# Limit theorems in asymptotic statistics for diffusions in volatility estimation

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## Statistical estimation of the volatility

**Today we will discuss:**

- **Asymptotic expansion of the realized volatility with a review of the martingale expansion**
- **Conditional martingale expansion**
- **Related topics**
  - **variance derivative**
  - **nonsynchronous covariance estimator**

**among many aspects of volatility estimation:**

1. **p-variation under irregular sampling**
2. **Mixed normal limit theorems under non-synchronous random sampling scheme**

3. **Asymptotic expansion for the non-synchronous covariance estimator in central limit case**
4. **Lead-lag estimation and non-synchronicity**
5. **Quasi-likelihood analysis for volatility Mixed normal type martingale expansion applied to the realized volatility**
6. **Mixed normal type martingale expansion applied to the realized volatility**
7. **Conditional inference and conditional asymptotic expansion**
8. **Volatility derivatives**
9. **Market microstructure**
10. **Implementation and data analysis**

# Higher-order asymptotics for the realized volatility

## Asymptotic expansion for the distribution: three principles*

- **Small $\sigma$ expansion**
  - **Watanabe (AP1987), Kusuoka and Stroock (JFA1991)**
  - **Applications to statistics:**
    **Y (PTRF1992,1993),**
    **Dermoune and Kutoyants (Stochastics1995),**
    **Sakamoto and Y (JMA1996, SISP1998),**
    **Uchida and Y (SISP2004),**
    **Masuda and Y (StatProbLet2004), .....**

- **Application to option pricing:**
  **Y (JJSS1992*),**
  **Kunitomo and Takahashi (MathFinance2001),**
  **Uchida and Y (SISP2004),**
  **Takahashi and Y (SISP2004, JJSS2005),**
  **Osajima (SSRN2007),**
  **Takahashi and Takehara (2009,2010),**
  **Andersen and Hutchings (SSRN2009),**
  **Antonov and Misirpashaev (SSRN2009),**
  **Chenxu Li (ColumbiaUniv2010),**
  **....**

* http://www.journalarchive.jst.go.jp/jnlpdf.php?cdjournal=jjss1970&cdvol=22&noissue=2&startpage=139&lang=ja&from=jnltoc

- **Mixing expansion:**
  - **Kusuoka and Y (PTRF2000), Y (PTRF2004)**
  - **Applications to statistics:**
    **Y (PTRF97),**
    **Sakamoto and Y (JJSS2003, AISM2004, AISM2009,**
    **JJSS2008, CommStat2010),**
    **Uchida and Y (SISP2006,SUTJMath2006),**
    **Kutoyants and Y (SISP2007), ....**
  - **Applications to finance: Masuda and Y (SPA2005)**
  - **Regenerative method: Fukasawa (PTRF2008)**

- **Distributional martingale expansion (Central limit)**
  - **Yoshida (PTRF1997)**
  - **Statistics: Y, Sakamoto and Y (SISP1998),**
  - **Finance: Fukasawa (FinanceStoch2009)**
- **Here we discuss the martingale expansion in mixed normal limit and its application.**

## Question: Quadratic form for a diffusion process

- **stochastic differential equation**

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dw_s.$$

- **quadratic form of the increments of** $X$**:**

$$U_n = \sum_{j=1}^{n} c(X_{t_{j-1}})(\Delta_j X)^2,$$

**where** $\Delta_j X = X_{t_j} - X_{t_{j-1}}$ **and** $t_j = j/n$**.**

- **Give the asymptotic expansion for the normalized error**

$$Z_n = \sqrt{n}(U_n - U_\infty),$$

**where** $U_\infty = \int_0^1 c(X_s)\sigma(X_s)^2 ds$**.**

## Method and Applications

- **We apply an asymptotic-expansion scheme for martingales with mixed normal limit (Y 2008)**
- **Applications are:**
  - **Realized volatility (discussed in this talk)**
  - **Estimators in QLA for the volatility**
  - **Variance derivatives**
  - **Scenario depending risk evaluation (Conditional inference)**

# Martingale expansion and formula

## Formula: Martingale expansion in mixed normal limits (Y 2008) (1)

**A sequence of $d$-dimensional functionals**

$$Z_n = M_n + W_n + r_n N_n.$$

- $M_n = M_1^n$ **for some $d$-dimensional continuous martingale $M^n = (M_t^n)_{t \in [0,1]}$ with respect to F.**
- $W_n$ **and** $N_n \in \mathcal{F}(\Omega; \mathbb{R}^d)$
- $(r_n)_{n\mathbb{N}}$ **is a sequence of positive numbers tending to zero as $n \to \infty$**
- $C_t^n = \langle M^n \rangle_t$ **and** $C_n = \langle M^n \rangle_1$**; they take values in $\mathbb{R}^d \otimes \mathbb{R}^d$.**

