Filtering with Exponential Criteria

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Outline

1. Introduction
   - Problems statement
   - Motivations and references
   - "Information State" approach
2. General Gaussian observation model
   - Conditional version of Cameron-Martin formula
   - LEG and RS filtering problem, result
   - Proof of the Cameron-Martin type formula
   - Particular cases
3. LEG and RS filtering problems, a bit more general

Outline

LEG problem in one slide

The model

- nonobservable signal process (X_t) with values in $\mathbb{R}^n$;
- observations (Y_t) from $\mathbb{R}^d$;
- exponential type payoff function L_T

The aim

To find $\bar{h}$ which minimizes the payoff function.

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The observation model
the precise definition

- **signal** $X = (X_t,\ t \geq 0)$— n-dimensional continuous Gaussian process with
  - mean function $m = (m_t, t \geq 0)$:

$$\mathbb{E}X_t = m_t$$

  - covariance function $\Gamma = (\Gamma(t,s), t \geq 0, s \geq 0)$:

$$\mathbb{E}(X_t - m_t)(X_s - m_s)' = \Gamma(t,s)\,, \quad t \geq 0\,,\ s \geq 0\,.$$

- **observation:**

$$Y_t = \int_0^t A(s)X_s ds + \widetilde{B}_t\,, \quad t \geq 0\,.$$

- where
  - $\widetilde{B}_t$ is $\mathbb{R}^d$ valued Wiener process, independent of X;
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where

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Two different cases

There are two different cases for LEG filtering problem:

- $\mu < 0$ - risk-preferring filtering problem,
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Our approach

- Solve the problem for $\mu < 0$ (it is easier).
- Extend results to the general case using the analytical properties.

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Recursive equation as a definition of the Risk-sensitive Filtering:

Exponential Criterium,II

$$\hat{h}(t) = \arg\min_{g\in\mathcal{Y}_t} \mathbb{E}\left[\mu \exp\left\{\frac{\mu}{2}(X_t-g)'Q_t(X_t-g) + \frac{\mu}{2}\int_0^t (X_s-\hat{h}(s))'Q_s(X_s-\hat{h}(s))ds\right\} \Big/ \mathcal{Y}_t\right],$$

where g is a $\mathcal{Y}_t$ measurable variable.

RS problem, result

Connection between two problems

Are they equal?

Q: Can we always take $\bar{h} = \hat{h}$?

A: Sometimes **yes**, sometimes **no** . . .

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Comparison with quadratic type payoff function

Quadratic Criterium

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Risk - Neutral Filtering

Q: What happens for the quadratic type payoff function?
A: Solutions of LQG and RS: $g=\bar{h}_T$, $\bar{h}_t=\hat{h}_t=\pi_t(X)$, where $\pi_t(X) := \mathbb{E}[X_t|\mathcal{Y}_t]$ (can be computed using Kalman filter).

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Kalman filter

Kalman filter

$\pi_t(X) = m_t + \int_0^t \bar{\gamma}(t,s)A'_s[dY_s - A_s\pi_s(X)ds],$

Filtering error

$\bar{\gamma}(t,s) = \Gamma(t,s) - \int_0^s \bar{\gamma}(t,r)[A'_r A_r]\bar{\gamma}'(s,r)dr,$

the variance of the filtering error—-

$\mathbb{E}[(X_t - \pi_t(X))(X_t - \pi_t(X))'/\mathcal{Y}_t] = \bar{\gamma}(t,t).$

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Outline

Motivations

- Robust estimation
- H_∞ estimations
- Estimation of probability to exceed the fixed level
- Theory of a system failure. Estimation of the parameters of a survival function with unobservable component.

History 1: control & partial observations

A. Bensoussan & J.H. van Schuppen, 1985

Markov observation model

$$\begin{cases} dX_t = (a_t X_t + u_t)dt + dB_t \\ dY_t = A_t X_t dt + d\widetilde{B}_t. \end{cases}$$

An important formula :

$$\mathbb{E}\left\{\exp\left(-\frac{1}{2}\int_0^T X_s^2\,ds\right)\Big|\mathcal{Y}_T\right\} = \exp\left\{-\frac{1}{2}\int_0^T \bar{\gamma}(s)\,ds\right\} \times$$

$$\times \exp\left\{-\frac{1}{2}\int_0^T V_s^2\,ds\right\} \cdot \exp\left\{\int_0^T A_s Z_s\,d\nu_s - \frac{1}{2}\int_0^T \|A_s Z_s\|^2\,ds\right\}$$

with $Z_t = V_t - \pi_t(X)$, and

$$dV_t = (a_t V_t + u_t)dt - \bar{\gamma}(t) V_t dt + \bar{\gamma}(t) A_t [dY_t - A_t V_t dt]$$

