Asymptotical distribution free test for parameter change in a diffusion model

(joint work with Y. Nishiyama)

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Overview

- We present a test for detecting if a change in the parameter in the drift of a diffusion process takes place.
- The test is based on the continuous observation of the process up to time T.
- The interest in this test is that it may be used for the most common family of diffusion process.
- The asymptotic distribution of the test statistics does not depend on the unknown parameter, so the test is asymptotically distribution free.
- It is also proved that the test is consistent against any alternative where the alternative means that, at a certain instant, the parameter specifying the drift coefficient changes.

Plan of the talk

- Parameter change problems and point change problems:
  - i.i.d observations, regression models, time series models
  - diffusion Process
    - change parameter (in time)
    - change point in space
- Goodness of fit test for diffusion
- Test for parameter change in ergodic diffusion process:
  - set up and conditions
  - study of the test statistic under H_0 and under H_1

Change point problem - i.i.d. case

- Originally, the problem was considered for i.i.d samples; see Hinkley (1971), Csörgő and Horváth (1997), Inclan and Tiao (1994), Chen and Gupta (2001).

Given a sequence of random variables $X_1, \ldots, X_n$ i.i.d. with with a common parametric distribution $F(\theta)$, $\theta \in \mathbb{R}^p$, then the general parameter change point problem is to test the following null hypothesis

$$H_0 : \theta_1 = \theta_2 = \ldots = \theta_n = \theta \text{ (unknown)}$$

versus the alternative

$$H_1 : \theta_1 = \ldots = \theta_k \neq \theta_{k+1} = \ldots = \theta_n$$

where k is an unknown position which have to be estimated.

Change point problem - i.i.d. case

- single change point

- multiple change points

- different models:

  - Gaussian (change in mean and change in variance), Exponential, Bernoulli

- different methods:

  - LRP, Bayesian, Information criteria approach, CUSUM approach, wavelet approximation

Change point problem - regression model

- The problem of change point moved quickly to regression models.
  - Separate regimes:

$$Y_i = a_0 + b_0 X_i + \varepsilon_i \quad i = 1, 2, \ldots, k$$

$$Y_i = a_1 + b_1 X_i + \varepsilon_i \quad i = k+1, \ldots, n$$

    Quandt (1958), Hinkley (1969) Brown, Durbin and Evans (1975), Chen (1998)

  - Two-phase linear regression. Koul and Qian (2002).

$$Y_i = (a_0 + b_0 X_i)\mathbf{1}_{\{X_i \leq r\}} + (a_1 + b_1 X_i)\mathbf{1}_{\{X_i > r\}} \varepsilon_i \quad i = 1, 2, \ldots, n$$

  - Two-phase non linear regression. Ciuperca (2004), Fujii (2008).

$$Y_i = S(X_i, \theta) + \varepsilon_i \quad i = 1, 2, \ldots, n$$

    where $X_i \in (0,1)$ and $S(x,\theta) = S_1(x)\mathbf{1}_{\{x \leq \theta\}} + S_2(x)\mathbf{1}_{\{x > \theta\}}$

Change point problem - time series analysis

The change point problem has been also considered for time series model:

- abrupt structural breaks

$$X_t = a_0 + a_1 X_{t-1} + b_0 \mathbf{1}_{\{t>\tau\}} + \varepsilon_t, \quad t = 1, \dots, T$$

- threshold models

$$X_t = a_1 X_{t-1} \mathbf{1}_{\{X_{t-1}<\theta\}} + a_2 X_{t-1} \mathbf{1}_{\{X_{t-1}\geq\theta\}} + \varepsilon_t \quad t = 1, \dots, T$$

In different models the problems are: test if there is a change point, estimate the threshold, estimate the time of change and the parameters involved.

Main approaches: least squares, MLE, Bayesian and CUSUM. See Picard (1985) Bai (1994, 1997), Kim *et al.* (2000), Brodsky and Darkhovsky (2000), Lee *et al.* (2003), Chan and Kutoyants (2010), Tong (2010).

