}{C(H,3)} t^{\alpha} \int_0^t (t-u)^{\alpha-1} u^{-\alpha} \psi_u du = \theta,$$

or

$$\rho_t + \frac{\sigma_2}{\sigma_1} \frac{1}{C(H,3)} \int_0^t (t-u)^{\alpha-1} \rho_u du = \frac{e_t}{\sigma_1}, \tag{17}$$

where $\rho_t = t^{-\alpha} \psi_t$, $e_t = \theta t^{-\alpha}$.

We solve (17) using successive approximations

$$\rho_t^{(n+1)} + \frac{\sigma_2}{\sigma_1}\frac{1}{C(H,3)}\int_0^t (t-u)^{\alpha-1}\rho_u^{(n)}du = \frac{e_t}{\sigma_1}. \tag{18}$$

Denote for simplicity $C := \frac{\sigma_2}{\sigma_1 C(H,3)}$ and start with $\rho_t^{(0)} = 0$, $\rho_t^{(1)} = \frac{e_t}{\sigma_1}$. Then we obtain from (18) that

$$\rho_t^{(2)} = (-1)\frac{C}{\sigma_1}\int_0^t (t-u)^{\alpha-1}e_u du + \frac{e_t}{\sigma_1}.$$

It is very simple now to prove by induction that for $n > 1$

$$\rho_t^{(n)} = \frac{1}{\sigma_1} \sum_{k=1}^{n-1} (-C)^k \int_0^t e_s (t-s)^{k\alpha - 1} \frac{\Gamma^k(\alpha)}{\Gamma(k\alpha)} ds + \frac{e_t}{\sigma_1},$$

and the solution $\rho_t = \lim_{n\to\infty} \rho_t^{(n)}$ evidently can be represented as a series

$$\rho_t = \frac{1}{\sigma_1} \sum_{n=1}^{\infty} (-C)^n \int_0^t e_s (t-s)^{n\alpha - 1} \frac{\Gamma^n(\alpha)}{\Gamma(n\alpha)} ds + \frac{e_t}{\sigma_1}.$$

Hence

$$\psi_t = t^\alpha \rho_t = \frac{\theta}{\sigma_1} \Gamma(1-\alpha) \sum_{n=1}^{\infty} (-C)^n \frac{\Gamma^n(\alpha)}{\Gamma((n-1)\alpha + 1)} t^{n\alpha} + \frac{\theta}{\sigma_1}. \qquad (19)$$

The series on the right-hand side of (19) can be expressed in terms of the Mittag–Leffler function $E_\rho(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n/\rho+1)}$,

$$A_t = -Ct^\alpha \pi (\sin \pi\alpha)^{-1} E_{1/\alpha}(-C\Gamma(\alpha)t^\alpha),$$

and in these terms $\psi_t = \frac{\theta}{\sigma_1}(A_t + 1)$. Therefore, the likelihood ratio for the mixed fractional Brownian model equals

$$\frac{dP_{\theta,1}(t)}{dP_{0,1}(t)} = \exp\left\{\int_0^t \psi_s dB_s^{(1)} - \frac{1}{2}\int_0^t \psi_s^2 ds\right\}$$

$$= \exp\left\{\frac{\theta}{\sigma_1}\int_0^t (A_s+1)dB_s^{(1)} - \frac{1}{2}\frac{\theta^2}{\sigma_1^2}\int_0^t (A_s+1)^2 ds\right\},$$

whence the maximum likelihood estimate for θ equals

$$\widehat{\theta}_T^1 = \sigma_1 \frac{\int_0^T (A(s)+1)dB_s^{(1)}}{\int_0^t (A(s)+1)^2 ds}$$

$$= \sigma_1 \frac{\int_0^T (A(s)+1)dB_s + \frac{\theta}{\sigma_1}\int_0^T (A(s)+1)^2 ds}{\int_0^t (A(s)+1)^2 ds} = \theta + \sigma_1 \frac{\int_0^t (A(s)+1)dB_s}{\int_0^T (A(s)+1)^2 ds}.$$