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History 2: LEG filtering, discrete time setting
J.L. Speyer, 1992

Discrete time observation model

$$\begin{cases} X_{t+1} &= a_t X_t + \varepsilon_t \\ Y_t &= A_t X_t + \tilde{\varepsilon}_t, \ (\varepsilon, \tilde{\varepsilon}) - i.i.d. \end{cases}$$

Exponential payoff function

$$L_T(h,\mu) = \mathbb{E}\left[\mu \exp\left\{\frac{\mu}{2}\sum_0^T (X_s - h(s))' Q_s (X_s - h(s))\right\}\right]$$

Answer - using dynamical programming

$$\begin{aligned}\hat{h}(T) &= \arg\min_{g \in \mathcal{Y}_T} \mathbb{E}\left[\mu \exp\left\{\frac{\mu}{2}(X_T - g)' Q_T (X_T - g)\right.\right. \\ &+ \left.\left.\frac{\mu}{2}\sum_0^{T-1}(X_s - \hat{h}(s))' Q_s (X_s - \hat{h}(s))\right\} \Big/ \mathcal{Y}_T\right],\end{aligned}$$

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Risk-Sensitive Filtering, definition by recursive equation

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History 3: Risk-Sensitive setting
R.J. Elliott, S. Dey, J.B.Moore, 1994

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"Information State" approach, first definitions

proposed by R.J. Elliott, S. Dey, J.B.Moore and .. for RS filtering problem

A **non normalized** random measure

$$\nu_t(dx) = \mathbb{E}\left[\mathbb{I}(X_t \in dx)\exp\left\{\frac{\mu}{2}\int_0^t (X_s - h(s))' Q_s (X_s - h(s))ds\right\} \Big/ \mathcal{Y}_t\right]$$

Information variable: definition

Information variable $\lambda_t(x)$ is the density of the measure $\nu_t(dx)$.

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Information variable, equation & solution
Markov case

Equation

$$d\lambda_t(x) = \left[\tfrac{1}{2}\lambda_{xx} - a_t(x\lambda)_x + \tfrac{\mu}{2}Q(x-h(t))^2\lambda\right] dt + A_t\lambda\, d\nu_t$$

Solution

$$\lambda_t(\mathbf{x}) = C\exp\left\{-\frac{1}{2}(\mathbf{x}-z_t^h)'\bar{\gamma}^{-1}(t)(\mathbf{x}-z_t^h)\right.$$
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with $C = [(2\pi)^n det(\bar{\gamma}(t))]^{-\frac{1}{2}}$

The equations for z^h and $\bar{\gamma}$ can be written also.

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The equations for z^h and $\bar{\gamma}$ can be written also.

Using information variable to solve LEG and RS
Markov case

Three steps towards the solution

- Integrate the density $\lambda_t(x)$.
- Minimize the quadratic form
- Obtain the answer
  - $\hat{h}_t = z^{\hat{h}}$
  - $\hat{h} = \bar{h}$

Remaining questions

- Probabilistic sense of involved functions z^h and $\bar{\gamma}$?
- What to do if the observation model is not Markovian?

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Conditional expectation of the exponent: definition

What we want to find (notation)

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Conditional expectation of the exponent: the answer

The Cameron-Martin type formula

For any $T > 0$

$$\mathcal{I}_T = C \exp\left\{ \frac{1}{2}(z_T^h - g)'[Id - \mu M\bar{\gamma}(T,T)]^{-1}\mu M(z_T^h - g) + \frac{\mu}{2}\int_0^T (z_s^h - h(s))'Q_s(z_s^h - h(s))ds + \frac{\mu}{2}\int_0^T tr(\bar{\gamma}(s,s)Q_s)ds \right\}$$

$$\times \exp\left\{ \int_0^T (z_s^h - \pi_s(X))'A_s' d\nu_s - \frac{1}{2}\int_0^T \|A_s(z_s^h - \pi_s(X))\|^2 ds \right\}.$$

with $C = [det(Id - \mu M\bar{\gamma}(T,T))]^{-\frac{1}{2}}$.

Proof of LEG

Greek and Latin letters in the previous formula

- Innovation $d\nu_t = dY_t - A_t \pi_t(X)dt$, $\nu_0 = 0$,
- $\bar{\gamma} = (\bar{\gamma}(t,s), 0 \leq s \leq t)$: unique solution of Riccati-Volterra equation

$$\bar{\gamma}(t,s) = \Gamma(t,s) - \int_0^s \bar{\gamma}(t,r)[A_r' A_r - \mu Q_r]\bar{\gamma}'(s,r)dr,$$

such that $\bar{\gamma}(t,t) \geq 0$.

