

Estimation of function in stationary noise

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Estimation of pseudoperiodic function

This talk is connected with nonparametric estimation of the function $s(t)$ as the observation process $Y(t)$ is given by

$$dY(t) = s(t)dt + dX(t), t \in [-T, T].$$

Here unknown function $s \in \mathcal{L}_* \subset \mathcal{L}$, \mathcal{L} is the Banach space with the norm $\|\cdot\|_{\mathcal{L}}$,

$$\|s\|_{\mathcal{L}}^2 = \sup_x \int_x^{x+1} |s(t)|^2 dt < \infty, \quad (1)$$

\mathcal{L}_* is the subset of the Stepanov space $\mathcal{L}(\Lambda)$ of pseudoperiodic functions

$$s(t) = \sum_{u \in \Lambda} a(u)e^{iut}, \text{ defined by } \sum_{u \in \Lambda} (1 + |u|)^{2\beta} |a(u)|^2 \leq C, \quad (2)$$

Λ is a countable subset of real line such that

$$\tau = \tau(\Lambda) = \inf_{u, v \in \Lambda, u \neq v} |u - v| > 0. \quad (3)$$

noise process $X(t)$ is the gaussian process with stationary increments with zero mean and with the spectral density f :

$$\mathbf{E} \left\{ \int \varphi(t) dX(t) \overline{\int \psi(t) dX(t)} \right\} = \int \widehat{\varphi}(u) \overline{\widehat{\psi}(u)} f(u) du,$$

where $\widehat{\varphi}$ is the Fourier transformation of function φ .

For an estimator \widehat{s}_T of unknown function s we denote

$$R_T(\widehat{s}_T, f) = \sup_{s \in \mathcal{L}_*} \mathbf{E}_{s,f} \|\widehat{s}_T - s\|_{\mathcal{L}}^2.$$

Let $R_T(f)$ be the minimax risk,

$$R_T(f) = \inf_{\widehat{s}_T} \sup_{s \in \mathcal{L}_*} \mathbf{E}_{s,f} \|\widehat{s}_T - s\|_{\mathcal{L}}^2.$$

We plan construct an estimator \widehat{s}_T such that for appropriate class \mathcal{K} of spectral densities and sufficiently large T

$$\sup_{f \in \mathcal{K}} \frac{R_T(\widehat{s}_T, f)}{R_T(f)} \leq C(\mathcal{K}, \tau, \mathcal{L}_*), \quad (4)$$

where the constant $C(\mathcal{K}, \tau, \mathcal{L}_*)$ depends only on class \mathcal{K} , the value

$$\tau = \tau(\Lambda) = \inf_{u, v \in \Lambda, u \neq v} |u - v| > 0,$$

and values β, C in the inequality

$$\sum_{u \in \Lambda} (1 + |u|)^{2\beta} |a(u)|^2 \leq C,$$

that define the parametric set \mathcal{L}_* . We prove for this estimator that under appropriate condition on spectral density f

$$R_T(\hat{s}_T, f) \asymp T^{-\frac{2\beta}{2\beta + \gamma + 1}},$$

where $\gamma > -1$ and depends on some characteristic of f .

Class \mathcal{K} of spectral densities

For function f and interval I , $0 \in I$, denote

$$f_I(t) = \frac{1}{|I|} \int_I f(t+x) dx, \text{ where } |I| \text{ is the length of } I.$$

In the case as $I = [-\varepsilon/2, \varepsilon/2]$ we shall write $f_\varepsilon(t)$ instead of $f_I(t)$.

The nonnegative function f satisfied to the Muckenhoupt condition if

$$\lambda(f) := \sup_I \frac{1}{|I|} \int_I f(x) dx \times \frac{1}{|I|} \int_I \frac{1}{f(x)} dx < \infty. \quad (5)$$

We suppose that

$$\lambda(\mathcal{K}) := \sup_{f \in \mathcal{K}} \lambda(f) < \infty \quad (6)$$

We need another condition on the regular behavior of sums

$$\Sigma_f(N) := \sum_{|u| \leq N} f_\varepsilon(u), \text{ as } N \rightarrow \infty.$$

We suppose that for some $\gamma > -1$ and $\varepsilon = N^{-(1+\gamma+\beta)}$ functions $f \in \mathcal{K}$ satisfy to the following inequality

$$\sum_{|u| \leq N} f_\varepsilon(u) \leq C(\mathcal{K})N^{1+\gamma}. \quad (7)$$

For example,

if $f(x) = (1 + |x|)^\alpha$, then $\gamma = \alpha$;

if $f(x) = |u - x|^\alpha, |u - x| \leq \delta$, for all $u \in \Lambda$, then $\gamma = -\frac{\alpha(2\beta + 1)}{\alpha + 1}$.

