

Sequential parameter estimators with guaranteed accuracy for delay differential equations

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Outline

- 1 Regression problem
- 2 Restriction to SDDEs
- 3 Problems with the delay
- 4 Analytical tools
- 5 Proof of the Theorem
- 6 General parameter set

The problem

We start with a linear regression model

$$dX(t) = \vartheta' a(t)dt + dW(t), \quad t \geq 0 \quad (1)$$

where

- $(W(t), t \geq 0)$ is an adapted one-dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$,
- ϑ is an unknown parameter from some subset Θ of R^{p+1} and
- $(a(t), t \geq 0)$ forms an observable adapted $(p+1)$ -dimensional cadlag process.

See e.g. Galtchouk, Konev (2001), Konev, Pergamenshchikov (1992)

The model (1) includes

Ornstein-Uhlenbeck-processes, linear stochastic differential equations (CARMA-processes) and **linear stochastic delay differential equations**.

Problem: Estimate ϑ based on observation of $X(t), t \geq 0$.

$$dX(t) = \vartheta' a(t)dt + dW(t), \quad t \geq 0,$$

A natural candidate for estimating ϑ is the least squares estimator (LSE)

$$\vartheta(T) = \left(\int_0^T a(t)a'(t)dt \right)^{-1} \int_0^T a(t)dX(t), \quad T > 0.$$

In this talk, we restrict ourselves to the example of special delay differential equations

$$dX(t) = \vartheta_0 X(t)dt + \vartheta_1 X(t-1)dt + dW(t). \quad (2)$$

The considered candidate for estimating ϑ now equals (for fixed $T > 0$)

$$\vartheta(T) = \left[\int_0^T (X(t), X(t-1))'(X(t), X(t-1))dt \right]^{-1} \int_0^T (X(t), X(t-1))'dX(t).$$

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We use the notation

$$I_T := \begin{pmatrix} \int_0^T X^2(t) dt & \int_0^T X(t)X(t-1) dt \\ \bullet & \int_0^T X^2(t-1) dt \end{pmatrix}$$

(Fishers information matrix), and

$$V_T := \begin{pmatrix} \int_0^T X(t) dX(t) \\ \int_0^T X(t-1) dX(t) \end{pmatrix},$$

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(Fishers information matrix), and

$$V_T^W := \begin{pmatrix} \int_0^T X(t) dW(t) \\ \int_0^T X(t-1) dW(t) \end{pmatrix},$$

to get

$$\vartheta(T) = (I_T)^{-1} V_T, \text{ and}$$

$$\vartheta(T) - \vartheta = (I_T)^{-1} V_T^W =$$

$$\left(\begin{array}{cc} \int_0^T X^2(t) dt & \int_0^T X(t)X(t-1) dt \\ \bullet & \int_0^T X^2(t-1) dt \end{array} \right)^{-1} \left(\begin{array}{c} \int_0^T X(t) dW(t) \\ \int_0^T X(t-1) dW(t) \end{array} \right).$$

We see, that

the asymptotic properties of $\vartheta(T)$ for $T \rightarrow \infty$ are determined by the asymptotic behavior of $X(T)$ for $T \rightarrow \infty$.

We define with $\text{tr} I_T = \int_0^T (X^2(t) + X^2(t-1)) dt$

$$\tau_\varepsilon := \inf\{T > 0 \mid \text{tr} I_T \geq \varepsilon^{-1}\},$$

$$\vartheta_\varepsilon := I_{\tau_\varepsilon}^{-1} V_{\tau_\varepsilon}.$$

Fix any number $q \geq 2$. We are interested in the quantity

$$E_{\vartheta} \|\vartheta_\varepsilon - \vartheta\|^q = E_{\vartheta} \|I_{\tau_\varepsilon}^{-1} V_{\tau_\varepsilon}^W\|^q \leq E_{\vartheta} \left[\|I_{\tau_\varepsilon}^{-1}\|^q \|V_{\tau_\varepsilon}^W\|^q \right].$$

Using the Burkholder-Gundy inequality we obtain for some constant b_q :

$$E_{\vartheta} \|V_{\tau_\varepsilon}^W\|^q \leq b_q \varepsilon^{-\frac{q}{2}}.$$

