Some results on identification of partially observed linear systems

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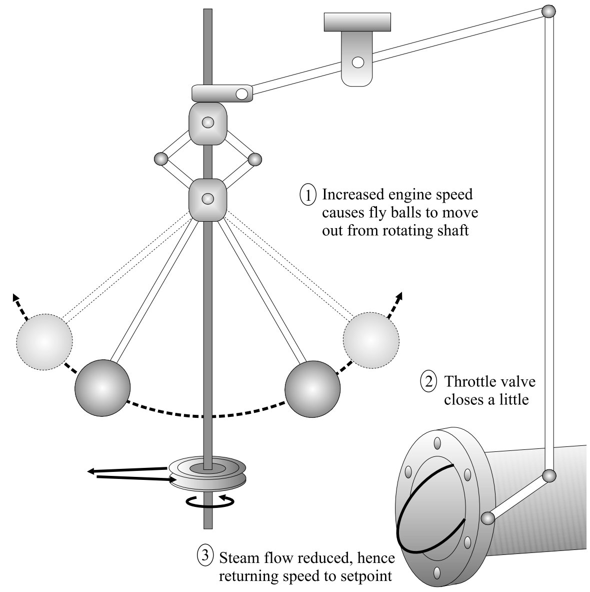
Introduction

Control engineering has taken several major steps forward at crucial times in history (e.g. the industrial revolution, the Second World War, the push into space, economic globalization, shareholder value thinking, etc.). Each of these steps has been matched by a corresponding burst of development in the underlying theory of control.

A key step forward in the development of control occurred during the industrial revolution. A major development at this time was Watt's fly ball governor (1788). This device regulated the speed of a steam engine by throttling the flow of steam. Devices like these remain in service to our days.


1 Increased engine speed causes fly balls to move out from rotating shaft
2 Throttle valve closes a little
3 Steam flow reduced, hence returning speed to setpoint

Early on, when the compelling concept of feedback was applied, engineers sometimes encountered unexpected results. For example, if we go back to Watt's fly ball governor, it was found that under certain circumstances these systems could produce self sustaining oscillations. Towards the end of the 19th century several researchers (including Maxwell) showed how these oscillations could be described via the properties of **ordinary differential equations**.

The developments around the period of the World War II were also matched by significant developments in Control Theory. For example, the pioneering works of Bode, Nyquist, Nichols, Evans and others appeared at this time. This resulted in simple graphical means for analyzing single-input single-output feedback control problems. These methods are now generally known by the generic term **Classical Control Theory**.

The 1960s saw the development of an alternative state space approach to control. This followed the publication of work by Wiener, Kalman (and others) on optimal estimation and control. This work allowed multi-variable problems to be treated in a unified fashion which had been difficult, if not impossible, in the classical framework. This set of developments is loosely termed **Modern Control Theory**.

By the 1980s these various approaches to control had reached a sophisticated level and emphasis then shifted to other related issues including the effect of model error on the performance of feedback controllers. This can be classified as the period of **Robust Control Theory**.

In parallel there has been substantial work on nonlinear control problems. This has been motivated by the fact that many real world control problems involve nonlinear effects. There have been numerous other developments including adaptive control, auto-tuning, intelligent control, etc.

Our objective is to focus on a model related to partially observed linear systems, where the function we would like to control is not observed directly, and to perform estimation of different functional characteristics in this model.

Setting-up

Assume that we observe a process $X = (X_t,\ 0 \le t \le T)$ satisfying the following system of stochastic differential equations:

$$\mathrm{d}X_t = h_t Y_t \,\mathrm{d}t + \varepsilon \,\mathrm{d}W_t, \quad X_0 = 0,$$
$$\mathrm{d}Y_t = g_t Y_t \,\mathrm{d}t + \varepsilon \,\mathrm{d}V_t, \quad Y_0 = y_0 \neq 0, \quad 0 \le t \le T,$$

where W_t and V_t, $0 \le t \le T$, are two independent Wiener processes. The process $Y = (Y_t,\ 0 \le t \le T)$ **cannot be observed** directly, but it is *the one we would like to control*.

In this model, we consider the problem of estimation of different functions on $0 \le t \le T$, in the asymptotics of a *small noise*, i.e., as $\varepsilon \to 0$. We propose some kernel-type estimators for our functions and study their properties.

The first observation to be made is that the functions h_t and g_t, $0 \leq t \leq T$, *cannot be estimated at the same time*.

