Estimation of scaling parameter for continuous processes

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Gaussian-like processes

- Our basic model is giving by the integral

$$X_t = X_0 + \int_0^t \sigma_s dG_s, \qquad t \geq 0,$$

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- Under Hölder continuity conditions on σ and G the above integral is well-defined in the Riemann-Stieltjes sense.

- The stochastic process X is assumed to be observed at time points $t_i = i\Delta_n$, $i = 0, \ldots, [t/\Delta_n]$ with $\Delta_n \to 0$.

The scaling parameter

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  for some $\alpha \in (0,2)$ and $\alpha' > 0$.

- Our aim is to estimate the scaling parameter

$$\alpha \in (0,2)$$

  from high frequency data $X_{i \quad n}$.

Estimation

- Our estimation procedure relies on the power variation statistic

$$V(X,p)_t^n = \Delta_n \tau_{\Delta_n}^{-p} \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p, \qquad \Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n},$$

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- In contrast to the semimartingale framework the statistic $V(X,p)^n_t$ is not feasible as the normalizing constant $\tau_{\ n}$ is unknown.

- Recall the identity

$$\tau^2_{\ n} = \text{const} \cdot \ ^{\alpha}_{n} + O(\ ^{\alpha+\alpha'}_{n}),$$

  where $\alpha \in (0,2)$ is the parameter of our interest.

Law of large numbers

- **Theorem:** Under certain regularity conditions on the variogram R we obtain the convergence

$$V(X,p)^n_t \xrightarrow{ucp} m_p \int_0^t |\sigma_s|^p ds$$

with $m_p = \mathbb{E}(|N(0,1)|^p)$.

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- For the corresponding CLT we require some stronger conditions.

Central limit theorem

Theorem: Let the volatility process σ be smooth enough and further assume that

$$R_t = \text{const} \cdot t^{\alpha} + O(t^{\alpha+\alpha'}) \qquad \text{as } t \to 0$$

for some $\alpha \in (0,3/2) \setminus \{1\}$ and $\alpha' > 0$. Then we deduce the stable convergence

$$n^{-1/2}\Big(V(X,p)_t^n - m_p \int_0^t |\sigma_s|^p ds\Big) \xrightarrow{\mathcal{D}_{st}} \rho \int_0^t |\sigma_s|^p dW'_s,$$

where W' is a new Brownian motion (independent of everything) and

$$\rho^2 = \lim_{n\to\infty} n^{-1}\mathrm{Var}(V(B^H,p)_1^n)$$

with B^H being a fBm with Hurst parameter $H = \alpha/2$.

Some remarks

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- Joint convergence for a family of powers $(p_1, \ldots, p_k)$ and frequencies $(d_1 \quad {}_n, \ldots, d_l \quad {}_n)$ is straightforward.

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- Joint convergence for a family of powers $(p_1, \ldots, p_k)$ and frequencies $(d_1 \quad {}_n, \ldots, d_l \quad {}_n)$ is straightforward.

- The restriction

$$\alpha \in (0, 3/2) \setminus \{1\}$$

is explained by the fact that for $\alpha \in (3/2, 2)$ we obtain a non-central limit theorem with a slower rate of convergence.

Ratio statistics

- Even though all asymptotic results are infeasible we can use the relationship

$$\tau^{2}{}_{n} = \text{const} \cdot {}_{n}^{\alpha} + O({}_{n}^{\alpha+\alpha'}),$$

to estimate $\alpha \in (0, 2)$. This implies that $\tau_{2}^{2}{}_{n} / \tau^{2}{}_{n} \to 2^{\alpha}$.

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- Our estimation method is based on the change of frequency:

$$R^n_t = \frac{\sum_{i=2}^{[t/\ \ n]} |X_{i\ \ n} - X_{(i-2)\ \ n}|^2}{\sum_{i=1}^{[t/\ \ n]} |X_{i\ \ n} - X_{(i-1)\ \ n}|^2} \xrightarrow{P} 2^{\alpha}.$$

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- A central limit theorem for the normalized statistic

$$\ _n^{-1/2} \left(\frac{\log R_t^n}{\log 2} - \alpha \right)$$

holds for $\alpha \in (0,3/2) \setminus \{1\}$ and $\alpha' > 1/2$. The latter condition is required to ensure that $\ _n^{-1/2}(\tau^2_{2\ \ n} / \tau^2_{\ n} - 2^\alpha) \to 0$.

Remark

- In practice it is more informative to consider a *power plot* to infer the parameter α. Consider the power variation ratio

$$R(q)_t^n = \frac{\sum_{i=2}^{[t/\quad n]} |X_{i\quad n} - X_{(i-2)\quad n}|^q}{\sum_{i=1}^{[t/\quad n]} |X_{i\quad n} - X_{(i-1)\quad n}|^q} \xrightarrow{P} 2^{q\alpha/2}.$$

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- Example: For $q \in (\underline{a}, \overline{a})$ the scaling parameter α can be estimated via

$$\hat{\alpha} = \frac{1}{\overline{a} - \underline{a}} \int_{\underline{a}}^{\overline{a}} \frac{2 \log R(q)_t^n}{q \log 2} dq \xrightarrow{P} \alpha.$$

Higher order di erences

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- It turns out that considering higher order di erences solves the problem. Let $^{(q)n}_{i} X$ denote the qth order di erence of X, e.g.

$$^{(2)n}_{i} X = X_{i \quad n} - 2X_{(i-1) \quad n} + X_{(i-2) \quad n}.$$

De ne the power variation via

$$V^{(q)}(X,p)^n_t = \quad {}_n(\tau^{(q)}_n)^{-p} \sum_{i=1}^{[t/ \quad n]} | \quad ^{(q)n}_{i} X|^p$$

with $(\tau^{(q)}_n)^2 = \mathbb{E}(\quad ^{(q)n}_{i} G)^2$.