## Formula: Martingale expansion in mixed normal limits (2)

- **The tangent variables:** $\overset{\circ}{C}_n = r_n^{-1}(C_n - C_\infty)$, $\overset{\circ}{W}_n = r_n^{-1}(W_n - W_\infty)$ $\overset{\circ}{F}_n = r_n^{-1}(F_n - F_\infty)$
- **Suppose**

  – $(M^n, N_n, \overset{\circ}{C}_n, \overset{\circ}{W}_n, \overset{\circ}{F}_n) \to^{d_s(\mathcal{F})} (M^\infty, N_\infty, \overset{\circ}{C}_\infty, \overset{\circ}{W}_\infty, \overset{\circ}{F}_\infty)$

  – $\mathcal{L}\{M_t^\infty | \mathcal{F}\} = N_d(0, C_t^\infty)$.
- **Random symbols $\underline{\sigma}$ and $\overline{\sigma}$ are defined with those ingredients (see below).**

# Formula: Martingale expansion in mixed normal limits (3)

- **Let**

$$\underline{\sigma}(z, iu, iv) = \frac{1}{2}\tilde{C}_\infty(z)^{j,k}(iu_j)(iu_k) + \tilde{W}_\infty(z)^j(iu_j) + \tilde{N}_\infty(z)^j(iu_j) + \tilde{F}_\infty(z)^l(iv_l)$$

**for** $u = (u_j) \in \mathbb{R}^d$ **and** $v = (v_l) \in \mathbb{R}^{d_1}$**, where** $\tilde{C}_\infty(z)$**,** $\tilde{W}_\infty(z)$ **and** $\tilde{F}_\infty(z)$ **are defined from** $\overset{\circ}{C}_\infty$**,** $\overset{\circ}{W}_\infty(z)$ **and** $\overset{\circ}{F}_\infty$**, respectively.**

- **The "adaptive" random symbol** $\underline{\sigma}$ **corresponding to the classical second-order correction term of Y. (1997).**

- **The "anticipative" random symbol** $\overline{\sigma}(iu, iv)$ **newly appeared in the mixed limit case.**

- **For example, for the quadratic form**

$$M_t^n = r_n^{-1} \sum_j \int_{t_{j-1}\wedge t}^{t_j \wedge t} \dot{K}^n(s) \oslash \Big( \int_{t_{j-1}}^{s} \ddot{K}^n(r) dw_r \Big) dw_s,$$

**the random symbol** $\overline{\sigma}(iu, iv)$ **becomes**

$$\bar{\sigma}(iu, iv) = \frac{1}{2} \mathbf{Tr}^* \int_0^1 \bar{K}^{\infty}(t,t)[iu] \otimes \sigma_{t,t}(iu, iv)\ \mu(dt),$$

**where**

$$\begin{aligned}
\sigma_{s,r}(iu, iv) = {} & \frac{1}{2} D_r C_s^{\infty}[u^{\otimes 2}] \otimes \Big( iD_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + iD_s F_\infty[v] \Big) \\
& + \Big( iD_r W_\infty[u] - \frac{1}{2} D_r C_\infty[u^{\otimes 2}] + iD_r F_\infty[v] \Big) \\
& \quad \otimes \Big( iD_s W_\infty[u] - \frac{1}{2} D_s C_\infty[u^{\otimes 2}] + iD_s F_\infty[v] \Big) \\
& + \Big( iD_r D_s W_\infty[u] - \frac{1}{2} D_r D_s C_\infty[u^{\otimes 2}] + iD_r D_s F_\infty[v] \Big).
\end{aligned}$$

## Formula: Martingale expansion in mixed normal limits (4)

**Set**

$$\sigma = \underline{\sigma} + \bar{\sigma}. \tag{1}$$

**With Watanabe's delta functional, let**

$$\begin{aligned} p_n(z,x) = & E\Big[\phi(z;W_\infty,C_\infty)\delta_x(F_\infty)\Big] \\ & +r_n E\Big[\sigma(z,\partial_z,\partial_x)^*\Big\{\phi(z;W_\infty,C_\infty)\delta_x(F_\infty)\Big\}\Big]. \end{aligned}$$