Change point problem - Diffusion processes

For continuous time observations, Lee, Nishiyama and Yoshida (2003) considered the model

$$\mathrm{d}X_t = S(X_t, \theta)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0, \quad t \geq 0$$

where θ takes different values before and after a time instant τ^*.

To test the hypothesis (change point in *time*)

H_0 : the true value $\theta_0 \in \Theta$ does not change

versus the alternative

$H_1 : H_0$ false

they study the CUSUM tests statistic and proved that under H_0 is asymptotically distribution free.

Change point problem - Diffusion processes

De Gregorio and Iacus (2008) considered the change point problem for the ergodic model

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sqrt{\theta}\sigma(X_t)\mathrm{d}W_t, \quad 0 \le t \le T, \quad X_0 = x_0,$$

observed at discrete time instants $t_i = i\Delta_n$, $i = 0, \dots, n$, $\Delta_n = t_{i+1} - t_i$ under the sampling scheme $\Delta_n \to 0$, $n \to \infty$, $n\Delta_n = T$, T fixed. The coefficients b and σ are supposed to be known. The change point problem is formulated as follows

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s)\mathrm{d}s + \sqrt{\theta_1}\int_0^t \sigma(X_s)\mathrm{d}W_s, & 0 \le t \le \tau^* \\ \\ X_{\tau^*} + \int_{\tau^*}^t b(X_s)\mathrm{d}s + \sqrt{\theta_2}\int_{\tau^*}^t \sigma(X_s)\mathrm{d}W_s, & \tau^* < t \le T \end{cases}$$

where $\tau^* \in (0, T)$ is the change point and θ_1, θ_2 two parameters to be estimated. The approach follows the lines of Bai (1994, 1997).

Change point problem - Diffusion processes

Iacus and Yoshida (2009) give a very general result. They considered

$$\mathrm{d}Y_t = b_t \mathrm{d}t + \sigma(\theta, X_t)\mathrm{d}W_t, \quad t \in [0, T],$$

The coefficient $\sigma(\theta, x)$ is known up to θ, while b_t may be unknown and unobservable.

The sample consists of (X_{t_i}, Y_{t_i}), $i = 0, 1, ..., n$, where $t_i = i\Delta$ for $\Delta = \Delta_n = T/n$, T is fixed. It is an high-frequency setup.

The process Y generating the data satisfies the stochastic integral equation

$$Y_t = \begin{cases} Y_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma(\theta_1^*, X_s)\mathrm{d}W_s & \text{for } t \in [0, \tau^*) \\ Y_{\tau^*} + \int_{\tau^*}^t b_s \mathrm{d}s + \int_{\tau^*}^t \sigma(\theta_2^*, X_s)\mathrm{d}W_s & \text{for } t \in [\tau^*, T]. \end{cases}$$

The focus is on the estimation of the change point and its properties.

Remark: clearly this model includes diffusion models by taking, e.g., $Y_t = X_t$ and $b_t = b(X_t)$.

Change point problem - Diffusion processes

Kutoyants (1994, 2004, 2010) considered change point (*in space*) problems for the drift of an ergodic diffusion process solution to

$$\mathrm{d}X_t = S(X_t, \theta)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0, \quad t \geq 0 \tag{1}$$

with the trend function discontinuous along the two points of the state space of X_t, say $x_*^{(1)}(\theta)$ and $x_*^{(2)}(\theta)$, $\theta \in [\alpha, \beta] \subset \mathbb{R}$ and the interest is in the estimation of θ.

The simplest of these models is the *simple switching model*

$$\mathrm{d}X_t = -\mathrm{sgn}(X_t - \theta)\mathrm{d}t + \mathrm{d}W_t, \quad X_0, \quad t \geq 0$$

The problem consist in estimating θ. It was proved that the BE and MLE are uniformly consistent and their limit distribution were given.