Asymptotic theory

Theorem: For all $\alpha \in (0,2)$ and $q \geq 1$ it holds that

$$V^{(q)}(X,p)_t^n \xrightarrow{ucp} m_p \int_0^t |\sigma_s|^p ds.$$

Under further assumptions on R and σ we obtain

$$n^{-1/2}\left(V^{(q)}(X,p)_t^n - m_p \int_0^t |\sigma_s|^p ds\right) \xrightarrow{\mathcal{D}_{st}} \rho^{(q)} \int_0^t |\sigma_s|^p dW'_s$$

for all $q \geq 2$. Here W' is a new Brownian motion (independent of everything) and

$$(\rho^{(q)})^2 = \lim_{n\to\infty} n^{-1}\mathrm{Var}(V^{(q)}(B^H,p)_1^n)$$

with B^H being a fBm with Hurst parameter $H = \alpha/2$.

A model with smooth drift

- Let us now consider the model

$$Z = X + Y,$$

where $X_t = X_0 + \int_0^t \sigma_s dG_s$ is our basic process and Y is a drift process with

$$Y \in C^r(\mathbb{R}_{\geq 0}) \qquad \text{a.s.}$$

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- It turns out that the higher order di erences have a second useful property: they make the power variation robust to certain smooth drift processes.

Robust asymptotic results

Theorem: Let $Z = X + Y$ and $Y \in C^r(\mathbb{R}_{\geq 0})$ a.s.

(i) If $r > \alpha/2$ then it holds that

$$V^{(q)}(Z,p)_t^n - V^{(q)}(X,p)_t^n \xrightarrow{ucp} 0$$

for all $\alpha \in (0,2)$, $p \geq 0$ and $q \geq 1$.

(ii) If $r - \alpha/2 > 1/2$ then it holds that

$$n^{-1/2}\Big(V^{(q)}(Z,p)_t^n - V^{(q)}(X,p)_t^n\Big) \xrightarrow{ucp} 0$$

for all $\alpha \in (0,2)$, $p \geq 0$ and $q \geq \min([r],1) + 1$.

Estimation with gaps

- Let us go back to the original model

$$X_t = X_0 + \int_0^t \sigma_s dG_s, \qquad t \geq 0.$$

- Assume now that the variogram R of G satisfies the relation

$$R_t = \text{const} \cdot t^{\alpha} + O(t^{\alpha+\alpha'}), \qquad \text{as } t \to 0,$$

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with $\alpha \in (0,2)$ and $\alpha' \in (0,1/2)$.

- As $\tau^2_{2\ \,n}/\tau^2_{\ \,n} - 2^\alpha = O(\ ^\alpha_n)$ the CLT for the ratio statistic R^n_t does not hold anymore, because the quantity

$$\ _n^{-1/2}(\tau^2_{2\ \,n}/\tau^2_{\ \,n} - 2^\alpha)$$

explodes as $\ _n \to 0$.

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- Let $M_n \to \infty$ with $M_n \quad {}_n \to 0$. De ne a new ratio statistic via

$$\overline{R}^n_t = \frac{\sum_{i=2}^{[t/M_n \quad {}_n]} |X_{iM_n \quad {}_n} - X_{iM_n \quad {}_n - 2}|^2}{\sum_{i=1}^{[t/M_N \quad {}_n]} |X_{iM_n \quad {}_n} - X_{iM_n \quad {}_n - 1}|^2}.$$

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- Choose M_n such that

$$(M_n \quad _n)^{-1/2}(\tau_2^2 \quad _n / \tau^2 \quad _n - 2^{\alpha}) \to 0.$$

Robust asymptotic results

Theorem: Assume that

$$R_t = \text{const} \cdot t^{\alpha} + O(t^{\alpha+\alpha'}) \qquad \text{as } t \to 0$$

for some $\alpha \in (0, 3/2) \setminus \{1\}$ and $\alpha' \in (0, 1/2)$.

(i) We deduce that $\overline{R}_t^n \xrightarrow{P} 2^{\alpha}$ and

$$(M_n \quad {}_n)^{-1/2} \left(\overline{R}_t^n - 2^{\alpha} \right) \xrightarrow{\mathcal{D}_{st}} \int_0^t f_s dW_s'$$

for a known process $(f_s)_{s \geq 0}$.

(ii) In the critical case $M_n \sim \quad {}_n^{2\alpha'-1}$ we obtain the convergence rate

$${}_n^{-\alpha'}.$$

This is indeed the optimal convergence rate.

Thank you!