For the demonstration of the consistency of the estimate $\widehat{\theta}^1_T$ with probability 1, it is sufficient to prove the divergence of the integral $\int_0^t (A(s)+1)^2 ds$ when $t \to \infty$. Note that $C < 0$ since $\frac{\sigma_1}{\sigma_2} < 0$, and

$$\sum_{n=1}^{\infty} (-C\Gamma(\alpha))^n \frac{t^{n\alpha}}{\Gamma((n-1)\alpha+1)} > \sum_{n=1}^{\infty} (-C\Gamma(\alpha))^n \frac{t^{n\alpha}}{\Gamma(n+1)} =$$

$$= \sum_{n=1}^{\infty} (-C\Gamma(\alpha))^n \frac{t^{n\alpha}}{n!} = \exp\{-C\Gamma(\alpha)t^{\alpha}\} \to \infty,$$

when $t \to \infty$ because $\alpha > 0$ and $-C\Gamma(\alpha) > 0$. Note that $\delta_t = t^{-\alpha}\psi_t$ satisfies conditions (ii)–(vi). So we have proved the following result.

Theorem 1. *The drift parameter maximum likelihood estimate of the linear Brownian–fractional-Brownian model* (9) *is consistent with probability* 1.

The Asymptotic Normality of the Maximum Likelihood Estimates

First, consider one of the limit theorems for the stochastic integrals w.r.t. the Wiener process $\{W_t, \mathcal{F}_t, t > 0\}$. Let $\{h(s), s \geq 0\}$ be an $\mathcal{F}_s$-adapted predictable function such that $\mathbb{E} \int_0^t h^2(s)ds$ is finite for any $t > 0$ and $\mathcal{F}_n(t) = \sigma\{h(s),\, W(s),\, s \leq nt\}$. Consider the sequence $Y_n(t) := \int_0^{nt} h(s)dW_s$. Evidently, $Y_n(t)$ are $\mathcal{F}_n(t)$-square-integrable martingales, $t \in [0, T]$, and their angle brackets equal $\langle Y_n \rangle (t) = \int_0^{nt} h^2(s)ds$. Assume the following:

(ix) there exists a sequence $\{A_n,\, n \geq 1\}$ such that $A_n \uparrow \infty$, $n \to \infty$ and for some $c_0 > 0$

$$\int_0^n h^2(s)ds \cdot A_n^{-2} \xrightarrow{P} c_0.$$

Consider the sequence of normalized square-integrable martingales $X_n(t) := A_n^{-1} \cdot \int_0^{nt} h(s)dW_s$. Then $\langle X_n \rangle (1) = \int_0^n h^2(s)ds \cdot A_n^{-2} \xrightarrow{P} c_0$, therefore X_n satisfy conditions of one of the limit theorems (Theorem 4.1) of Liptser, Shiryaev, if we consider the set of convergence points consisting of one point $t = 1$. By using this theorem we obtain the following result:

Lemma 2. *Let condition (ix) hold. Then the random variable*

$$Z_n := \int_0^n h(s)dW_s \cdot (A_n)^{-1}$$

weakly converges to the random variable $c_0^{-1/2}N(0,1)$.
Consider the estimate $\widehat{\theta}_n$ of the form (7). We see that for the pure fractional diffusion model $h(s) = A(s) + 1$ and is nonrandom. Therefore we obtain from Lemma 2 that

$$\left(\int_0^n (A(s)+1)^2 ds\right)^{1/2} (\widehat{\theta}_n^1 - \theta) \to N(0,1).$$

Moreover, under the assumption
(x) there exists a sequence $\{A_n,\ n \geq 1\}$ such that $A_n \uparrow \infty$, $n \to \infty$ and

$$\int_0^n s^{2\alpha}(J_s')^2 ds \cdot A_n^{-2} \xrightarrow{P} \varphi_0, n \to \infty,$$

we have a weak convergence

$$\varphi_0^{1/2} A_n(\widehat{\theta}_n - \theta) \to N(0,1).$$

In this sense we say that the estimate $\widehat{\theta}_n$ are asymptotically normal.

Now we consider an "opposite" situation where the components of the diffusion model — a fBm B^H and a Wiener process B — are independent.

The Estimates of the Drift Parameter in the Mixed Brownian–Fractional-Brownian Diffusion Model Where B_t and B_t^H are Independent

Let the diffusion equation contain stochastic differentials with respect to fBm and the Wiener process,

$$dX_t = \theta X_t\, dt + \sigma_1 X_t\, dB_t + \sigma_2 X_t\, dB_t^H,$$

$X_{t=0} = X_0 \in \mathbb{R}$, $0 \le t \le T,\ T > 0$, $\{\theta, \sigma_1, \sigma_2\} \subset \mathbb{R} \setminus \{0\}$, where the processes B_t and B_t^H are independent. Evidently, we can rewrite the solution of our simple linear equation as

$$X_t = X_0 \exp\{\theta t + \sigma_1 B_t + \sigma_2 B_t^H - 1/2\sigma_1^2 t\}.$$