- $z^h = (z_s^h, s \geq 0)$ unique solution of Ito-Volterra equation

$$z_t^h = m_t + \int_0^t \bar{\gamma}(t,s)\mu Q_s[z_s^h - h(s)]ds$$

$$+ \int_0^t \bar{\gamma}(t,s)A_s'[dY_s - A_s z_s^h ds].$$

Greek and Latin letters in the previous formula

- Innovation $d\nu_t = dY_t - A_t\pi_t(X)dt$, $\nu_0 = 0$,
- $\bar{\gamma} = (\bar{\gamma}(t,s), 0 \leq s \leq t)$: unique solution of Riccati-Volterra equation

$$\bar{\gamma}(t,s) = \Gamma(t,s) - \int_0^s \bar{\gamma}(t,r)[A_r' A_r - \mu Q_r]\bar{\gamma}'(s,r)dr,$$

such that $\bar{\gamma}(t,t) \geq 0$.

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Auxiliary observations, I

$\bar{Y}_t = (Y_t^1, Y_t^2)$, such that

$$\begin{cases} dY_t^1 = dY_t, \ Y_0^1 = 0 \\ dY_t^2 = Q_t^{\frac{1}{2}} X_t dt + d\bar{B}_t, \ Y_0^2 = 0, \end{cases}$$

where $\bar{B}_t$ is a standard Brownian motion, independent of $(X, \widetilde{B})$.

Auxiliary observations, II

$d\xi_t = (X_t - h(t))' Q_t^{\frac{1}{2}} dY_t^2, \ \xi_0 = 0.$

Bayes formula and Girsanov theorem

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Probabilistic sense of γ and z, result

Probabilistic sense

- $z_t^h = \bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t)$
  with
  - $\bar{\pi}_t(X) = \mathbb{E}(X/\bar{\mathcal{Y}}_t)$ –the conditional expectation of X
  - $\bar{\gamma}_{X\xi}(t) = \mathbb{E}[(X_t - \bar{\pi}_t(X))(\xi_t - \bar{\pi}_t(\xi))/\overline{\mathcal{Y}}_t]$ - the conditional covariance
- $\bar{\gamma}(t,t) = \mathbb{E}[(X_t - \bar{\pi}_t(X))(X_t - \bar{\pi}_t(X))'/\overline{\mathcal{Y}}_t]$ — the variance of the filtering error

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Outline

LEG Filtering Problem, Result

Consider $\bar{h} = (\bar{h}(t),\ t \geq 0)$: solution of LEG filtering problem

LEG, definition

$\bar{h} =$
$$\arg\min_{h \in \mathcal{H}} \mathbb{E}\left[\mu \exp\left\{\frac{\mu}{2}\int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds\right\}\right].$$

LEG, solution

$$\bar{h}(t) = m_t + \int_0^t \bar{\gamma}(t,s) A_s' [dY_s - A_s \bar{h}(s) ds],$$

finding γ: Riccati-Volterra equation

$$\bar{\gamma}(t,s) = \Gamma(t,s) - \int_0^s \bar{\gamma}(t,r)[A_r' A_r - \mu Q_r]\bar{\gamma}'(s,r) dr,$$

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RS problem, result

Consider $\hat{h} = (\hat{h}(t),\ t \geq 0)$: solution of RS filtering problem

▸ The second problem: Risk Sensitive Filtering

RS, solution

$$\hat{h}(t) = m_t + \int_0^t \bar{\gamma}(t,s)A'_s[dY_s - A_s\hat{h}(s)ds],$$

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Proof of LEG

The following uniform lower bound holds:

$$E\left[\mu \exp\left\{\frac{\mu}{2}\int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds\right\}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\exp\left\{\frac{\mu}{2}\int_0^T (X_s - h(s))' Q_s (X_s - h(s)) ds\right\} \Big/ \mathcal{Y}_T\right]\right]$$

Conditional expectation of the exponent: the answer

$$\geq \mu \exp\left\{\frac{\mu}{2}\int_0^T tr(\bar{\gamma}(s,s) Q_s) ds\right\}$$

$$\times \mathbb{E}\left[\exp\left\{\int_0^T (z_s^h - \pi_s(X))' A_s' d\nu_s - \frac{1}{2}\int_0^T \|A_s(z_s^h - \pi_s(X))\|^2 ds\right.\right.$$