Threshold Ornstein-Uhlenbeck process

$$dX_t = -\rho_1 X_t \mathbf{1}_{\{X_t<\theta\}} - \rho_2 X_t \mathbf{1}_{\{X_t\geq\theta\}} + \sigma dW_t, \quad X_0, \quad 0 \leq t \leq T$$

Different estimation problems were considered

- ρ_1 and ρ_2 known, θ unknown. BE and MLE are T consistent, BE is asymptotically efficient.

- All parameter unknown. BE and MLE fro θ are T consistent, BE is asymptotically efficient. BE MLE for ρ_1 and ρ_2 are $\sqrt{T}$-consistent . MLE need some attention.

- misspecification

  $$dX_t = -\rho_1 X_t \mathbf{1}_{\{X_t<\theta\}} - \rho_2 X_t \mathbf{1}_{\{X_t\geq\theta\}} + h(X_t)dt + \sigma dW_t, \quad X_0, \quad 0 \leq t \leq T$$

  h is an unknown function

Change point problem - Diffusion processes

Many other change point problems have been analyzed and studied:

- Discontinuous signal in a white Gaussian noise: Ibragimov and Khasminskii (1975) and (1981, Chapter 7.2);
- Change-point type models for dynamical systems with small noise: Kutoyants (1980) and (1994, Chapter 5);
- Change-point type model of delay equations: Küchler and Kutoyants (2000);
- Discontinuous periodic signal in a time inhomogeneous diffusion: Höpfner and Kutoyants (2009).

The result presented by Dachian (see Dachian, 2009 and D & N, 2010) connects, in some sense, some of this change point problems.

In fact it was proved that the limit likelihood ratio process of some of this estimation problems, converges to the limit likelihood ratio process $Z_0(u)$ (that appear in some other problems) as the distance of the threshold goes to 0. With the necessary rescaling.

Goodness of fit test - diffusion process

Goodness of fit tests play an important role in theoretical and applied statistics.

Such test are useful if they are distribution free or asymptotically distribution free.

We have

$$dX_t = S(X_t)dt + \sigma(X_t)dW_t, \qquad X_0, t \le 0$$

and we wish to test

$$H_0 : S = S_0$$

against any alternative

$$H_0 : S \neq S_0$$

Goodness of fit test - diffusion process

Suppose that the null hypothesis is simple: the observation X^T comes from the threshold models with known θ_0.

To test this hypothesis Kutoyants (2010) propose the statistics

$$\mathbb{V}_T^2(\theta_0) = T\int_{-\infty}^{+\infty} H(\theta_0, x)(\hat{F}_T(x) - F(\theta_0, x))^2 dF(\theta_0, x)$$

where $\hat{F}_T(x) = \frac{1}{T}\int_{-\infty}^{x} \mathbf{1}_{X_t \le x} dt$ is the empirical distribution function.

It holds that $\mathbb{V}_T^2(\theta_0)$ converge in distribution, under H_0 to $\int_0^{+\infty} W(s)^2 e^{-s} ds$. So the test is asymptotically distribution free.

The test can be generalized for composite null hypothesis, with the value of θ is unknown. $\mathbb{V}_T^2(\hat{\theta}_T)$ can be used, where $\hat{\theta}_T$ is some consistent estimator for θ, and the test is again asymptotically distribution free.

Some generalizations can be found in Kutoyants (2009).

Goodness of fit test for ergodic diffusion

N & N (2009) propose a test statistics based on the *score marked empirical process*, defined as:

$$V_T(x) = \frac{1}{\sqrt{T}} \int_0^T \mathbf{1}_{(-\infty,x]}(X_t) \frac{1}{\sigma(X_t)} (\mathrm{d}X_t - S_0(X_t)\mathrm{d}t)$$

To test the simple hypothesis

$$H_0 : S = S_0$$

against any alternative $S_1 \neq S_0$, for the ergodic diffusion process

$$dX_t = S(X_t)dt + \sigma(X_t)dW_t, \quad X_0 \quad t \geq 0$$

observed on the continuous time interval $[0, T]$

It has been proved that the test proposed is asymptotically distribution free and consistent.