So, we suppose that we observe both components. Let, as before, θ be the parameter to be estimated. We shall try to represent the estimate of θ via the components B_t and B_t^H because it seems to be impossible to represent it via the whole process X_t. Let P_θ be the basic probability measure corresponding to the process X. We introduce probability measures $P_{0,i}, i = 1, 2$ and $P_{\theta,i}, i = 1, 2$ as follows. The probability measures $P_{0,1}(t)$ and $P_{\theta,1}(t)$ are determined by

$$\frac{dP_{\theta,1}(t)}{dP_{0,1}(t)} = \exp\left\{\int_0^t \psi_s \, dB_s^{(1)} - \frac{1}{2}\int_0^t \psi_s^2 \, ds\right\}$$

for a nonrandom function ψ_s such that $\int_0^t \psi_s^2 \, ds < \infty$ and

$$E \exp\left\{\int_0^t \psi_s \, dB_s^{(1)} - \frac{1}{2}\int_0^t \psi_s^2 \, ds\right\} = 1.$$

Here the process $B_t^{(1)}$ is composed accordingly to the Girsanov theorem,

$$B_t^{(1)} := B_t + \int_0^t \psi_s \, ds \tag{20}$$

and $B_t^{(1)}$ is a standard Wiener process with respect to the probability measure $P_{0,1}(t)$.

The probability measures $P_{0,2}$ and $P_{\theta,2}(t)$ satisfy the relation

$$\frac{dP_{\theta,2}(t)}{dP_{0,2}(t)} = \exp\left\{\int_0^t s^{\alpha}\delta_s\, dB_s^{(2)} - \frac{1}{2}\int_0^t s^{2\alpha}\delta_s^2\, ds\right\},$$

where δ_s satisfies the relation $\int_0^t l_H(t,s)|\delta_s|\, ds < \infty$, $t \in [0, T]$, admits the following representation:

$$\int_0^t l_H(t,s)\varphi_s\, ds = \widetilde{\alpha}\int_0^t \delta_s\, ds, \tag{21}$$

the Wiener process $B_t^{(2)}$ is defined from the equation

$$\int_0^t l_H(t,s)\, dB_s^{H,2} = \widetilde{\alpha}\int_0^t s^{-\alpha}\, dB_s^{(2)},$$

and the process

$$B_t^{H,2} := B_t^H + \int_0^t \varphi_s\, ds$$

is a fractional Brownian motion on $[0, T]$ with respect to the measure $P_{0,2}(t)$.

So, the total drift coefficient equals

$$\sigma_1 \int_0^t \psi_s \, ds + \sigma_2 \int_0^t \varphi_s \, ds = \theta t,$$

or, if we suppose that the functions ψ and φ are continuous,

$$\sigma_1 \psi_t + \sigma_2 \varphi_t = \theta. \tag{22}$$

Since B_t and B_t^H are independent, the final probability measure $P_0(t)$ is the product of the measures $P_{0,1}(t)$ and $P_{0,2}(t)$. Thus the final likelihood ratio is

$$\frac{dP_\theta(t)}{dP_0(t)} = \exp\left[\left\{\int_0^t \psi_s \, dB_s^{(1)} - \frac{1}{2}\int_0^t \psi_s^2 \, ds\right\} \times \left\{\int_0^t s^\alpha \delta_s \, dB_s^{(2)} - \frac{1}{2}\int_0^t s^{2\alpha}\delta_s^2 \, ds\right\}\right]$$

$$= \exp\left\{\int_0^t \psi_s \, dB_s^{(1)} + \int_0^t s^\alpha \delta_s \, dB_s^{(2)} - \frac{1}{2}\int_0^t \left(\psi_s^2 + s^{2\alpha}\delta_s^2\right) \, ds\right\}. \tag{23}$$

Solving equations (21) and (22) with respect to the functions ψ_t and δ_t, respectively, we obtain

$$\psi_t = \frac{1}{\sigma_1}(\theta - \sigma_2\varphi_t), \tag{24}$$

$$\delta_t = \widehat{\alpha}\left(\int_0^t l_H(t,s)\varphi_s ds\right)'_t. \tag{25}$$