Otherwise we do not have a bound for

$$E_{\vartheta} \|I_{\tau_\varepsilon}^{-1}\|^q.$$

Note: In the O-U-case $\vartheta_1 = 0$ we have

$$I_T = \int_0^T X^2(t)dt \text{ and thus } \text{tr } I_T = I_T,$$

thus

$$I_{\tau(\varepsilon)}^{-1} = \left[\int_0^{\tau_\varepsilon} X^2(t)dt \right]^{-1} = \varepsilon.$$

Moreover it holds

$$\begin{aligned} E_\vartheta [\vartheta(\tau_\varepsilon) - \vartheta]^2 &= E_\vartheta \left[\left(\int_0^{\tau_\varepsilon} X^2(t)dt \right)^{-1} \int_0^{\tau_\varepsilon} X(t)dW(t) \right]^2 = \\ \varepsilon^2 E_\vartheta \left[\int_0^{\tau_\varepsilon} X(t)dW(t) \right]^2 &= \varepsilon^2 \cdot E_\vartheta \int_0^{\tau_\varepsilon} X^2(t)dt = \varepsilon^2 \cdot \frac{1}{\varepsilon} = \varepsilon \end{aligned}$$

But for the delay equation we have more than one parameter and thus

$$\text{tr } I_T \neq I_T.$$

Exit: Choose an increasing sequence $(c_n > 0, n \geq 1)$ tending to infinity and define

$$\tau_\varepsilon(n) := \inf\{T > 0 \mid \text{tr } I_T \geq \varepsilon^{-1} c_n\},$$

then it holds $\tau_\varepsilon(n) \uparrow \infty$ for $n \rightarrow \infty$ and for $\varepsilon \downarrow 0$, .

Note that $\text{tr } I_{\tau_\varepsilon(n)} = \varepsilon^{-1} c_n$ and put

$$\vartheta_\varepsilon(n) := I_{\tau_\varepsilon(n)}^{-1} V_{\tau_\varepsilon(n)}.$$

To control the behaviour of

$$E_\vartheta \|I_{\tau_\varepsilon(n)}^{-1}\|^q$$

we randomize the argument n as follows:

- Choose $(c_n > 0, n \geq 1)$ such that $\sum_{n \geq 1} c_n^{-\frac{q}{2}} < \infty$,
- define the random variables $\beta_\varepsilon(n) := \varepsilon c_n^{-1} \|I_{\tau_\varepsilon(n)}^{-1}\|^{-1} > 0$,
- introduce $S_\varepsilon(N) := \sum_{n=1}^N \beta_\varepsilon^q(n)$, (increasing with N).
- put $\rho := b_q \sum_{n \geq 1} c_n^{-\frac{q}{2}}$,
- finally $v_\varepsilon := \inf\{N \geq 1 : S_\varepsilon(N) \geq \rho\}$.

Now the sequential estimation plan $(T_\varepsilon, \vartheta_\varepsilon)$ is constructed as

$$T_\varepsilon := \tau_\varepsilon(v_\varepsilon),$$

$$\vartheta_\varepsilon := S_\varepsilon^{-1}(v_\varepsilon) \sum_{n=1}^{v_\varepsilon} \beta_\varepsilon^q(n) \vartheta_\varepsilon(n), \quad \varepsilon > 0.$$

(Random convex linear combination, remember $\vartheta_\varepsilon(n) = I_{\tau_\varepsilon(n)}^{-1} V_{\tau_\varepsilon(n)}$)

Theorem

Assume Θ is the set of parameters $\vartheta = (\vartheta_0, \vartheta_1)'$, under which the equation

$$dX(t) = \vartheta_0 X(t)dt + \vartheta_1 X(t-1)dt + dW(t)$$

admits a stationary solution. Then it holds:

1. For any $\varepsilon > 0$ and every $\vartheta \in \Theta$ the sequential estimation plan $(T_\varepsilon, \vartheta_\varepsilon)$ of ϑ is closed: $T_\varepsilon < \infty$ P_ϑ - a.s.,
2. for any $\varepsilon > 0$ it holds

$$\sup_{\vartheta \in \Theta} \|\vartheta_\varepsilon - \vartheta\|_q^2 \leq \varepsilon,$$
3. $\limsup_{\varepsilon \rightarrow 0} \varepsilon \cdot T_\varepsilon < \infty$ P_ϑ - a.s.,
4. the estimator ϑ_ε is strongly consistent:

$$\lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon = \vartheta \quad P_\vartheta \text{ - a.s..}$$