Some other approaches can be found in Dachian & Kutoyants (2008) and Kutoyants (2009).

Test for parameter change in ergodic diffusion process

Preliminaries - Existence

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $\{\mathcal{A}_t\}_{t\geq 0}$ a filtration of $\mathcal{A}$

Let $\{S(\cdot,\theta) : \theta \in \Theta\}$ be a family of $\mathbb{R}$-valued measurable functions on $\mathbb{R}$ indexed by $\Theta \subset \mathbb{R}^k$

Let $\sigma : \mathbb{R} \to (0,\infty)$ be a measurable known function.

Suppose that $S(\cdot,\theta)$ and $\sigma(\cdot)$ are such that there exists a solution X^θ to the stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t S(X_s,\theta)ds + \int_0^t \sigma(X_s)dW_s, \quad t \geq 0, \tag{2}$$

where $W = \{W_s : \ s \geq 0\}$ is a standard Wiener process and the initial value X_0 is independent of W_t, $t \geq 0$.

Preliminaries - Ergodicity

The *scale function* of the diffusion (2) is defined by

$$p_\theta(x) = \int_0^x \exp\left\{-2\int_0^y \frac{S(v,\theta)}{\sigma^2(v)}\,\mathrm{d}v\right\}\mathrm{d}y$$

The *speed measure* of the diffusion (2) is defined by

$$m_\theta(A) = \int_A \frac{1}{\sigma(x)^2}\exp\left(2\int_0^x \frac{S(y,\theta)}{\sigma(y)^2}dy\right)dx, \qquad A \in \mathcal{B}(\mathbb{R})$$

Let us suppose that X^θ is regular, $\lim_{x\to\pm\infty} p_\theta(x) = \pm\infty$ and that $m_\theta(\mathbb{R}) < \infty$.

Then, the solution process X^θ is ergodic with the invariant distribution function $F_\theta(\cdot)$ given by $F_\theta(x) = m_\theta((-\infty, x])/m_\theta(\mathbb{R})$

So it holds for any $dF_\theta(x)$-integrable function g that, with probability one,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(X_t)dt = \int_{\mathbb{R}} g(z)dF_\theta(z).$$

Test: definition of the problem

Now let us consider the SDE

$$X_t = X_0 + \int_0^t S(X_s, \theta) ds + \int_0^t \sigma(X_s) dW_s, \quad t \geq 0,$$

where θ may change at a certain point $s \in [0, T]$.

We wish to test:

H_0: there exists a certain $\theta_0 \in \Theta$ such that $\theta = \theta_0$ for all $s \in [0, T]$;

H_1: there exist two different values $\theta_0 \neq \theta_1$ both belonging to Θ, and a certain $u_* \in (0, 1)$, such that $\theta = \theta_0$ for $s \in [0, Tu_*]$ and $\theta = \theta_1$ for $s \in (Tu_*, T]$.

We will construct a test statistic which is asymptotically distribution free under H_0, and consistent under H_1

Conditions - 1

We suppose that S is two times differentiable with respect to θ and the derivatives satisfy the following conditions:

$$\int_{\mathbb{R}} \frac{||\dot{S}(z,\theta_0)||}{\sigma(z)} dF_{\theta_0}(z) < \infty, \quad \forall \theta_0 \in \Theta, \tag{3}$$

and

$$\int_{\mathbb{R}} \frac{\sup_\theta ||\ddot{S}(z,\theta)||}{\sigma(z)} dF_{\theta_0}(z) < \infty, \quad \forall \theta_0 \in \Theta. \tag{4}$$

The log-likelihood function of the process (2), observed up to time T, is given by

$$L_T(\theta) = \int_0^T \frac{S(X_t,\theta)}{\sigma^2(X_t)} dX_t - \frac{1}{2}\int_0^T \frac{S^2(X_t,\theta)}{\sigma^2(X_t)} dt. \tag{5}$$

We define $\widehat{\theta}_T$ the maximizer of (5) over Θ.