Substituting equalities (24) and (25) into likelihood ratio (23), we get at the point $t = T$ that

$$\frac{dP_\theta(T)}{dP_0(T)} = \exp\left\{\frac{1}{\sigma_1}\int_0^T (\theta - \sigma_2\varphi_s)\, dB_s^{(1)} + \widehat{\alpha}\int_0^T s^\alpha \left(\int_0^s l_H(s,u)\varphi_u\, du\right)'_s dB_s^{(2}\right.$$

$$\left. - \frac{1}{2}\int_0^T \left[\frac{1}{\sigma_1^2}(\theta - \sigma_2\varphi_s)^2 + s^{2\alpha}\widehat{\alpha}\left(\left(\int_0^s l_H(s,u)\varphi_u\, du\right)'_s\right)^2\right] ds\right\}. \tag{26}$$

If follows from (26) that the maximum likelihood estimate $\widehat{\theta}^1_T$ of the parameter θ satisfies the equality

$$\frac{1}{\sigma_1}\int_0^T dB_s^{(1)} - \frac{1}{\sigma_1^2}\int_0^T (\theta - \sigma_2\varphi_s)\, ds = 0,$$

which can be rewritten as follows:

$$\sigma_1 B_T^{(1)} + \sigma_2 \int_0^T \varphi_s\, ds - \theta T = 0.$$

This gives us the following estimate of the parameter θ:

$$\widehat{\theta}^1_T = \frac{\sigma_1 B_T^{(1)}}{T} + \frac{\sigma_2 \int_0^T \varphi_s\, ds}{T}. \tag{27}$$

Now we solve equation (22) with respect to the function φ_t and substitute it into equation (27):

$$\widehat{\theta}_T^1 = \theta + \frac{\sigma_1}{T}\left(B_T^{(1)} - \int_0^T \psi_s\, ds\right). \tag{28}$$

Substituting (20) into (28) yields

$$\widehat{\theta}_T^1 = \theta + \sigma_1 \frac{B_T}{T}. \tag{29}$$

It is evident that the estimate (29) of parameter θ^1_T is strongly consistent. We can construct another estimate of the parameter θ. The function δ_t is expressed via φ_t by equality (21). Denote also $\zeta_t := \left(\int_0^t l_H(t,s)\psi_s\, ds\right)'_t$. Then

$$\begin{aligned}\delta_t &= \widehat{\alpha}\left(\int_0^t l_H(t,s)\varphi_s\, ds\right)'_t = \frac{1}{\sigma_2}\widehat{\alpha}\left(\int_0^t l_H(t,s)(\theta - \sigma_1\psi_s)\, ds\right)'_t \\ &= \widehat{\alpha}\left(\frac{\theta}{\sigma_2}\left(\int_0^t l_H(t,s)\, ds\right)'_t - \frac{\sigma_1}{\sigma_2}\zeta_t\right) \\ &= \widehat{\alpha}\Big(\frac{\theta}{\sigma_2}C(H)t^{-2\alpha} - \frac{\sigma_1}{\sigma_2}\zeta_t\Big), \end{aligned} \tag{30}$$

where $C(H) = C_H^{(5)}(1-2\alpha)B_1 C_H^{(5)}, B_1 = B\,(1-\alpha, 1-\alpha)$.

Using equality (30) for likelihood ratio (23), taking the logarithms, differentiating with respect to θ, and equating the derivative to zero, we obtain at the point $t = T$

$$\int_0^T s^{-\alpha}\, dB_s^{(2)} - \widehat{\alpha} \int_0^T \left(\frac{\theta C(H)}{\sigma_2} s^{-2\alpha} - \frac{\sigma_1}{\sigma_2} \zeta_s \right) ds = 0,$$

or

$$\int_0^T s^{-\alpha}\, dB_s^{(2)} - \widehat{\alpha}^3 \theta \frac{C(H)}{\sigma_2} T^{1-2\alpha} + \widehat{\alpha} \frac{\sigma_1}{\sigma_2} \int_0^T l_H(T, s) \psi_s\, ds = 0.$$

This implies another estimate for the parameter θ:

$$\widehat{\theta}_T^2 = \frac{\sigma_2 \widetilde{\alpha} \int_0^T s^{-\alpha} \, dB_s^{(2)} + \sigma_1 \int_0^T l_H(T,s) \psi_s \, ds}{C_H^{(5)} B_1 T^{1-2\alpha}}. \tag{31}$$