To sketch the proof, we need some analytical properties of the considered delay equation.

$$X(t) = x_0(t)X_0(0) + \vartheta_1 \int_{-1}^0 x_0(t-s-1)X_0(s) ds + \int_0^t x_0(t-s)dW(s), \quad t \geq 0. \quad (3)$$

where $x_0(\cdot)$ is the *fundamental solution* of the deterministic delay equation

$$\dot{x}_0(t) = \vartheta_0 x_0(t) + \vartheta_1 x_0(t-1), \quad x_0(s) = \mathbf{1}_{\{0\}}(s), \quad s \in [-1, 0]. \quad (4)$$

Asymptotic properties of $X(t)$ for $T \rightarrow \infty$ are determined by the asymptotics of

$x_0(t)$ for $t \rightarrow \infty$.

Let us study the asymptotic properties of $x_0(t)$ for $t \rightarrow \infty$.

Laplace transform of equation (4):

- $$\lambda \hat{x}_0(\lambda) = \vartheta_0 \hat{x}_0(\lambda) + \vartheta_1 e^{-\lambda} \hat{x}_0(\lambda), \quad \operatorname{Re} \lambda > 0,$$

- $$\hat{x}_0(\lambda) = \frac{1}{\lambda - \vartheta_0 - \vartheta_1 e^{-\lambda}}, \quad \operatorname{Re} \lambda > 0,$$

- Characteristic function

$$h(\lambda) := \lambda - \vartheta_0 - \vartheta_1 e^{-\lambda}, \quad \lambda \in K,$$

- Set Λ of zeros of the characteristic function

$$\Lambda := \{\lambda \in K \mid h(\lambda) = 0\}$$

-

$$v_0 := \max\{\operatorname{Re} \lambda \mid \lambda \in \Lambda\} < \infty, \quad v_k := \max\{\operatorname{Re} \lambda \mid \lambda \in \Lambda, \operatorname{Re} \lambda < v_{k-1}\}.$$

Applying the inverse Laplace transform and Cauchy's residual theorem it can be shown

Lemma

For $c \in (v_2, v_1)$ the fundamental solution $x_0(\cdot)$ of (4) can be represented in the form

$$x_0(t) = \psi_0(t) \exp[v_0 t] + \psi_1(t) \exp[v_1 t] + o(\exp(\gamma t)) \quad (5)$$

for some $\gamma < c$.

Here $\psi_0(t)$ equals

1. $\frac{1}{v_0 - \vartheta_0 + 1}$ if $v_0 \in \Lambda, m(v_0) = 1,$
2. $2t + \frac{2}{3}$ if $v_0 \in \Lambda, m(v_0) = 2,$
3. $\frac{2(v_0 - \vartheta_0 + 1)}{(v_0 - \vartheta_0 + 1)^2 + \xi_0^2} \cos(\xi_0 t) + \frac{2\xi_0}{(v_0 - \vartheta_0 + 1)^2 + \xi_0^2} \sin(\xi_0 t)$

if $v_0 \notin \Lambda.$

(ξ_0 is the imaginary part of the $\lambda \in \Lambda$ with $Re(\lambda) = v_0$).

Asymptotics of $x_0(T)$ for $T \rightarrow \infty$ is determined by the element λ from the set Λ with greatest real part v_0 .

Conclusion: As we will see later, the asymptotic properties of $(\vartheta(T) - \vartheta)$ for $T \rightarrow \infty$ depend on the two values v_0 and v_1 .

The set Θ of parameters $\vartheta = (\vartheta_0, \vartheta_1)'$, under which the equation

$$dX(t) = \vartheta_0 X(t)dt + \vartheta_1 X(t-1)dt + dW(t)$$

admits a stationary solution, is the set where $\vartheta_0 < 0$ holds. This is the area N in the subsequent figure.