$*$ **For more information, see ISM Research Memorandum 1125 (2010).**

## Return to Quadratic form for a diffusion process

## Question: Quadratic form for a diffusion process

- **stochastic differential equation**

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- **quadratic form of the increments of $X$:**

$$U_n = \sum_{j=1}^{n} c(X_{t_{j-1}})(\Delta_j X)^2,$$

**where** $\Delta_j X = X_{t_j} - X_{t_{j-1}}$ **and** $t_j = j/n$**.**

- **Give the asymptotic expansion for the normalized error**

$$Z_n = \sqrt{n}(U_n - U_\infty),$$

**where** $U_\infty = \int_0^1 c(X_s)\sigma(X_s)^2 ds$**.**

## Stochastic expansion

$$Z_n = M_1^n + \frac{1}{\sqrt{n}} N_n,$$

**where**

$$M_t^n = \sqrt{n} \sum_{j=1}^{n} 2 c_{t_{j-1}} \sigma_{t_{j-1}}^2 \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} dw_r dw_s,$$

**and**

$$\begin{aligned} N_n = & \ 6n \sum_{j=1}^{n} c_{t_{j-1}} \sigma_{t_{j-1}} \sigma_{t_{j-1}}^{[1]} \int_{t_{j-1}}^{t_j} \int_{t_{j-1}}^{t} \int_{t_{j-1}}^{s} dw_u dw_s dw_t \\ & +2 \sum_{j=1}^{n} c_{t_{j-1}} b_{t_{j-1}} \sigma_{t_{j-1}} \int_{t_{j-1}}^{t_j} dw_t \end{aligned}$$

$$
\begin{aligned}
&+2n\sum_{j=1}^{n} c_{t_{j-1}}\sigma_{t_{j-1}}\sigma^{[1]}_{t_{j-1}}\int_{t_{j-1}}^{t_j}(t-t_{j-1})dw_t \\
&+2n\sum_{j=1}^{n} c_{t_{j-1}}\sigma_{t_{j-1}}b_{t_{j-1}}\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t} dw_s dt \\
&+n^{-1}\sum_{j=1}^{n} c_{t_{j-1}}b^2_{t_{j-1}} + n^{-1}\sum_{j=1}^{n} c_{t_{j-1}}\sigma_{t_{j-1}}b^{[1]}_{t_{j-1}} \\
&-n\sum_{j=1}^{n} c^{[1]}_{t_{j-1}}\sigma^2_{t_{j-1}}\int_{t_{j-1}}^{t_j}\int_{t_{j-1}}^{t} dw_s dt \\
&-\frac{1}{2n}\sum_{j=1}^{n} c^{[0]}_{t_{j-1}}\sigma^2_{t_{j-1}} - \frac{1}{n}\sum_{j=1}^{n} c^{[1]}_{t_{j-1}}\sigma_{t_{j-1}}\sigma^{[1]}_{t_{j-1}} + o_M(1).
\end{aligned}
$$

**Here $o_M(1)$ denotes a term of $o(1)$ as $n \to \infty$ with respect to $\mathbb{D}_{s,p}$-norms of any order.**

**We wrote $b_t$ for $b(X_t)$ and $\sigma_t$ for $\sigma(X_t)$. The Itô decomposition of $\sigma_t = \sigma(X_t)$ is denoted by**

$$\sigma_t = \sigma_0 + \int_0^t \sigma_s^{[1]} dw_s + \int_0^t \sigma_s^{[0]} ds.$$

**Though $\sigma_s^{[1]}$ and $\sigma_s^{[2]}$ have a simple expression with $b$, $\sigma$ and $X_s$, those symbols are convenient to simplify the notation. This rule will be applied for other functionals.**

# Reference variable

**For a reference variable, we will consider**

$$F_n = \frac{1}{n}\sum_{j=1}^{n} \beta(X_{t_{j-1}}) \quad \textbf{or} \quad F_n = F_\infty := \int_0^1 \beta(X_t)dt.$$

## Nondegeneracy

**Let** $a(x) = c(x)\sigma(x)^2$**. Let**

$$V_0(x_1, x_2) = \begin{bmatrix} b(x_1) - \frac{1}{2}\sigma(x_1)\partial_{x_1}\sigma(x_1) \\ \beta(x_1) \end{bmatrix} \quad \text{and} \quad V_1(x_1, x_2) = \begin{bmatrix} \sigma(x_1) \\ 0 \end{bmatrix}$$

**for** $x_1 \in \mathbb{R}$ **and** $x_2 \in \mathbb{R}^{d_1}$**. The Lie algebra generated by**

$$V_1,\ [V_i, V_j]\ (i, j = 0, 1),\ [V_i, [V_j, V_k]]\ (i, j, k = 0, 1), \ldots.$$