The **reason** is as follows: Even in the situation where there is no noise we have

$$\mathrm{d}x_t = h_t y_t \,\mathrm{d}t, \quad x_0 = 0,$$
$$\mathrm{d}y_t = g_t y_t \,\mathrm{d}t, \quad y_0 \neq 0, \quad 0 \leq t \leq T.$$

The second equation gives $y_t = y_0 \exp\left\{\int_0^t g_s \mathrm{d}s\right\}$ and therefore

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = f_t, \quad x_0 = 0, \qquad \text{where} \quad f_t = h_t y_t = h_t \, y_0 \exp\left\{\int_0^t g_s \mathrm{d}s\right\}.$$

Hence, we can only estimate f_t, the latter being a mixture of g_t and h_t. This is why we have to consider estimation of the functions f_t, g_t, and h_t separately, one by one. **We assume that all unknown functions are at least continuous.**

Estimation of f_t

Suppose that $f_t \in \mathcal{F}$, where $\mathcal{F}$ is the set of uniformly continuous functions bounded by a constant.

Introduce the estimator

$$\hat{f}_{t,\varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) \mathrm{d}X_s$$

Heuristics: Since in the unperturbed model we have $f_t = \dot{x}_t$, we use a "*slow derivative*" of X_t as an estimator of f_t. This "slow derivative" is the above kernel-type estimator.

Here and in the sequel, the kernel $K(\cdot)$ is bounded and satisfies the "*standard conditions*": K has compact support $[A, B]$ with $A < 0$, $B > 0$, and

$$\int_A^B K(u)\, \mathrm{d}u = 1.$$

The first result is the following:

Proposition: *Let* $\varphi_\varepsilon \to 0$ *and* $\varepsilon^2\varphi_\varepsilon^{-1} \to 0$ *as* $\varepsilon \to 0$. *Then for any* $0 < a < b < T$ *we have*

$$\lim_{\varepsilon\to 0} \sup_{f\in\mathcal{F}} \sup_{a\le t\le b} \mathbf{E}_f \left(\hat{f}_{t,\varepsilon} - f_t\right)^2 = 0.$$

Remark. As usual, this convergence can be made uniform on $[0, T]$ by a special choice of one-sided kernels.

Suppose that the functions h_t and g_t are such that the function f_t is k-times continuously differentiable and the k-th derivative satisfies the Hölder condition of order $\alpha \in (0,1]$:

$$\left| f_t^{(k)} - f_s^{(k)} \right| \leq L \left| t - s \right|^{\alpha} .$$

We denote by $\mathcal{F}_\beta$, where $\beta = k + \alpha$, the class of such functions. The kernel $K(\cdot)$ satisfies the following standard conditions in addition to those already stated:

$$\int_A^B K(u)\, u^l \mathrm{d}u = 0, \qquad l = 1, \ldots, k.$$

We take now $\varphi_\varepsilon = \varepsilon^{\frac{2}{2\beta+1}}$.

Proposition: *There exists a constant $C > 0$ such that*

$$\lim_{\varepsilon \to 0} \sup_{f \in \mathcal{F}_\beta} \sup_{a \leq t \leq b} \varepsilon^{-\frac{4\beta}{2\beta+1}} \mathbf{E}_f \left(\hat{f}_{t,\varepsilon} - f_t \right)^2 \leq C.$$

This rate of convergence is optimal in the following sense:

Proposition: *There exists a constant* $c > 0$ *such that*

$$\varliminf_{\varepsilon \to 0} \inf_{\bar{f}_{t,\varepsilon}} \sup_{f \in \mathcal{F}_\beta} \varepsilon^{-\frac{4\beta}{2\beta+1}} \mathbf{E}_f \left(\bar{f}_{t,\varepsilon} - f_t \right)^2 \geq c.$$

Here, $\bar{f}_{t,\varepsilon}$ is an arbitrary estimator of f_t. Therefore an estimator whose the rate of convergence would be better than that of $\hat{f}_{t,\varepsilon}$ does not exist.

The proof of this bound requires the Kalman-Bucy filter for writing the likelihood ratio.

Estimation of h_t

Let us come back to our model

$$\mathrm{d}X_t = h_t Y_t \,\mathrm{d}t + \varepsilon \,\mathrm{d}W_t, \quad X_0 = 0,$$
$$\mathrm{d}Y_t = g_t Y_t \,\mathrm{d}t + \varepsilon \,\mathrm{d}V_t, \quad Y_0 = y_0 \neq 0, \quad 0 \le t \le T.$$

Suppose now that g_t is a known bounded function and we would like to estimate h_t which is unknown. Then the estimator is as follows:

$$\hat{h}_{t,\varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) y_s^{-1} \,\mathrm{d}X_s.$$

Heuristics: In the unperturbed model we have $h_t = \dot{x}_t / y_t$. This hints us to use the "slow derivative" of X_t is the form of kernel-type estimator, and y_s appears in the denominator.

