Drift parameter estimation in models with fractional Brownian motion by discrete observations

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Stochastic differential equations driven by a fractional Brownian motion have been a subject of an active research for the last two decades. Main reason is that such equations seem to be one of the most suitable tools to model long-range dependence in many applied areas, such as physics, finance, biology, network studies etc.

This talk deals with statistical estimation of drift parameter for a stochastic differential equation with fBm by discrete observation of its solution. We propose three new estimators and prove their strong consistency under the so-called "high-frequency data" assumption that the horizon of observations tends to infinity, while the interval between them goes to zero. Moreover, we obtain almost sure upper bounds for the rate of convergence of the estimators. The estimators proposed go far away from being maximum likelihood estimators, and this is their crucial advantage, because they keep strong consistency but they are not complicated technically and are convenient for the simulations.

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\left\{B_t^H, t \geq 0\right\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with the covariance

$$\mathrm{E}\left[\, B_t^H B_s^H \,\right] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

It is well known that B^H has a modification with almost surely continuous paths (even Hölder continuous of any order up to H), and further we will assume that it is continuous itself.
In what follows we assume that the Hurst parameter $H \in (1/2, 1)$ is fixed.

In this case, the integral with respect to the fBm B^H will be understood in the generalized Lebesgue–Stieltjes sense. Its construction uses the fractional derivatives, defined for $a < b$ and $\alpha \in (0,1)$ as

$$(D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x) - f(u)}{(x-u)^{1+\alpha}} du \right),$$

$$(D_{b-}^{1-\alpha} g)(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x) - g(u)}{(u-x)^{2-\alpha}} du \right).$$

Provided that $D_{a+}^{\alpha} f \in L_1[a,b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_\infty[a,b]$, where $g_{b-}(x) = g(x) - g(b)$, the generalized Lebesgue-Stieltjes integral $\int_a^b f(x) dg(x)$ is defined as

$$\int_a^b f(x) dg(x) = e^{i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f)(x) (D_{b-}^{1-\alpha} g_{b-})(x) dx. \tag{1}$$

It follows from Hölder continuity of B^H that for $\alpha \in (1-H, 1)$ $D_{b-}^{1-\alpha} B_{b-}^H \in L_\infty[a,b]$ a.s. Then for a function f with $D_{a+}^{\alpha} f \in L_1[a,b]$ we can define integral with respect to B^H through (1):

$$\int_a^b f(x)\, dB^H(x) := e^{i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} B_{b-}^H)(x)\, dx. \qquad (2)$$

Consider the stochastic differential equation driven by fractional Brownian motion B^H:

$$\begin{aligned} dX_t &= \theta a(t, X_t)dt + b(t, X_t)dB_t^H, \quad 0 \le t \le T, \quad T > 0, \\ X\big|_{t=0} &= X_0 \in \mathbb{R}. \end{aligned} \tag{3}$$

Here $\theta \in \mathbb{R}$ is unknown parameter to be estimated.

Assume that the following conditions hold:

(I) there exist positive constants C_1, C_2 such that for all $t \in [0, T]$, $x, y \in \mathbb{R}$

$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \le C_1 |x - y|,$$
$$|a(t,x)| + |b(t,x)| \le C_2(1 + |x|);$$

(II) there exist constants $C_3 > 0$ and $\rho \in \left(\frac{1}{H} - 1, 1\right)$ such that for all $t \in [0, T]$, $x, y \in \mathbb{R}$

$$\left|b'_x(t,x) - b'_y(t,y)\right| \le C_3 |x - y|^\rho;$$

(III) there exist constants $C_4 > 0$ and $\mu \in (1 - H, 1)$ such that for all $t, s \in [0, T]$, $x \in \mathbb{R}$

$$|b(t,x) - b(s,x)| + \left|b'_x(t,x) - b'_x(s,x)\right| \le C_4 |t - s|^\mu.$$

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(III) there exist constants $C_4 > 0$ and $\mu \in (1 - H, 1)$ such that for all $t, s \in [0, T]$, $x \in \mathbb{R}$

$$\left|b(t,x) - b(s,x)\right| + \left|b'_x(t,x) - b'_x(s,x)\right| \le C_4 \left|t - s\right|^{\mu}.$$

According to [Nualart,Rascanu(2002), Theorem 2.1], under the conditions (I)–(III) there exists a unique solution X of the stochastic equation (3).