**at** $(x_1, x_2)$ **is denoted by Lie**$[V_0; V_1](x_1, x_2)$**.**

**Assume that supp**$(X_0)$ **is compact. Moreover, for nondegeneracy, we assume**

**[H1]** $\inf_{x \in \mathbb{R}} |a(x)| > 0$.

**[H2]** **Lie**$[V_0; V_1](X_0, 0) = \mathbb{R}^{1+d_1}$ **a.s.**

## 2nd order specification*

$$(M_\infty, \mathring{C}_\infty, N_\infty) =^d \Big( \int_0^1 \sqrt{2} a(X_s) dB_s,$$
$$\int_0^1 \frac{4\sqrt{2}}{3} a(X_s)^2 dB_s + \int_0^1 \frac{4}{3} a(X_s)^2 dB'_s,$$
$$\int_0^1 q_s dB''_s + \int_0^1 h_s ds \Big),$$

**where $(B, B', B'')$ is a three-dimensional standard Wiener process, independent of $\mathcal{F}$, defined on the extension $\bar{\Omega}$, and**

$$h_t = c_t b_t^2 + c_t b_t^{[1]} \sigma_t - \frac{1}{2} c_t^{[0]} \sigma_t^2 - c_t^{[1]} \sigma_t \sigma_t^{[1]}.$$

# Adaptive random symbol

**The adaptive random symbol** $\underline{\sigma}(z, iu, iv)$ **is given by**

$$\underline{\sigma}(z, iu, iv) = \frac{2z}{3} \int_0^1 a(X_s)^3 ds \Big( \int_0^1 a(X_s)^2 ds \Big)^{-1} (iu)^2 + iu \int_0^1 h_t dt.$$

**NB** $a(x) = c(x)\sigma(x)^2$**.**

# Anticipative random symbol (1)

**The random symbol $\sigma_{s,r}(iu, iv)$ admits the expression**

$$\begin{aligned}
&\sigma_{s,r}(iu, iv) \\
= \ & u^2 \int_r^s \alpha'(X_t) D_r X_t dt \Big( - u^2 \int_s^1 \alpha'(X_t) D_s X_t dt + i \int_s^1 \beta'(X_t)[v] D_s X_t dt \Big) \\
&+ \Big( - u^2 \int_r^1 \alpha'(X_t) D_r X_t dt + i \int_r^1 \beta'(X_t)[v] D_r X_t dt \Big) \\
&\cdot \Big( - u^2 \int_s^1 \alpha'(X_t) D_s X_t dt + i \int_s^1 \beta'(X_t)[v] D_s X_t dt \Big) \\
&+ \Big( - u^2 \int_s^1 \{ \alpha''(X_t) D_r X_t D_s X_t + \alpha'(X_t) D_r D_s X_t \} dt \\
&+ i \int_s^1 \{ \beta''(X_t)[v] D_r X_t D_s X_t + \beta'(X_t)[v] D_r D_s X_t \} dt \Big)
\end{aligned}$$

**for $r \leq s$, where the prime $'$ stands for the derivative in $x_1 \in \mathbb{R}$.**

**The processes $D_s X_t$ and $D_r D_s X_t$ are determined according to routine; for example, $D_s X_t$ satisfies the equation**

$$D_s X_t = \sigma(X_s) + \int_s^t \beta'(X_t) D_s X_t dt + \int_s^t \sigma'(X_t) D_s X_t dw_t$$

**for $s \leq t$. $D_r D_s X_t$ admits a similar equation.**

# Anticipative random symbol (2)

**Now we obtain the anticipative random symbol**

$$\bar{\sigma}(iu, iv) = \int_0^1 iu\, a(X_s)\sigma_{s,s}(iu, iv)\, ds$$

**with**

$$\begin{aligned}\sigma_{s,s}(iu, iv) = & \Big( - u^2 \int_s^1 \alpha'(X_t) D_s X_t dt + i \int_s^1 \beta'(X_t)[v] D_s X_t dt \Big)^2 \\ & -u^2 \int_s^1 \{\alpha''(X_t)(D_s X_t)^2 + \alpha'(X_t) D_s D_s X_t\} dt \\ & +i \int_s^1 \{\beta''(X_t)[v](D_s X_t)^2 + \beta'(X_t)[v] D_s D_s X_t\} dt\end{aligned}$$

# Recall Formula:

## The full random symbol:

$$\sigma = \underline{\sigma} + \bar{\sigma}.$$

## The asymptotic expansion:

$$p_n(z,x) = E\Big[\phi(z;W_\infty,C_\infty)\delta_x(F_\infty)\Big] + r_n E\Big[\sigma(z,\partial_z,\partial_x)^*\Big\{\phi(z;W_\infty,C_\infty)\delta_x(F_\infty)\Big\}\Big].$$