The estimator $\hat{h}_{t,\varepsilon}$ has properties similar to those of $\hat{f}_{t,\varepsilon}$. In particular, if $\varphi_\varepsilon = \varepsilon^{\frac{2}{2\beta+1}}$, then

$$\lim_{\varepsilon\to 0} \sup_{h\in\mathcal{F}_\beta} \sup_{a\le t\le b} \varepsilon^{-\frac{4\beta}{2\beta+1}} \mathbf{E}_h \left(\hat{h}_{t,\varepsilon} - h_t\right)^2 \le C.$$

Estimation of y_t

Suppose that h_t is a known function bounded away from zero. We would like to estimate $y_t = y_0 \exp\left\{\int_0^t g_s \mathrm{d}s\right\}$, where g_t is unknown. Then it can be shown that the estimator

$$\hat{y}_{t,\varepsilon} = \frac{1}{\varphi_\varepsilon} \int_0^T K\left(\frac{s-t}{\varphi_\varepsilon}\right) h_s^{-1}\, \mathrm{d}X_s$$

has similar properties to those of $\hat{f}_{t,\varepsilon}$, i.e., it is uniformly consistent and has asymptotically optimal rate of convergence under a smoothness condition ($g_t \in \mathcal{F}_{\beta-1}$): If $\varphi_\varepsilon = \varepsilon^{\frac{2}{2\beta+1}}$, then

$$\lim_{\varepsilon\to 0} \sup_{g\in\mathcal{F}_{\beta-1}} \sup_{a\le t\le b} \varepsilon^{-\frac{4\beta}{2\beta+1}} \mathbf{E}_g \left(\hat{y}_{t,\varepsilon} - y_t\right)^2 \le C.$$

Estimation of g_t

Suppose now that h_t is a known function bounded away from zero. We would like to estimate the function g_t from observations $X = (X_t,\ 0 \le t \le T)$. If we consider the unperturbed model, we will see that

$$g_t = \frac{y_t'}{y_t}.$$

Therefore if we construct good estimators $\hat{y}_{t,\varepsilon}'$ and $\hat{y}_{t,\varepsilon}$ of y_t' and y_t, respectively, then their ratio can be a consistent estimator of g_t. Since the convergence of moments is required, we introduce truncation and put

$$\hat{g}_{t,\varepsilon} = \frac{\hat{y}_{t,\varepsilon}'}{\hat{y}_{t,\varepsilon}}\ 1\!\!\mathrm{I}_{\{\hat{y}_{t,\varepsilon} > \kappa\}}$$

where the constant $\kappa > 0$ is sufficiently small.

Suppose that $g_t \in \mathcal{F}_{\beta-1}$ and introduce the estimator of the derivative

$$\hat{y}'_{t,\varepsilon} = \frac{1}{\varphi_\varepsilon^2} \int_0^T Q\left(\frac{s-t}{\varphi_\varepsilon}\right) h_s^{-1}\,\mathrm{d}X_s$$

where the kernel $Q(\cdot)$ has compact support $[A_*, B_*]$, $A_* < 0$, $B_* > 0$, and satisfies the conditions

$$\int_{A_*}^{B_*} Q(u)\, u\,\mathrm{d}u = 1, \quad \int_{A_*}^{B_*} Q(u)\, u^l \mathrm{d}u = 0, \quad l = 0, 2, \ldots, k.$$

Then it can be shown that

$$\sup_{a \le t \le b} \varepsilon^{-\frac{4\beta-4}{2\beta+1}} \mathbf{E}_g \left(\hat{y}'_{t,\varepsilon} - y'_t\right)^2 \le C,$$

where we put $\varphi_\varepsilon = \varepsilon^{2/(2\beta+1)}$.

Suppose that $y_0 > 0$ and put

$$\kappa = \frac{1}{2} \inf_{g \in \mathcal{F}_{\beta-1}} \inf_{0 \le t \le T} y_t.$$

The rate of convergence of the estimator $\hat{g}_{t,\varepsilon}$ is the same:

Proposition: *There exists a constant* $C > 0$ *such that*

$$\sup_{a \le t \le b} \varepsilon^{-\frac{4\beta-4}{2\beta+1}} \mathbf{E}_g \left(\hat{g}_{t,\varepsilon} - g_t\right)^2 \le C.$$

It can be shown that this rate is optimal in the minimax sense.

Note that if $|g_t| \le D$, then we can take

$$\kappa = \frac{y_0}{2} e^{-DT}.$$

Conclusions

We have shown different possibilities of estimation in partially observed linear models.

For all estimators, the rate of convergence is established, and this rate is shown to be the optimal one.

In some sense, the model we have considered is close to the state space models in Modern Control Theory.

Useful information

The Barcelona International Conference on Applied Statistics (BAS2011) will be held in Barcelona on September 26–30, 2011.

Visit the conference Web site:

http://jornades.uab.cat/bas/

Thank you!