Conditions - 2

We suppose that for every $\theta_i \in \Theta$, θ_i is the unique local and global minimizer of

$$\theta \mapsto g(\theta, \theta_i) = \int_{\mathbb{R}} \frac{(S(x,\theta) - S(x,\theta_i))^2}{\sigma^2(x)} dF_{\theta_i}(x), \tag{6}$$

over Θ. Actually, we suppose that the function

$$\theta \mapsto \frac{\partial}{\partial \theta} g(\theta, \theta_i) = \int_{\mathbb{R}} \frac{(S(x,\theta) - S(x,\theta_i))\dot{S}(x,\theta)}{\sigma^2(x)} dF_{\theta_i}(x), \tag{7}$$

is zero if and only if $\theta = \theta_i$. Hereafter, we suppose that the order of integration and differentiation is exchangeable. Let θ^* be the minimizer of

$$\theta \mapsto G(\theta, \theta_0, \theta_1) = u_* g(\theta, \theta_0) + (1 - u_*) g(\theta, \theta_1) \tag{8}$$

over Θ. Here θ_0, θ_1 and u_* are the same as specified under H_1.

Later on we will suppose that $\sqrt{T}(\widehat{\theta}_T - \theta_0) = O_{\mathbf{P}}(1)$ under H_0 and that $\widehat{\theta}_T \to^p \theta^*$ under H_1.

Comments on conditions

• Condition $\sqrt{T}(\widehat{\theta}_T - \theta_0) = O_{\mathbf{P}}(1)$ under H_0 is really standard in estimation theory for ergodic diffusion (see for example Kutoyants, 2004)

• $\widehat{\theta}_T \to^p \theta^*$ under H_1 is true. This can be proved because under H_1, the following almost sure convergence holds

$$\frac{1}{T}L_T(\theta) \to \int_{\mathbb{R}} \frac{S(z,\theta_0)^2}{\sigma^2(z)} dF_{\theta_0}(z) + \int_{\mathbb{R}} \frac{S(z,\theta_1)^2}{\sigma^2(z)} dF_{\theta_1}(z) - G(\theta,\theta_0,\theta_1),$$

and under some mild conditions this convergence is uniform in $\theta \in \Theta$. (see for example van der Vaart and Wellner, 1996)

Definition of the test statistic

Here on we suppose that all the conditions stated in previous slides hold.

In order to construct a statistic for this testing problem, we introduce the random field

$$\{\widehat{V}_T(u,x) : (u,x) \in [0,1] \times \mathbb{R}\}$$

given by

$$\widehat{V}_T(u,x) = \frac{1}{\sqrt{T}} \int_0^T (\mathbf{1}_{\{s \le Tu\}} - u) \mathbf{1}_{\{X_s \le x\}} \frac{1}{\sigma(X_s)} (dX_s - S(X_s, \widehat{\theta}_T) ds),$$

where $\widehat{\theta}_T$ satisfies

$\sqrt{T}(\widehat{\theta}_T - \theta_0) = O_{\mathsf{P}}(1)$ under H_0.

Main result

The main result is the following.

Theorem (i) Under H_0, if $\sqrt{T}(\widehat{\theta}_T - \theta_0) = O_{\mathbf{P}}(1)$, it holds that

$$\sup_{u,x} |\widehat{V}_T(u,x)| \to^d \sup_{(s,t)\in[0,1]^2} |B^\circ(s,t)|$$

where B° is a centered Gaussian random field with the covariance

$$\mathbf{E}[B^\circ(s_1,t_1)B^\circ(s_2,t_2)] = (s_1 \wedge s_2 - s_1 s_2)(t_1 \wedge t_2).$$