# Asymptotic expansion: the quadratic form

**Theorem 1. Suppose that** [H1] **and** [H2] **are satisfied. Then for any positive numbers** $M$ **and** $\gamma$**,**

$$\sup_{f\in\mathcal{E}(M,\gamma)}\left|E\big[f(Z_n,F_n)\big]-\int_{\mathbb{R}^{1+d_1}}f(z,x)p_n(z,x)dzdx\right|=o\left(\frac{1}{\sqrt{n}}\right)$$

**as** $n\to\infty$**, where** $\mathcal{E}(M,\gamma)$ **is the set of measurable functions** $f:\mathbb{R}^{1+d_1}\to\mathbb{R}$ **satisfying** $|f(z,x)|\le M(1+|z|+|x|)^\gamma$ **for all** $(z,x)\in\mathbb{R}\times\mathbb{R}^{d_1}$**.**

**Example 1.** Quadratic form of the diffusion process, $F_n=F_\infty$. Suppose that [H1] and [H2] are satisfied

# Other topics

# Variance derivative

**Asymptotic expansion for** $\sqrt{n}\big(h(RV_n) - h(\langle X \rangle)\big)$ **is given by the signed measure**

$$q_n = (T_n)_* \, p_n$$

**by**

$$T_n(z, x) = h'(x)z + \frac{1}{2\sqrt{n}} h''(x) z^2.$$

## Asymptotic expansion for the nonsynchronous covariance estimator

- Realized volatility
  - first-order : many
  - second-order (normal limit) : martingale expansion (Y97)
  - second-order (mixed normal limit) : martingale expansion
- Nonsynchronous covariance estimator
  - first-order : Hayashi-Y (SPA)
  - second-order (normal limit) : Dalalyan-Y (AIHP)
  - second-order (mixed normal limit) : !?

# Conditional asymptotic expansion

## Conditional asymptotic expansion

**It is possible to give the asymptotic expansion of the conditional law $\mathcal{L}\{Z_n|F_n\}$ in the same framework we applied for the expansion of the joint law $\mathcal{L}\{(Z_n, F_n)\}$.**

# Conditional asymptotic expansion

**Approximation to the (local) conditional density of $\mathcal{L}\{Z_n|F_n = x\}$:**

$$
\begin{aligned}
&p_n(z|x)\\
&= E\Big[\phi(z;W_\infty, C_\infty)\delta_x(F_\infty)\Big]/p^{F_n}(x)\\
&\quad +r_n E\Big[\sigma(z,\partial_z,\partial_x)^*\Big\{\phi(z;W_\infty, C_\infty)\delta_x(F_\infty)\Big\}\Big]/p^{F_n}(x)
\end{aligned}
$$

**and**

$$
p^{F_n}(x) = p^{F_\infty}(x) - r_n\partial_{x_l}\Big(E\Big[\mathring{F}_\infty^{\,l}|F_\infty = x\Big]p^{F_\infty}(x)\Big).
$$

# Conditional asymptotic expansion

**Let**

$$\Delta_n(f|x) = \left| E\left[f(Z_n, F_n)\middle|F_n = x\right] - \int f(z,x)p_n(z|x)\,dz\right|.$$

**Under the assumption, we can choose a continuous version** $p^{F_n}$ **of the density of** $\mathcal{L}\{F_n\}$**. Let** $\mathcal{S}_\infty = \{x \in \mathbb{R}^{d_1}; p^{F_\infty}(x) > 0\}$**. Let** $\bar{Q}_n(B|x) = \int_B h_n^0(z,x)dz$ **for** $B \in \mathbb{B}_d$**. For a measure** $\nu$**,** $f \in \mathcal{F}(\mathbb{R}^d)$ **and** $r > 0$**, let**

$$\omega(f; r, \nu) = \int_{\mathbb{R}^d} \sup_{z:|z|\le r} |f(y+z) - f(y)|\nu(dy).$$

# Conditional asymptotic expansion

**Theorem 2.** **Under certain non-degeneracy conditions, for any positive numbers $M$, $\gamma$ and $L_0$, there exists a constant $C$ such that**

$$\Delta_n(f|x) \leq C\left\{\omega(f; r_n^{L_0}, \bar{Q}_n(\cdot,x)^+) + \bar{o}(r_n)\right\}/p^{F_n}(x) \quad (x \in \mathcal{S}_\infty)$$

**for all $f \in \mathcal{E}(M,\gamma)$.**

**In particular, it gives a conditional limit theorem for the conditional distribution $\mathcal{L}\left\{Z_n \,\middle|\, F_n = x\right\}$.**