$X(\cdot)$ is ergodic if $\vartheta \in \Theta$. We get in particular

$$\lim_{t \rightarrow \infty} t^{-1} \int_0^t X^2(s)ds = \int_0^\infty x_0^2(s)ds < \infty \quad P_\vartheta - \text{a.s.} \quad (6)$$

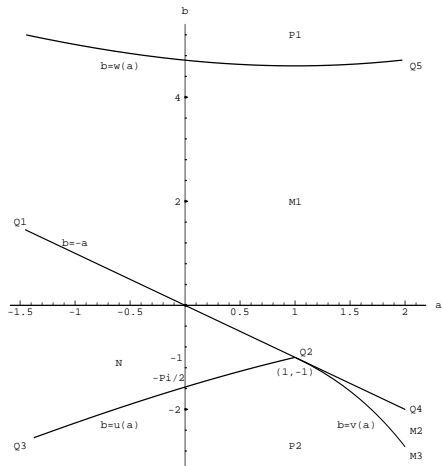
and therefore it follows

$$\lim_{T \rightarrow \infty} \frac{I_T}{T} = \lim_{T \rightarrow \infty} T^{-1} \begin{pmatrix} \int_0^T X^2(s)ds & \int_0^T X(s)X(s-1)ds \\ \int_0^T X(s)X(s-1)ds & \int_0^T X^2(s-1)ds \end{pmatrix} =$$

$$\begin{pmatrix} \int_0^{\infty} x_0^2(s) ds & \int_0^{\infty} x_0(s)x_0(s+1) ds \\ \int_0^{\infty} x_0(s)x_0(s+1) ds & \int_0^{\infty} x_0^2(s) ds \end{pmatrix} =: I_{\infty}. \quad (7)$$

(I_{∞} is nonsingular)

$$\lim_{T \rightarrow \infty} T^{-1} \operatorname{tr} I_T = 2 \int_0^{\infty} x_0^2(s) ds = \operatorname{tr} I_{\infty} \quad (8)$$



1. Finiteness of the stopping time T_ε :

We have (note that $\text{tr } I_{\tau_\varepsilon(n)} = \varepsilon^{-1} c_n$)

$$\lim_{n \rightarrow \infty} \frac{\tau_\varepsilon(n)}{\varepsilon^{-1} c_n} = \lim_{n \rightarrow \infty} \left[\text{tr} \frac{I_{\tau_\varepsilon(n)}}{\tau_\varepsilon(n)} \right]^{-1} = \left[\lim_{T \rightarrow \infty} \text{tr} \left[\frac{I_T}{T} \right] \right]^{-1} = [\text{tr } I_\infty]^{-1} =: \alpha > 0$$

$$\beta_\varepsilon(n) := (\varepsilon^{-1} c_n \|I_{\tau_\varepsilon(n)}^{-1}\|)^{-1} =$$

$$\left[\left(\frac{\text{tr } I_{\tau_\varepsilon(n)}}{\tau_\varepsilon(n)} \right) (\tau_\varepsilon(n)) \|I_{\tau_\varepsilon(n)}^{-1}\| \right]^{-1} \rightarrow_{n \rightarrow \infty} [\text{tr } I_\infty \|I_\infty^{-1}\|]^{-1} =: \beta > 0$$

$$\frac{\varepsilon^{-1} c_n}{\tau_\varepsilon(n)} \beta_\varepsilon^2(n) \rightarrow_{n \rightarrow \infty} \alpha^{-1} \beta^2 = [\text{tr } I_\infty \|I_\infty^{-1}\|^2]^{-1} =: C_* > 0$$

Because $\beta_\varepsilon(n) \rightarrow_{n \rightarrow \infty} [(\text{tr} l_\infty) \|l_\infty^{-1}\|]^{-1} = \beta > 0$ we get

$$\sum_{n=1}^{\infty} \beta_\varepsilon^q(n) = \infty$$

Taking into account $S_\varepsilon(N) = \sum_{n=1}^N \beta_\varepsilon^q(n)$, it follows, that P_ϑ – a.s.

$$v_\varepsilon := \inf\{N \geq 1 : S_\varepsilon(N) \geq \rho\}$$

is finite and, consequently, that

$$T_\varepsilon := \tau_\varepsilon(v_\varepsilon)$$

is finite.

2. Accuracy of the estimator ϑ_ε :

$$\begin{aligned} \|\vartheta_\varepsilon - \vartheta\|_q^2 &:= (E_\vartheta(\|\vartheta_\varepsilon - \vartheta\|)^q) = \\ &\left(E_\vartheta \left[S_\varepsilon^{-1}(v_\varepsilon) \left\| \sum_{n=1}^{v_\varepsilon} \beta_\varepsilon^q(n) (\vartheta_\varepsilon(n) - \vartheta) \right\| \right]^q \right)^{\frac{2}{q}} \leq \\ &\left(E_\vartheta \left[S_\varepsilon^{-1}(v_\varepsilon) \sum_{n=1}^{v_\varepsilon} \beta_\varepsilon^q(n) \|\vartheta_\varepsilon(n) - \vartheta\| \right]^q \right)^{\frac{2}{q}} \end{aligned}$$