Now we substitute the expression (24) for the function ψ_t into relation (31) and obtain with $C(H,1) = C_H^{(5)} B_1$ that

$$\widehat{\theta}^2_T = \theta - \frac{\sigma_2}{C(H,1)T^{1-2\alpha}}\left[\int_0^T l_H(T,s)\varphi_s\, ds - \widetilde{\alpha}\int_0^T s^{-\alpha} dB_s^{(2)}\right].$$

Recall that $\widetilde{\alpha}\int_0^T s^{-\alpha}\,dB_s^{(2)} = \int_0^T l_H(T,s)\,dB_s^{H,2}$.
Further,

$$\int_0^T l_H(T,s)\varphi_s\,ds - \int_0^T l_H(T,s)\,dB_s^{H,2} = -\int_0^T l_H(T,s)\,dB_s^H.$$

So, the second estimate of the parameter θ is given by

$$\widehat{\theta}_T^2 = \theta + \frac{\sigma_2}{C(H,1)T^{1-2\alpha}}\int_0^T l_H(T,s)\,dB_s^H,$$

or

$$\widehat{\theta}_T^2 = \theta + \frac{\sigma_2\widetilde{\alpha}}{C(H,1)}\frac{\int_0^T s^{-\alpha}\,d\widetilde{B}_s}{T^{1-2\alpha}}, \tag{32}$$

where $\widetilde{B}_s$ is some Wiener process. The strong consistency of the estimate $\widehat{\theta}_T^2$ is also clear.

Now we compare the estimates $\widehat{\theta}^1_T$ and $\widehat{\theta}^2_T$. First we compute the variances of the remainder terms in formulae (29) and (32) and compare $\sigma_1^2 T^{-1}$ and $\sigma_2^2 C(H,1)^{-2} T^{2\alpha-1}$. Since $H \in \left(\frac{1}{2}, 1\right)$, it is obvious that there exists a number N such that $\sigma_1^2 T^{-1} < \sigma_2^2 C(H,1)^{-2} T^{2\alpha-1}$ for all $T > N$. It means that the variance of the deviation between the estimate $\widehat{\theta}^1_T$ and true value is smaller than that of the corresponding deviation between the estimate $\widehat{\theta}^2_T$ and the true value. It this sense, the estimate $\widehat{\theta}^1_T$ is better than $\widehat{\theta}^2_T$.

Local Asymptotic Normality and Asymptotic Efficiency of the Estimate of the Drift Parameter in a Linear Brownian Diffusion Model

Consider (only for comparison with the fractional case, see below) a pure linear Brownian model

$$dX_t^\theta = \frac{1}{T^\beta}\theta X_t\,dt + cX_t\,dB_t,\ X_{t=0} = X_0,\ c \in \mathbb{R}\setminus\{0\},\ t\in[0,T],\ \beta\in\left(\frac{1}{2},1\right]$$

Put $\Theta = (0,\infty)$ and let $\theta\in\Theta$. According to Definition 2.1 of Ibragimov, Khas'minski, a family of measures $P_\theta(t)$ has the property of local asymptotic normality (LAN) at the point $\theta\in\Theta$ as $t\to\infty$, if

$$Z_{t,\theta}(u) := \frac{dP_{\theta+A(t,\theta)u}(t)}{dP_\theta(t)} = \exp\left\{u\xi_{t,\theta} - \frac{1}{2}u^2 + \zeta_t(u,\theta)\right\} \qquad (33)$$

for some function $A(t,\theta)$ and any number $u \in \mathbb{R}$, where $\xi_{t,\theta} \Rightarrow N(0,1)$ as $t \to \infty$ with respect to the measure $P_\theta(t)$, and $\zeta_t(u,\theta) \overset{P_\theta(t)}{\to} 0$, $t \to \infty$, for all numbers $u \in \mathbb{R}$. We say in this case that the LAN property holds for the family of measures $P_\theta(t)$ as $t \to \infty$ at the point θ.

Theorem 2. *The LAN property holds for the family of measures $P_\theta(t)$ as $t \to \infty$ at any point $\theta \in \Theta$.*

Proof. We change the probability measure $P_\theta(t)$, which corresponds to the process X_t^θ for the measure $P_0(t)$. Then the drift $\theta X_t\, dt$ disappears and we obtain

$$X_t^0 = X_0 + c \int_0^t X_s^0 \, d\widehat{B}_t,$$

where $\widehat{B}_t = B_t + t\theta/(cT^\beta)$ is a Wiener process w.r.t. the measure $P^0(t)$.