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Outline

The General Filtering Theorem

- Fixe $\mu = -1$, $M = 0$.
- Denote by $G_t = \exp\left\{-\frac{1}{2}\int_0^t [(X-h)'Q(X-h)]ds\right\}$.
- We want to find $\pi_T(G)$.
- General filtering theorem:

$$\pi_t(G) = \exp\left\{-\frac{1}{2}\int_0^t \alpha_2(s)ds + \int_0^t [\alpha_1(s) - \pi_s(X)]' A_s' d\nu_s - \frac{1}{2}\int_0^t \|A_s(\alpha_1(s) - \pi_s(X))\|^2 ds\right\},$$

- with

$$\alpha_1(t) = \frac{\pi_t(XG)}{\pi_t(G)},$$

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Bayes formula and Girsanov theorem

▸ Auxiliary filtering problem

- Introduce the new measure $\tilde{P}_t$:

$$\frac{d\tilde{P}_t}{dP} = \exp\left\{-\int_0^t (X_s - h(s))' Q_s^{\frac{1}{2}}\, d\bar{B}_s - \frac{1}{2}\int_0^t (X_s - h(s))' Q_s (X_s - h(s)) ds\right\}.$$

- Thanks to our construction:
  - $e^{-\xi_t} = G_t \rho_t$, with $\rho_t = \frac{d\tilde{P}_t}{dP}$
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$$\frac{\mathbb{E}(X_t e^{-\xi_t})/\overline{\mathcal{Y}}_t)}{\mathbb{E}(e^{-\xi_t})/\overline{\mathcal{Y}}_t)} = \frac{\tilde{\mathbb{E}}(X_t G)/\overline{\mathcal{Y}}_t)}{\tilde{\mathbb{E}}(G)/\overline{\mathcal{Y}}_t)} = \frac{\pi_t(XG)}{\pi_t(G)} = \alpha_1(t),$$

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Gaussian properties

Using a conditionally Gaussian properties of (X, ξ) wrt $\overline{\mathcal{Y}}_t$

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Outline

Applying our solution to particular cases

Methodology

- Write equations for
  - the variance of the filtering error $\bar{\gamma}_{xx}(t)$;
  - for the difference $z_t^h = \bar{\pi}_t(X) - \bar{\gamma}_{X\xi}(t)$.
- Solve the LEG filtering problem: take $\bar{h} = z_t^{\bar{h}}$.

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$(B, \widetilde{B}) = ((B_t, \widetilde{B}_t), t \geq 0)$ is a $(n + d)$-dimensional standard Brownian motion.

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Markov observation model, independent noises

Observation model

$$\begin{cases} dX_t = a_t X_t dt + \sigma_1(t) dB_t, \\ dY_t = A_t X_t dt + d\widetilde{B}_t. \end{cases}$$

The solution of LEG filtering problem

$$\begin{cases} d\bar{h}_t = a_t \bar{h}_t dt + \bar{\gamma}_{xx}(t) A'_s [dY_t - A_t \bar{h}_t dt], \\ \frac{d\bar{\gamma}_{xx}(t)}{dt} = a_t \bar{\gamma}_{xx}(t) + \bar{\gamma}_{xx}(t) a'_t - \bar{\gamma}_{xx}(t)[-\mu Q_t + A'_t A_t] \bar{\gamma}_{xx}(t) \\ +\sigma_1(t)\sigma'_1(t). \end{cases}$$

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Correlated signal - observation noises

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Ito-Volterra type signal

Observation model

$$\begin{cases} X_t = \int_0^t a(t,s)X_s ds + \int_0^t K(t,s)dB_s \\ dY_t = A_t X_t dt + d\widetilde{B}_t, \end{cases}$$

with a nice kernel K.

Solution of the LEG filtering problem

$$\begin{cases} \bar{h}_t = \int_0^t a(t,s)\bar{h}_s ds + \int_0^t \bar{\gamma}(t,s)A'_s[dY_s - A_s\bar{h}_s ds] \\ \bar{\gamma}(t,s) = \int_0^s [a(t,r)\bar{\gamma}(s,r) + \bar{\gamma}(t,r)a(s,r)'] - \\ - \int_0^s \bar{\gamma}(t,r)[-\mu Q_r + A'_r A_r]\bar{\gamma}(s,r)dr + \int_0^s K(t,r)K(s,r)dr \end{cases}$$

Ornstein-Uhlenbeck type noises

Observation model

$$\begin{cases} dX_t = a_t X_t dt + dG_t \\ dY_t = A_t X_t dt + d\widetilde{G}_t, \end{cases}$$

where $\widetilde{G}_t$ and G_t two independent Ornstein-Uhlenbeck processes:

noises

$$\begin{cases} dG_t = \beta_t G_t dt + dB_t \\ d\widetilde{G}_t = \widetilde{\beta}_t \widetilde{G}_t dt + d\widetilde{B}_t \end{cases}$$