So, we denote by $\mathcal{K}(\lambda, C, \gamma), \gamma > -1$ the class of nonnegative functions such that

$$\lambda(f) = \sup_I \frac{1}{|I|} \int_I f(x) dx \times \frac{1}{|I|} \int_I \frac{1}{f(x)} dx < \lambda,$$

and

$$\sum_{|u| \leq N} f_\varepsilon(u) \leq C N^{1+\gamma}, \text{ if } \varepsilon = N^{-(1+\gamma+\beta)}.$$

The main result

We denote

$$Y[\varphi] = \int \overline{\varphi(t)} dY(t).$$

and set for fix T

$$\varphi_u(t) = e^{iut}, \quad \varphi_u^T(t) = e^{iut} \mathbf{1}_{[-T, T]}(t).$$

If $\{\psi_v^T, v \in \lambda\}$ is the conjugate system with respect to the system $\{\varphi_v^T, v \in \lambda\}$:

$$\frac{1}{2T} \int_{-T}^T \varphi_u^T(t) \overline{\psi_v^T(t)} dt = \delta_{u,v},$$

then

$$s(t) = \sum_{u \in \Lambda} \left\{ \int s(r) \overline{\psi_u^T(r)} dr \times \varphi_u(t) \right\}, \quad \text{if } s(t) = \sum_{u \in \Lambda} a(u) \varphi_u(t).$$

Here we assume that

$$\sum_{u \in \Lambda} |a(u)|^2 < \infty, \text{ and } \tau = \tau(\Lambda) = \inf_{u, v \in \Lambda, u \neq v} |u - v| > 0.$$

The value

$$\hat{a}_T(u) = Y[\psi_u^T] = a(u) + X[\psi_u^T], \text{ where } X[\psi] = \int \overline{\psi(t)} dX(t)$$

may be considered as estimator for the coefficient $a(u)$.

For the appropriately chosen $N = N(T)$, the estimator

$$\hat{s}_T(t) = \sum_{u \in \Lambda, |u| \leq N} \hat{a}_T(u) \varphi_u(t)$$

can be considered as estimator for function $s(t)$.

Theorem 1. Suppose that unknown function $s \in \mathcal{L}$ and spectral density $f \in \mathcal{K}(\lambda, C, \gamma)$. Then, for

$$N(T) = T^{\frac{1}{1+\gamma+2\beta}}, R_T(\hat{s}_T, f) \asymp T^{-\frac{2\beta}{2\beta+\gamma+1}}.$$

Theorem 2. Under the condition of theorem 1, for $T > T_0$

$$\sup_{f \in \mathcal{K}} \frac{R_T(\widehat{s}_T, f)}{R_T(f)} \leq C(\mathcal{K}, \tau, \mathcal{L}_*), \quad (8)$$

The reasons choice of the class \mathcal{K}

Really we have the problem of estimating the vector $a = \{a(u), u \in \Lambda\}$ of coefficients $a(u)$ of function

$$s(t) = \sum_{u \in \Lambda} a(u) e^{iut}$$

on observations

$$y_u = a(u) + x_u, \text{ where } y_u = Y[\psi_u^T], \ x_u = X[\psi_u^T],$$

as $\sum_{u \in \Lambda} (1 + |u|)^{2\beta} |a(u)|^2 \leq C$.

At the case when variables x_u are independent gaussian, it is a well studied problems. In our model gaussian variables x_u are dependent.

But under the Muckenhoupt condition

$$\lambda(f) := \sup_I \frac{1}{|I|} \int_I f(x) dx \times \frac{1}{|I|} \int_I \frac{1}{f(x)} dx < \infty. \quad (9)$$

the system $\{x_u, u \in \Lambda\}$ is a basis in the space $L^2(dP)$. It is sufficient for some of our analytical needs. Under this condition

$$\inf \mathbf{E} \left(x_u - \sum_{v \neq u} c_v x_v \right) > (1 - \rho^2) \mathbf{E}(x_u)^2, \text{ where } \rho^2 < 1.$$

The reasons choice of the space \mathcal{L}

It is well known that under the condition

$$\tau(\Lambda) = \inf_{u, v \in \Lambda, u \neq v} |u - v| > 0,$$

Banach norm $\|s\|_{\mathcal{L}}$

$$\|s\|_{\mathcal{L}}^2 = \sup_x \int_x^{x+1} |s(t)|^2 dt \quad (10)$$

is equivalent to the Hilbert norm $\|s\|_*$,

$$\|s\|_*^2 = \frac{1}{2T} \int_{-T}^T |s(t)|^2 dt,$$

in the space $\mathcal{L}(\Lambda)$ of pseudoperiodic functions.