Consider the likelihood ratio corresponding to this change of measure with $\varphi_s = \theta/(cT^\beta)$:

$$\frac{dP_\theta(t)}{dP_0(t)} = \exp\left\{\int_0^t \frac{\theta}{cT^\beta}\, d\widehat{B}_s - \frac{1}{2}\int_0^t \frac{\theta^2}{(cT^\beta)^2}\, ds\right\}$$

$$= \exp\left\{\frac{\theta}{cT^\beta}\widehat{B}_t - \frac{1}{2}\frac{\theta^2}{(cT^\beta)^2} t\right\}.$$

Now we consider the linear model with a parameter θ shifted by $A(t)u$. The likelihood ratio for such a change of measure is of the form

$$\frac{P_{\theta+A(t)u}(t)}{dP_0(t)} = \exp\left\{\frac{1}{cT^\beta}(\theta + A(t)u)\widehat{B}_t - \frac{1}{2(cT^\beta)^2}(\theta + A(t)u)^2 t\right\}$$

and

$$\frac{dP_{\theta+A(t,\theta)u}(t)}{dP_\theta(t)} = \frac{dP_{\theta+A(t,\theta)u}(t)}{dP_0(t)} \cdot \left(\frac{dP_\theta(t)}{dP_0(t)}\right)^{-1}$$

$$= \exp\left\{\frac{1}{cT^\beta}(\theta + A(t)u)\widehat{B}_t - \frac{1}{2(cT^\beta)^2}(\theta + A(t)u)^2 t - \frac{\theta}{cT^\beta}\widehat{B}_t - \frac{1}{2}\frac{\theta^2}{(cT^\beta)^2}t\right\}$$

$$= \exp\left\{\frac{uA(t)}{cT^\beta}\widehat{B}_t - \frac{1}{2}u^2\frac{A^2(t)}{(cT^\beta)^2}t - \frac{A(t)u\theta}{(cT^\beta)^2}t\right\}.$$

Set $A(t) := cT^\beta/\sqrt{t}$. Then

$$\frac{dP_{\theta+A(t,\theta)u}(t)}{dP_\theta(t)} = \exp\left\{u\frac{\widehat{B}_t}{\sqrt{t}} - \frac{1}{2}u^2 - \frac{u\theta\sqrt{t}}{cT^\beta}\right\}.$$

Since $\widehat{B}_t/\sqrt{t} \Rightarrow N(0,1)$ under both the measures $P_0(t)$ and $P_\theta(t)$ and, in addition, $u\theta\sqrt{t}/(cT^\beta) \to 0$ as $t \to \infty$ for $T \geq t$ and $\alpha > \frac{1}{2}$, the above definition implies the LAN property for the family $P_\theta(t)$ as $t \to \infty$ and at any point $\theta \in \Theta$. □

Consider now the asymptotic efficiency of the estimate of parameter θ. According to definition (11.3), introduced in the monograph of Ibragimov, Khas'minski, an estimate $\{\theta_t, t > 0\}$ of a parameter θ is asymptotically efficient under the LAN property for the cost function $\omega(A^{-1}(t,\theta)x)$ at the point θ if

$$\lim_{\delta\to 0} \varliminf_{t\to\infty} \sup_{|\theta'-\theta|<\delta} E_{P_{\theta'}(t)}\omega\left(A^{-1}(t,\theta)(\theta_t - \theta')\right) = E\omega(N(0,1)).$$

Let $\omega \in W$, where W is the class of functions defined on Θ and satisfying the conditions:

1) $\omega(u) \geq 0$, $\omega(0) = 0$, ω is a Borel function, continuous at zero and not identically zero;
2) $\omega(u) = \omega(-u)$,
3) the set $\{u\colon \omega(u) < c\}$ is convex for any $c > 0$.

Further we consider the cost function $\omega\left(A^{-1}(t,\theta)x\right) \in W_p$, where $W_p \subset W$ is the class of functions of W that have a dominant polynomial. Consider the maximum likelihood estimate of the parameter θ in a linear Brownian model

$$\widehat{\theta}_t = \frac{cT^\beta}{t}\widehat{B}_t = \frac{cT^\beta}{t}\left(B_t + \frac{1}{cT^\beta}\theta t\right) = \theta + \frac{cT^\beta}{t}B_t.$$

To prove the asymptotic efficiency of the estimate $\widehat{\theta}_t$ we use Theorem 1.3 of Chapter III from Ibragimov, Khas'minski. According to this theorem, the estimate $\widehat{\theta}_t$ is asymptotically efficient in the sense mentioned above if the following conditions hold:

