Estimation of function in stationary noise

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Statistique Asymptotique des Processus stochastiques VIII,
Mars 21 - 24, Le Mans, France

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Estimation of pseudoperiodic function

This talk is connected with nonparametric estimation of the function $s(t)$ as the observation process $Y(t)$ is given by

$$dY(t) = s(t)dt + dX(t), t \in [-T, T].$$

Here unknown function $s \in \mathscr{L}_* \subset \mathscr{L}$, $\mathscr{L}$ is the Banach space with the norm $\| \cdot \|_{\mathcal{L}}$,

$$\|s\|_{\mathcal{L}}^2 = \sup_x \int_x^{x+1} |s\,(t)\,|^2\,dt < \infty, \tag{1}$$

$\mathscr{L}_*$ is the subset of the Stepanov space $\mathscr{L}(\Lambda)$ of pseudoperiodic functions

$$s(t) = \sum_{u \in \Lambda} a(u) e^{iut}, \text{ defined by } \sum_{u \in \Lambda} (1 + |u|)^{2\beta} |a(u)|^2 \le C, \tag{2}$$

Λ is a countable subset of real line such that

$$\tau = \tau(\Lambda) = \inf_{u,v \in \Lambda,\, u \neq v} |u - v| > 0. \tag{3}$$

noise process $X(t)$ is the gaussian process with stationary increments with zero mean and with the spectral density f:

$$\mathbf{E}\left\{\int \varphi(t)dX(t)\overline{\int \psi(t)dX(t)}\right\} = \int \widehat{\varphi}(u)\overline{\widehat{\psi}(u)}\, f(u)\, du,$$

where $\widehat{\varphi}$ is the Fourier transformation of function φ.

For an estimator $\widehat{s}_T$ of unknown function s we denote

$$R_T\left(\widehat{s}_T, f\right) = \sup_{s\in\mathscr{L}_*} \mathbf{E}_{s,f}\left\|\widehat{s}_T - s\right\|^2_{\mathscr{L}}.$$

Let $R_T(f)$ be the minimax risk,

$$R_T(f) = \inf_{\widehat{s}_T}\sup_{s\in\mathscr{L}_*} \mathbf{E}_{s,f}\left\|\widehat{s}_T - s\right\|^2_{\mathscr{L}}.$$

We plan construct an estimator $\widehat{s}_T$ such that for appropriate class $\mathscr{K}$ of spectral densities and sufficiently large T

$$\sup_{f\in\mathscr{K}} \frac{R_T\left(\widehat{s}_T, f\right)}{R_T(f)} \le C\left(\mathscr{K}, \tau, \mathscr{L}_*\right), \tag{4}$$

where the constant $C\left(\mathscr{K}, \tau, \mathscr{L}_*\right)$ depends only on class $\mathscr{K}$, the value

$$\tau = \tau(\Lambda) = \inf_{u,v\in\Lambda,\, u\neq v} |u-v| > 0,$$

and values β, C in the inequality

$$\sum_{u\in\Lambda} (1+|u|)^{2\beta} |a(u)|^2 \leq C,$$

that define the parametric set $\mathscr{L}_*$. We prove for this estimator that under appropriate condition on spectral density f

$$R_T\left(\widehat{s}_T, f\right) \asymp T^{-\frac{2\beta}{2\beta+\gamma+1}},$$

where $\gamma > -1$ and depends on some characteristic of f.

Class $\mathscr{K}$ of spectral densities

For function f and interval $I,\ 0 \in I$, denote

$$f_I(t) = \frac{1}{|I|} \int\limits_I f(t+x)\,dx,\ \text{where } |I| \text{ is the length of } I.$$

In the case as $I = [-\varepsilon/2, \varepsilon/2]$ we shall write $f_\varepsilon(t)$ instead of $f_I(t)$.

The nonnegative function f satisfied to the Muckenhoupt condition if

$$\lambda(f) := \sup_I \frac{1}{|I|} \int\limits_I f(x)\,dx \times \frac{1}{|I|} \int\limits_I \frac{1}{f(x)}\,dx < \infty. \tag{5}$$

We suppose that

$$\lambda(\mathscr{K}) := \sup_{f \in \mathscr{K}} \lambda(f) < \infty \tag{6}$$

We need another condition on the regular behavior of sums

$$\Sigma_f(N) := \sum_{|u| \le N} f_\varepsilon(u),\ \text{as } N \to \infty.$$

We suppose that for some $\gamma > -1$ and $\varepsilon = N^{-(1+\gamma+\beta)}$ functions $f \in \mathscr{K}$ satisfy to the following inequality

$$\sum_{|u|\leq N} f_\varepsilon(u) \leq C(\mathscr{K}) N^{1+\gamma}. \tag{7}$$

For example,

$$\text{if } f(x) = (1+|x|)^\alpha \text{ , then } \gamma = \alpha;$$

$$\text{if } f(x) = |u-x|^\alpha, |u-x| \leq \delta, \text{ for all } u \in \Lambda \text{ , then } \gamma = -\frac{\alpha(2\beta+1)}{\alpha+1}.$$