Applying Hölders inequality

$$\sum a_n b_n \leq \left(\sum a_n^{\frac{q}{q-1}} \right)^{\frac{q-1}{q}} \left(\sum b_n^q \right)^{\frac{1}{q}}$$

with

$$a_n = \beta_\varepsilon^{q-1}(n) \text{ and } b_n = \beta_\varepsilon(n) \|\vartheta_\varepsilon(n) - \vartheta\|,$$

2. Accuracy of the estimator ϑ_ε :

$$\begin{aligned} \|\vartheta_\varepsilon - \vartheta\|_q^2 &:= (E_\vartheta(\|\vartheta_\varepsilon - \vartheta\|)^q) = \\ &\left(E_\vartheta \left[S_\varepsilon^{-1}(v_\varepsilon) \left\| \sum_{n=1}^{v_\varepsilon} \beta_\varepsilon^q(n) (\vartheta_\varepsilon(n) - \vartheta) \right\| \right]^q \right)^{\frac{2}{q}} \leq \\ &\left(E_\vartheta \left[S_\varepsilon^{-1}(v_\varepsilon) \sum_{n=1}^{v_\varepsilon} \beta_\varepsilon^q(n) \|\vartheta_\varepsilon(n) - \vartheta\| \right]^q \right)^{\frac{2}{q}} \end{aligned}$$

Applying Hölders inequality

we obtain by using $S_\varepsilon(N) = \sum_{n=1}^N \beta_\varepsilon^q(n)$ the inequalities

$$\begin{aligned} \|\vartheta_\varepsilon - \vartheta\|_q^2 &\leq \left(E_\vartheta S_\varepsilon^{-q}(v_\varepsilon) (\sum_{n \leq v_\varepsilon} \beta_\varepsilon^q(n))^{q-1} \sum_{n \leq v_\varepsilon} \beta_\varepsilon^q(n) \|\vartheta_\varepsilon(n) - \vartheta\|^q \right)^{\frac{2}{q}} \leq \\ &\left(E_\vartheta S_\varepsilon^{-1}(v_\varepsilon) \sum_{n \geq 1} \beta_\varepsilon^q(n) \|\vartheta_\varepsilon(n) - \vartheta\|^q \right)^{\frac{2}{q}} \end{aligned}$$

Using the definitions

$$\rho := b_q \sum_{n \geq 1} c_n^{-\frac{q}{2}}, \quad v_\varepsilon := \inf\{N \geq 1 : S_\varepsilon(N) \geq \rho\},$$

$$\beta_\varepsilon(n) := \varepsilon c_n^{-1} \|I_{\tau_\varepsilon}^{-1}\|^{-1}, \quad S_\varepsilon(N) := \sum_{n=1}^N \beta_\varepsilon^q(n)$$

and the inequalities $E_\vartheta \|V_{\tau_\varepsilon}^W\|^q \leq b_q (c_n \varepsilon^{-1})^{\frac{q}{2}}$ and $S_\varepsilon^{-1}(v_\varepsilon) \leq \rho^{-1}$ we get

$$\begin{aligned} \|\vartheta_\varepsilon - \vartheta\|_q^2 &\leq \left(E_\vartheta S_\varepsilon^{-1}(v_\varepsilon) \sum_{n \geq 1} \beta_\varepsilon^q(n) \|\vartheta_\varepsilon(n) - \vartheta\|^q \right)^{\frac{2}{q}} \leq \\ &\left(\rho^{-1} \sum_{n \geq 1} E_\vartheta \beta_\varepsilon^q(n) \|I_{\tau_\varepsilon}^{-1}\|^q \|V_{\tau_\varepsilon(n)}^W\|^q \right)^{\frac{2}{q}} = \\ &\varepsilon^2 \left(\rho^{-1} \sum_{n \geq 1} \frac{1}{c_n^q} E_\vartheta \|V_{\tau_\varepsilon(n)}^W\|^q \right)^{\frac{2}{q}} \leq \\ &\varepsilon \left(\rho^{-1} b_q \sum_{n \geq 1} c_n^{-\frac{q}{2}} \right)^{\frac{2}{q}} = \varepsilon. \end{aligned}$$