In addition, suppose that the following conditions hold:

(IV) $b(t,x) \neq 0$;

(V) $a, b \in C([0,\infty) \times \mathbb{R})$.

Denote $\alpha = H - \frac{1}{2}$, $\widetilde{\alpha} = (1-2\alpha)^{-1}$, $C_H = \left(\frac{\Gamma(2-2\alpha)}{2H\Gamma(1-\alpha)^3\Gamma(\alpha+1)}\right)^{\frac{1}{2}}$, $l_H(t,s) = C_H s^{-\alpha}(t-s)^{-\alpha} I_{\{0<s<t\}}$, $\psi(t,x) = \frac{a(t,x)}{b(t,x)}$, $\varphi(t) = \psi(t, X_t)$, $I(t) = \int_0^t l_H(t,s)\varphi(s)ds$.

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Under the conditions (I), (III)–(V) $\varphi(t), t \in [0,T]$ is a continuous process with probability 1. Hence, it is Lebesgue integrable and for each $t \in [0,T]$ there exists an integral $\int_0^t l_H(t,s)\varphi(s)ds$.

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Consider the new process $\widehat{B}_t^H := B_t^H + \theta \int_0^t \varphi(s)ds$.

Suppose that the following assumptions hold.

(VI) There exists such function δ that belongs to $L_1[0,t]$ for all $t \in [0,T]$ a. s. and satisfies the equation

$$\theta \int_0^t l_H(t,s)\varphi(s)ds = (\widetilde{\alpha})^{-1/2} \int_0^t \delta_s ds;$$

(VII) $\mathrm{E} \int_0^t s^{2\alpha}\delta_s^2 ds < \infty,\ t \in [0,T]$;

(VIII) $\mathrm{E} \exp\left\{L_t - \frac{1}{2}\langle L\rangle_t\right\} = 1$, where $L_t = \int_0^t s^{\alpha}\delta_s d\widehat{B}_s$, and $\widehat{B}$ is Wiener process with respect to probability measure $\mathrm{P}_0(t)$ corresponding to the zero drift such that

$$\int_0^t l_H(t,s)\, d\widehat{B}_s^H = \widetilde{\alpha}^{-1/2} \int_0^t s^{-\alpha}\, d\widehat{B}_s.$$

We can present likelihood ratio as a function of the observed process X_t.

$$L_t = \int_0^t s^{\alpha} \delta_s d\widehat{B}_s = \int_0^t s^{2\alpha} \delta_s dY_s, \tag{4}$$

where

$$Y_s = \int_0^s u^{-\alpha} d\widehat{B}_u = \widetilde{\alpha}^{1/2} \int_0^s l_H(s,u) b^{-1}(u, X_u) dX_u, \tag{5}$$

$$\delta_s = \theta \widetilde{\alpha}^{1/2} \left(\int_0^s l_H(s,u) \varphi(u) du \right)' = C_H \theta \widetilde{\alpha}^{1/2} \left(\int_0^s (s-u)^{-\alpha} u^{-\alpha} \varphi(u) du \right)' \tag{6}$$

or

$$\delta_s = C_H \theta \widetilde{\alpha}^{1/2} \left(\frac{\varphi(s)}{s^{2\alpha}} + \alpha \int_0^s \frac{s^{-\alpha}\varphi(s) - u^{-\alpha}\varphi(u)}{(s-u)^{\alpha+1}} du \right). \tag{7}$$

According to [Mishura (2008), formula (6.3.13)], the maximum-likelihood estimator has the form

$$\hat{\theta}_t^{(1)} = \frac{\widetilde{\alpha}^{-1/2} \int_0^t s^{\alpha} l'(s) d\widehat{B}_s}{\int_0^t s^{2\alpha} (l'(s))^2 ds} = \frac{\theta L_t}{\int_0^t s^{2\alpha} \delta_s^2 ds}. \tag{8}$$

Using (4), (5), (7) and the definition of the kernel $l_H(t,s)$ we can write

$$\hat{\theta}_t^{(1)} = \frac{\theta \int_0^t s^{2\alpha} \delta_s dY_s}{\int_0^t s^{2\alpha} \delta_s^2 ds}$$

$$= \frac{\int_0^t \left(\varphi(s) + \alpha s^{2\alpha} \int_0^s \frac{s^{-\alpha}\varphi(s) - u^{-\alpha}\varphi(u)}{(s-u)^{\alpha+1}} du\right) d\left(\int_0^s v^{-\alpha}(s-v)^{-\alpha} b^{-1}(v, X_v) dX_v\right)}{\int_0^t s^{2\alpha} \left(\frac{\varphi(s)}{s^{2\alpha}} + \alpha \int_0^s \frac{s^{-\alpha}\varphi(s) - u^{-\alpha}\varphi(u)}{(s-u)^{\alpha+1}} du\right)^2 ds}$$