(a) $\lim_{t\to\infty} A^{-1}(t,\theta_2)A(t,\theta_1) = B(\theta_1,\theta_2)$ exists, the convergence is uniform in $\theta_i \in \Theta$ and $B(\theta_1,\theta_2)$ is continuous in θ_1;

(b) $\zeta_t(\theta) := A^{-1}(t,\theta)(\widehat{\theta}_t - \theta) \Rightarrow N(0,1)$ uniformly in $\theta_i \in \Theta$ as $t \to \infty$ with respect to the measure $P_\theta(t)$;

(c) for any $N > 0$ random variables $|A^{-1}(t,\theta)(\widehat{\theta}_t - \theta)|^N$, are $P_\theta(t)$-integrable for any $\theta \in \Theta$ uniformly in $t > t_0(N)$.

Condition (a) holds in our case because $A(t) = \frac{cT^\beta}{\sqrt{t}}$ does not depend on θ. Now we check condition (b):

$$\zeta_t(\theta) = A^{-1}(t,\theta)\big(\widehat{\theta}_t - \theta\big) = \frac{\sqrt{t}}{cT^\beta}\frac{cT^\beta}{t}B_t = B_t\frac{1}{\sqrt{t}} \Rightarrow N(0,1)$$

under both the measures $P_0(t)$ and $P_\theta(t)$. Condition (c) now is evident. Thus the estimate $\widehat{\theta}_t$ is asymptotically efficient as $t \to \infty$.

Local Asymptotic Normality and Asymptotic Efficiency of the Estimate of the Drift Parameter in a Linear Fractional Brownian Diffusion Model

Now consider a pure linear fractional Brownian model

$$dX_t = \frac{1}{T^{\beta}}\theta X_t\, dt + X_t\, dB_t^H, X_{t=0} = X_0, \theta \in \Theta, t \in [0, T], \beta \in (1-H, 1].$$

It will be clear later that in this model it is sufficient to consider $\beta \in \left(1-H, \frac{1}{2}\right)$. Now $\varphi_t = \theta/T^{\beta}$. Then

$$\widetilde{\alpha}\int_0^t \delta_s\, ds = \int_0^t l_H(t,s)\frac{\theta}{T^{\beta}}\, ds = \frac{\theta}{T^{\beta}}C(H,1)t^{1-2\alpha}, \delta_t = (\theta/T^{\beta})C(H,1)t^{-2\alpha}$$

Therefore $\widehat{\theta}_t = T^{\beta}\widetilde{\alpha}\int_0^t s^{-\alpha}\, d\widehat{B}_s C(H,1)^{-1}t^{2\alpha-1}$, where

$$\widetilde{\alpha}\int_0^t s^{-\alpha}\, d\widehat{B}_s$$

$$= \widetilde{\alpha}\int_0^t s^{-\alpha}\, dB_s + \frac{\theta}{T^{\beta}}C(H,1)t^{1-2\alpha}.$$

In other words,

$$T^{\beta}\widetilde{\alpha}\int^{t} s^{-\alpha}\, dB$$

Theorem 3. *The LAN property holds for the family $P_\theta(t)$ as $t \to \infty$ at any point $\theta \in \Theta$.*

Proof. We change the probability measure $P_\theta(t)$ for the measure $P_0(t)$. As a result, the drift $\theta X_t\, dt$ disappears. The corresponding likelihood ratio is given by

$$\frac{dP_\theta(t)}{dP_0(t)} = \exp\left\{ \int_0^t s^\alpha \delta_s \, d\widehat{B}_s - \frac{1}{2}\int_0^t s^{2\alpha}\delta_s^2 \, ds \right\}$$

$$= \exp\left\{ \frac{\theta C(H,1)\widetilde{\alpha}}{T^\beta} \int_0^t s^{-\alpha}\, d\widehat{B}_s - \frac{1}{2T^{2\beta}}(\theta C(H,1))^2 t^{1-2\alpha} \right\}.$$

Now we consider the linear model with parameter θ shifted by $A(t)u$ and denote for simplicity $K = C(H)$.

$$\frac{P_{\theta+A(t)u}(t)}{dP_0(t)} = \exp\left\{\frac{(\theta + A(t)u)K}{T^\beta}\int_0^t s^{-\alpha}\, d\widehat{B}_s - \frac{1}{2T^{2\beta}}((\theta+A(t)u)K)^2\frac{t^{1-2\alpha}}{1-2\alpha}\right.$$