Change of the observation model

Another form of the same observation model:

$$\begin{cases} d\widetilde{X}_t = \widetilde{a}_t \widetilde{X}_t dt + \widetilde{\sigma}_1(t) dB_t + \widetilde{\sigma}_2(t) d\widetilde{B}_t \\ dY_t = \widetilde{A}_t \widetilde{X}_t dt + d\widetilde{B}_t, \end{cases}$$

with

$$\widetilde{X}_t' = \begin{pmatrix} X_t & \widetilde{G}_t & G_t \end{pmatrix}, \widetilde{\sigma}_1' = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}, \widetilde{\sigma}_2' = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

$$\widetilde{a}_t = \begin{pmatrix} a(t) & 0 & \beta_t \\ 0 & \widetilde{\beta}_t & 0 \\ 0 & 0 & \beta_t \end{pmatrix} \text{ and } \widetilde{A}_t = \begin{pmatrix} A_t & \widetilde{\beta}_t & 0 \end{pmatrix}.$$

Change of criterium

Another form of the same criterium

$$L_T(h,\mu) \quad = \quad \mathbb{E}\left[\mu \exp\left\{\frac{\mu}{2}\int_0^T (\widetilde{X}_s - h(s))'\widetilde{Q}_s(\widetilde{X}_s - h(s))ds\right\}\right]$$

with $\widetilde{Q}_t = \begin{pmatrix} Q_t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The solution of the LEG filtering problem: $\bar{h}^1$ is the first component of $\bar{h}_t$ from "Correlated noises" model.

LEG and RS filtering problems, a bit more general

For a given positive symmetric deterministic matrices $\Omega_s, 0 \leq s \leq T$ let us denote by $\Phi_t(h) = (X_t', h_t')\Omega_t \begin{pmatrix} X_t \\ h_t \end{pmatrix}$.

"LEG setting"

$$\bar{h} = \arg\min_{h \in \mathcal{H}} \mathbb{E}\left[\mu \exp\left\{\frac{\mu}{2}\int_0^T \Phi_s(h)\, ds\right\}\right],$$

where h is a $(\mathcal{Y}_s)$ adapted process.

"RS setting"

$$\hat{h}(t) = \arg\min_{g \in \mathcal{Y}_t} \mathbb{E}\left[\mu \exp\left\{\frac{\mu}{2}\Phi_t(g) + \frac{\mu}{2}\int_0^t \Phi_s(\hat{h})\, ds\right\} \Big/ \mathcal{Y}_t\right],$$

LEG and RS filtering problems, a bit more general

For a given positive symmetric deterministic matrices $\Omega_s, 0 \leq s \leq T$ let us denote by $\Phi_t(h) = (X'_t, h'_t)\Omega_t \begin{pmatrix} X_t \\ h_t \end{pmatrix}$.

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Equality of two solutions, yes

The question

Does the equality $\bar{h} = \hat{h}$ hold ?

One possible answer

Yes for degenerated matrices Ω :
$\Omega_{1,1} = \Omega_{2,2} = -\Omega_{1,2} = -\Omega'_{2,1} = Q.$

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Equality of two solutions,no

LEG problem, solution

$\Omega = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $A = 1$, $\mu = -1$ and $X_t = B_t$.

$$\bar{h}(t) = \int_0^t \bar{H}(T, t, s)\, dY_s,$$

where

$$\bar{H}(T, t, s) = \frac{\sinh(\sqrt{3}s)\cosh(T-t)}{\sqrt{\alpha_t \alpha_s}},$$

$$\alpha_t = \frac{\sqrt{3}+1}{2}\cosh(T+(\sqrt{3}-1)t) + \frac{\sqrt{3}-1}{2}\cosh(T-(\sqrt{3}+1)t).$$

$\bar{h}(t)$ depends of T.

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RS problem, solution

$$\hat{h}(t) = \int_0^t \hat{H}(t,s)\, dY_s,$$

where

$$\hat{H}(t,s) = \frac{\sinh(\sqrt{3}s)\sqrt[3]{\cosh(\sqrt{3}t)}}{\sqrt{3}(\cosh(\sqrt{3}t))^{\frac{2}{3}}},$$

and so $\hat{h}$ and $\bar{h}$ are different.

Open questions

A couple of unexplored cases

1. The discrete time setting
2. Equality of the solutions of the two problems for non-Gaussian models