(ii) Under H_1, if $\widehat{\theta}_T \to^p \theta^*$, it holds that

$$\mathbf{P}\left(\sup_{u,x} |\widehat{V}_T(u,x)| > K\right) \to 1, \quad \forall K > 0.$$

Martingale approach

Our approach is based on considering first of all the random field

$$V_T(u,x) = \frac{1}{\sqrt{T}} \int_0^T (\mathbf{1}_{\{s \le Tu\}} - u)\mathbf{1}_{\{X_s \le x\}} \frac{1}{\sigma(X_s)} (dX_s - S(X_s, \theta_0) ds)$$

The random field V_T converges weakly, as T goes to infinity, in $\ell^\infty([0,1] \times \mathbb{R})$ to the centered Gaussian random field

$$\tilde{B}^\circ = \{B^\circ(u, F_{\theta_0}(x)) : (u,x) \in [0,1] \times \mathbb{R}\}$$

where

$$B^\circ = \{B^\circ(s,t) : (s,t) \in [0,1] \times [0,1]\}$$

is a centered Gaussian random field with the covariance

$$\mathbf{E}[B^\circ(s_1,t_1) B^\circ(s_2,t_2)] = (s_1 \wedge s_2 - s_1 s_2)(t_1 \wedge t_2).$$

Indeed V_T under H_0 can be written as $V_T(u,x) = M_T^{T,(u,x)}$ where

$$M_t^{T,(u,x)} = \frac{1}{\sqrt{T}} \int_0^t (\mathbf{1}_{\{s \le Tu\}} - u)\mathbf{1}_{\{X_s \le x\}} \frac{1}{\sigma(X_s)} (dX_s - S(X_s, \theta_0) ds), \qquad t \in [0,T].$$

Uniform convergence

The crucial point of our approach then will be to prove that

$$\sup_{u,x} |\widehat{V}_T(u,x) - V_T(u,x)| \to^p 0$$

Indeed under H_0, if $\sqrt{T}(\widehat{\theta}_T - \theta_0) = O_{\mathbf{P}}(1)$, then $\sup_{u,x} |\widehat{V}_T(u,x) - V_T(u,x)| \to^p 0$, as $T \to \infty$.

The proof is essentially based on the following development of $S(x,\theta)$ around θ_0. In fact we can write, for a good choice of $\tilde{\theta}$,

$$S(x,\theta) = S(x,\theta_0) + \dot{S}(x,\theta_0)'(\theta - \theta_0) + (\theta - \theta_0)'\ddot{S}(x,\tilde{\theta})(\theta - \theta_0).$$

where we have denoted the transpose with a prime.

The test in practice

If we wish to test if there is a change of the parameter in the model

$$X_t = X_0 + \int_0^t S(X_s, \theta)ds + \int_0^t \sigma(X_s)dW_s, \quad t \geq 0,$$

we can reject H_0 at a fixed level $0 < \alpha < 1$ if

$$\sup_{u,x} |\hat{V}_T(u,x)| > c_\alpha$$

where the test statistic is computed over the observation of the process for $t \in [0, T]$.

The critical value c_α is given by $\mathbf{P}(\sup_{(s,t)} |B^\circ(s,t)| > c_\alpha) = \alpha$

The distribution of the limit process is well known (see for example Brownrigg (2005))

Conclusions and further developments

A test for detecting if a change in the parameter in the drift of a diffusion process takes place has been presented. The test is based on the continuous observation of the process up to time T. It has been proved that it is asymptotically distribution free and consistent.

- Goodness of fit test - composite null hypothesis. The drift coefficient depend on a location and a scale parameter that have to be estimated. Find asymptotical distribution free test.

- Goodness of fit test - contiguous alternatives. We could consider the problem of testing $H_0 : S = S_0$ *versus* $H_1 : S = S_0 + \frac{h}{\varphi(T)}$, and we want to study the asymptotic properties of likelihood ratio under H_1.

- Change parameter test – discrete sampling scheme

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