The likelihood ratio for this model is of the form

$$\frac{dP_{\theta+A(t,\theta)u}(t)}{dP_\theta(t)} = \frac{dP_{\theta+A(t,\theta)u}(t)}{dP_0(t)} \cdot \left(\frac{dP_\theta(t)}{dP_0(t)}\right)^{-1}$$

$$= \exp\left\{\frac{K}{T^\beta}A(t)u\left(\int_0^t s^{-\alpha}\, d\widehat{B}_s - \frac{1}{2}A(t)u\frac{K}{T^\beta}\frac{t^{1-2\alpha}}{1-2\alpha} - \theta\frac{K}{T^\beta}\frac{t^{1-2\alpha}}{1-2\alpha}\right)\right\}.$$

Set $A(t) := T^{\beta}\widetilde{\alpha}/Kt^{1-H}$. Then the likelihood ratio obtains the form

$$\frac{dP_{\theta+A(t,\theta)u}(t)}{dP_{\theta}(t)} = \exp\left\{\widetilde{\alpha}u\frac{\int_0^t s^{-\alpha}d\widehat{B}_s}{t^{1-H}} - \frac{1}{2}u^2 - \frac{u\theta Kt^{1-H}}{T^{\beta}\widetilde{\alpha}}\right\}.$$

Since

$$\widetilde{\alpha}\frac{\int_0^t s^{-\alpha}\, d\widehat{B}_s}{t^{1-H}} \Rightarrow N(0,1)$$

and

$$\frac{u\theta K t^{1-H}}{T^{\beta}\widetilde{\alpha}} \to 0 \quad \text{as } t \to \infty,$$

the LAN property holds for the family $P_\theta(t)$ as $t \to \infty$ at any point $\theta \in \Theta$. □

Now we check the asymptotic efficiency of the estimate $\widehat{\theta}_t$. Consider conditions (a)-(c). Two of them, (a) and (c), are evident. To check (b) we use the following relations:

$$\zeta_t(\theta) = A^{-1}(t,\theta)(\widehat{\theta}_t - \theta) = \frac{C(H)t^{1-H}}{T^{\beta}\widetilde{\alpha}} \frac{T^{\beta}\int_0^t s^{-\alpha}\,dB_s}{C(H,1)t^{1-2\alpha}} = \frac{\left(\int_0^t s^{-\alpha}dB_s\right)\widetilde{\alpha}}{t^{1-H}} \Rightarrow I$$

Therefore, the estimate $\widehat{\theta}_t$ of the parameter θ is asymptotically efficient as $t \to \infty$.

Sequential estimation

Return to the model (2) and suppose that conditions (i)–(viii) hold. Consider for any $h > 0$ the stopping time

$$\tau(h) = \inf\{t > 0 : \int_0^t \chi_s^2 d\langle M\rangle_s = h\}.$$

Under conditions (vi)–(vii) $\tau(h) < \infty$ a.s. and $\int_0^{\tau(h)} \chi_s^2 d\langle M\rangle\}_s = h$. The sequential maximum likelihood estimate has a form

$$\hat{\theta}_{\tau(h)} = \frac{\int_0^{\tau(h)} \chi_s dZ_s}{h} = \theta + \frac{\int_0^{\tau(h)} \chi_s dM_s^H}{h}. \tag{34}$$

The process $\int_0^{\tau(h)} \chi_s dM_s^H$ is the square-integrable martingale and we immediately obtain that the estimate $\hat{\theta}_{\tau(h)}$ is unbiased. Also, the results of Liptser, Shiryaev Stat.R.Proc. are applicable directly to (34), and we obtain that the estimate $\hat{\theta}_{\tau(h)}$ is efficient, $E(\hat{\theta}_{\tau(h)} - \theta)^2 = \frac{1}{h}$, and for any estimates of the form $\hat{\theta}_\tau = \frac{\int_0^\tau \chi_s dZ_s}{\int_0^\tau \chi_s^2 d\langle M^H \rangle_s} = \theta + \frac{\int_0^\tau \chi_s dM_s^H}{\int_0^\tau \chi_s^2 d\langle M^H \rangle_s}$ with $\tau < \infty$ a.s. and $E \int_0^\tau \chi_s^2 d\langle M^H \rangle_s \leq h$ we have that $E(\hat{\theta}_{\tau(h)} - \theta)^2 \leq E(\hat{\theta}_\tau - \theta)^2$.

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