3. The asymptotics of T_ε :

Analogously to the formulas above one can show that P_ϑ -a.s. it holds

$$\lim_{\varepsilon \rightarrow 0} \beta_\varepsilon(n) = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{c_n} \|I_{\tau_\varepsilon(n)}^{-1}\|^{-1} = [(\text{tr} I_\infty) \|I_\infty^{-1}\|]^{-1} = \beta > 0.$$

$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-1} c_n}{\tau_\varepsilon(n)} \beta_\varepsilon^2(n) = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{c_n \tau_\varepsilon(n)} \|I_{\tau_\varepsilon(n)}^{-1}\|^{-2} = [\text{tr} I_\infty \|I_\infty^{-1}\|^2]^{-1} = \alpha^{-1} \beta^2 > 0,$$

If $\beta_* < \beta$, ε small enough and n sufficiently large, we get

$$S_\varepsilon(n) = \sum_{k=1}^n \beta_\varepsilon^q(k) \geq \beta_*^q n \geq \rho$$

Thus for all ε small enough and all n sufficiently large, we have

$$v(\varepsilon) = \inf\{m \geq 1 : S_\varepsilon(m) \geq \rho\} \leq n.$$

3. The asymptotics of T_ε :

Analogously to the formulas above one can show that P_ϑ -a.s. it holds

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$$\lim_{\varepsilon \rightarrow 0} \frac{\varepsilon^{-1} c_n}{\tau_\varepsilon(n)} \beta_\varepsilon^2(n) = \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{c_n \tau_\varepsilon(n)} \|I_{\tau_\varepsilon(n)}^{-1}\|^{-2} = [\text{tr} I_\infty \|I_\infty^{-1}\|^2]^{-1} = \alpha^{-1} \beta^2 > 0,$$

Because $\frac{\tau_\varepsilon(n)}{\varepsilon^{-1} c_n} \rightarrow_{\varepsilon \rightarrow 0} [\text{tr} I_\infty]^{-1}$ for every $n \geq 1$, we get

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon \tau_\varepsilon(v_\varepsilon) < \infty,$$

and therefore

$$\limsup_{\varepsilon \rightarrow 0} \varepsilon T_\varepsilon = \limsup_{\varepsilon \rightarrow 0} \varepsilon \tau_\varepsilon(v_\varepsilon) < \infty.$$

4. Consistency of the estimator $\vartheta(\varepsilon)$:

The Maximum-Likelihood-estimator is consistent:

$$\lim_{T \rightarrow \infty} \|\vartheta(T) - \vartheta\| \leq \limsup_{T \rightarrow \infty} \|T \cdot I_T^{-1}\| \cdot \left\| \frac{V_T^W}{T} \right\| = 0$$

Because $\tau_\varepsilon(n) \rightarrow \infty$ as $\varepsilon \rightarrow 0$ the estimators $\vartheta_\varepsilon(n) = \vartheta(\tau_\varepsilon(n))$ and their weighted arithmetical mean ϑ_ε are also consistent.

What happens if the parameter ϑ is outside of N ?

The terms $T^{-1}I_T$ never converge for $T \rightarrow \infty$. But

1. in some cases $\vartheta \in \Theta_1(v_0 > 0, v_0 \notin \Lambda, \text{ area } P_2)$ we have

$$\exp[-2v_0 T]I_T - I_\infty(T) \rightarrow 0,$$

where $I_\infty(T)$ are some (explicit known) random, periodic nonsingular matrices,

2. in some cases $\vartheta \in \Theta_2(v_0 > 0, v_0 \in \Lambda, m(v_0) = 1, v_1 > 0, M_2 \cup P_1)$ we have

$$\begin{pmatrix} \exp[-2v_0 T] & 0 \\ 0 & \exp[-2v_1 T] \end{pmatrix} (VI_T - I_\infty(T)) \rightarrow 0,$$

where

$$V := \begin{pmatrix} 1 & -\exp[-2v_0 T] \\ 0 & 1 \end{pmatrix}$$

and $I_\infty(T)$ are some (explicit known) random, possibly periodic, nonsingular matrices.

3. in some cases $\vartheta \in \Theta_3$ ($v_0 > 0, v_0 \in \Lambda, m(v_0) = 1, v_1 < 0$ area M_1) we have

$$\begin{pmatrix} e^{-2v_0 T} & 0 \\ 0 & T^{-1} \end{pmatrix} (VI_T - I_\infty(T)) \rightarrow 0,$$

where I_∞ is a random (explicitly known) nonsingular matrix.