So, we denote by $\mathscr{K}(\lambda, C, \gamma)$, $\gamma > -1$ the class of nonnegative functions such that

$$\lambda(f) = \sup_I \frac{1}{|I|} \int_I f(x)\, dx \times \frac{1}{|I|} \int_I \frac{1}{f(x)}\, dx < \lambda,$$

and

$$\sum_{|u|\leq N} f_\varepsilon(u) \leq C\, N^{1+\gamma}, \text{ if } \varepsilon = N^{-(1+\gamma+\beta)}.$$

The main result

We denote

$$Y[\varphi] = \int \overline{\varphi(t)}\, dY(t).$$

and set for fix T

$$\varphi_u(t) = e^{iut},\ \varphi_u^T(t) = e^{iut}\,\mathbb{1}_{[-T,T]}(t).$$

If $\left\{\psi_v^T, v \in \lambda\right\}$ is the conjugate system with respect to the system $\left\{\varphi_v^T, v \in \lambda\right\}$:

$$\frac{1}{2T}\int\limits_{-T}^{T} \varphi_u^T(t)\overline{\psi_v^T(t)}\, dt = \delta_{u,v},$$

then

$$s(t) = \sum_{u\in\Lambda}\left\{\int s(r)\overline{\psi_u^T(r)}\, dr \times \varphi_u(t)\right\},\ \text{if } s(t) = \sum_{u\in\Lambda} a(u)\varphi_u(t).$$

Here we assume that

$$\sum_{u\in\Lambda} |a(u)|^2 < \infty, \text{ and } \tau = \tau(\Lambda) = \inf_{u,v\in\Lambda,\, u\neq v} |u-v| > 0.$$

The value

$$\widehat{a}_T(u) = Y[\psi_u^T] = a(u) + X[\psi_u^T], \text{ where } X[\psi] = \int \overline{\psi(t)}\, dX(t)$$

may be considered as estimator for the coefficient $a(u)$.

For the appropriately chosen $N = N(T)$, the estimator

$$\widehat{s}_T(t) = \sum_{u\in\Lambda,\, |u|\leq N} \widehat{a}_T(u)\varphi_u(t)$$

can be considered as estimator for function $s(t)$.

Theorem 1. Suppose that unknown function $s \in \mathscr{L}$ and spectral density $f \in \mathscr{K}\,(\lambda, C, \gamma)$. Then, for

$$N(T) = T^{\frac{1}{1+\gamma+2\beta}}, R_T\left(\widehat{s}_T, f\right) \asymp T^{-\frac{2\beta}{2\beta+\gamma+1}}.$$

Theorem 2. Under the condition of theorem 1, for $T > T_0$

$$\sup_{f\in\mathscr{K}} \frac{R_T\left(\widehat{s}_T, f\right)}{R_T\left(f\right)} \le C\left(\mathscr{K}, \tau, \mathscr{L}_*\right), \tag{8}$$

The reasons choice of the class $\mathscr{K}$

Really we have the problem of estimating the vector $a = \{a(u),\ u \in \Lambda\}$ of coefficients $a(u)$ of function

$$s(t) = \sum_{u\in\Lambda} a(u)e^{iut}$$

on observations

$$y_u = a(u) + x_u,\ \text{where } y_u = Y[\psi_u^T],\ x_u = X[\psi_u^T],$$

as $\sum\limits_{u\in\Lambda} (1 + |u|)^{2\beta}|a(u)|^2 \le C.$

At the case when variables x_u are independent gaussian, it is a well studied problems. In our model gaussian variables x_u are dependent.

But under the Muckenhoupt condition

$$\lambda(f) := \sup_I \frac{1}{|I|} \int\limits_I f(x)\,dx \times \frac{1}{|I|} \int\limits_I \frac{1}{f(x)}\,dx < \infty. \tag{9}$$

the system $\{x_u,\ u \in \Lambda\}$ is a basis in the space $L^2(dP)$. It is sufficient for some our analytical needs. Under this condition

$$\inf \mathbf{E}\left(x_u - \sum_{v\neq u} c_v x_v\right) > (1-\rho^2)\mathbf{E}(x_u)^2,\ \text{where } \rho^2 < 1.$$

The reasons choice of the space $\mathscr{L}$

It is well known that under the condition

$$\tau(\Lambda) = \inf_{u,v\in\Lambda,\, u\neq v} |u-v| > 0,$$

Banach norm $\|s\|_{\mathcal{L}}$

$$\|s\|_{\mathcal{L}}^2 = \sup_x \int\limits_x^{x+1} |s\,(t)\,|^2\,dt \tag{10}$$

is equivalent to the Hilbert norm $\|s\|_*$,

$$\|s\|_*^2 = \frac{1}{2T} \int_{-T}^{T} |s(t)|^2 \, dt,$$

in the space $\mathscr{L}(\Lambda)$ of pseudoperiodic functions.