In every of these cases, a sequential plan $(T_j(\varepsilon), \vartheta_j(\varepsilon))$ can be constructed with similar properties as $(T_\varepsilon, \vartheta_\varepsilon)$, taking into account the specific kind of convergence of the normalized (I_T) to a limit matrix. Put

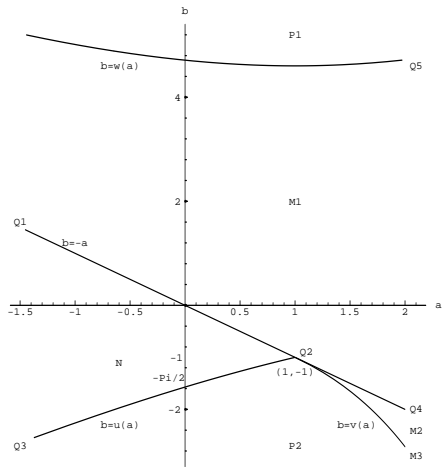
$$\Theta^* := \Theta \cup \Theta_1 \cup \Theta_2 \cup \Theta_3.$$

The remaining cases $\vartheta \in \mathbb{R}^2 \setminus \Theta^*$ form a set of Lebesgue measure zero. Because the true value of ϑ is not known before, we define

$$j^* := \arg \min T_j(\varepsilon), \quad j = 0, 1, 2, 3,$$

$$\vartheta^*(\varepsilon) := \vartheta_{j^*}(\varepsilon)$$

where $T_0(\varepsilon) := T_\varepsilon$ and $\vartheta_0(\varepsilon) := \vartheta_\varepsilon$. Finally we get



Theorem

1. For every $\varepsilon > 0$ and every $\vartheta \in \Theta^*$ the sequential estimation plans $(T^*(\varepsilon), \vartheta^*(\varepsilon))$ of ϑ are closed:

$$(T^*(\varepsilon) < \infty \text{ } P_{\vartheta} - \text{a.s.}).$$

2. For any $\varepsilon > 0$ it holds

$$\sup_{\vartheta \in \Theta^*} \|\vartheta^*(\varepsilon) - \vartheta\|_q^2 \leq \varepsilon;$$

3. For every $\vartheta \in \Theta^*$ the estimator $\vartheta^*(\varepsilon)$ is strongly consistent:

$$\lim_{\varepsilon \rightarrow 0} \vartheta^*(\varepsilon) = \vartheta \text{ } P_{\vartheta} - \text{a.s.}$$

There are different asymptotic properties of $T^*(\varepsilon)$ for $\varepsilon \rightarrow 0$ in every of the cases $\vartheta \in \Theta_i, i = 0, 1, 2, 3$

Some References

- [Brockwell,P.J. \(2001\)](#) Levy-driven CARMA processes *Ann. Inst. Stat. Math.* vol.53, p. 113–124,
- [Galtchouk,L., Konev,V.\(2001\)](#) On sequential estimation of parameters in semimartingale regression models with continuous time parameter *The Annals of Statistics*, vol.29, p.1508–1536,
- [Gushchin,A.A.,Küchler,U.\(1999\)](#) Asymptotic inference for a linear stochastic differential equation with time delay. *Bernoulli*, vol.5,p. 1059–1098,
- [Konev,V.V., Pergamenshchikov,S.M. \(1992\)](#) Sequential estimation of the parameters of linear unstable stochastic systems with guaranteed accuracy. *Problems of Inform. Trans.*, vol.28, p. 35–48, (in Russian).
- [Küchler,U., Vasiliev,V.A.\(2010\)](#) On guaranteed parameter estimation of a multiparameter linear regression process. *Automatica, Journal of IFAC, Elsevier*, 46, p. 637-646.
- (Ext. version:Preprint 7, Math. Inst.of Humboldt Univ. Berlin,1-60,2009)
- [Küchler,U., Vasiliev,V.A.,\(2005\)](#)Sequential identification of linear dynamic systems with memory. *Statist. Inference for Stochastic Processes*,8,p. 1-24.
- [Lai,T.L.,Wei,C.Z. \(1982\)](#) Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems.*Ann. Statist.*, vol.10, p. 154–166.

Thank you for your attention!