Remark

According to [Mishura (2008), Theorem 6.3.3], under assumptions (I)–(VIII) and $\int_0^\infty s^{2\alpha}(l'(s))^2 ds = \infty$ a. s. we have the convergence $\hat{\theta}_T^{(1)} \xrightarrow{\text{P 1}} \theta$, $T \to \infty$.

Let $t_k^n = \frac{k}{2^n}$, $k = 0, 1, 2, \dots, 2^{2n}$. We can define a discretized version of the maximum-likelihood estimator

$$\hat{\theta}_n^{(2)} := \frac{\sum_{k=0}^{2^{2n}-1} \left(\varphi(t_k^n) + \alpha (t_k^n)^{2\alpha} \sum_{i=1}^{k-1} \frac{(t_k^n)^{-\alpha}\varphi(t_k^n) - (t_i^n)^{-\alpha}\varphi(t_i^n)}{(t_k^n - t_i^n)^{\alpha+1} 2^n} \right) \left(\widetilde{Y}_{t_{k+1}^n} - \widetilde{Y}_{t_k^n} \right)}{\sum_{k=0}^{2^{2n}-1} (t_k^n)^{2\alpha} \left(\frac{\varphi(t_k^n)}{(t_k^n)^{2\alpha}} + \alpha \sum_{i=1}^{k-1} \frac{(t_k^n)^{-\alpha}\varphi(t_k^n) - t_i^{-\alpha}\varphi(t_i^n)}{(t_k^n - t_i^n)^{\alpha+1}} \frac{1}{2^n} \right)^2 \frac{1}{2^n}} \tag{9}$$

where

$$\widetilde{Y}_{t_k} = \sum_{i=1}^{k-1} (t_i^n)^{-\alpha} (t_k^n - t_i^n)^{-\alpha} b^{-1}(t_i^n, X_{t_i^n}) \left(X_{t_{i+1}^n} - X_{t_i^n} \right).$$

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where

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In the general case formula (9) is not suitable for applications because it involves a lot of weakly singular kernels and it is quite impossible to get its convergence to the true value of the parameter. But even if we get the convergence, the simulation error will be so great that annihilate our efforts in discretization. In order to avoid this technical difficulties, we start with the simplest case.

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$$dX_t = \theta b(X_t)dt + b(X_t)dB_t^H. \tag{10}$$

In this case the maximum-likelihood estimator can be written as follows:

$$\hat{\theta}_t^{(1)} = \frac{\int_0^t s^{-\alpha}(t-s)^{-\alpha} b^{-1}(X_s)dX_s}{\mathrm{B}(1-\alpha, 1-\alpha)t^{1-2\alpha}}. \tag{11}$$

Now we consider an estimator

$$\hat{\theta}_n^{(3)} = \frac{\sum_{k=1}^{2^{2n}-1} (t_k^n)^{-\alpha}(2^n - t_k^n)^{-\alpha} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\mathrm{B}(1-\alpha, 1-\alpha)2^{n(1-2\alpha)}},$$

where $t_k^n = \frac{k}{2^n}$, $k = 0, 1, \ldots, 2^{2n}$. This estimator is a discretized version of the estimator (11).

Theorem

Suppose that there exist positive constants C_1, C_2, C_3, C_5 and $\rho \in (1/H - 1, 1]$, such that

(a) $|b(x) - b(y)| \le C_1 |x - y|$ *for all* $x, y \in \mathbb{R}$,

(b) $C_5 \le |b(x)| \le C_2(1 + |x|)$ *for all* $x \in \mathbb{R}$,

(c) $|b'(x) - b'(y)| \le C_3 |x - y|^\rho$ *for all* $x, y \in \mathbb{R}$

Then $\hat{\theta}_n^{(3)} \xrightarrow{\text{P 1}} \theta, n \to \infty$. *Moreover, for any* $\beta \in (1/2, H)$ *and* $\gamma > 1/2$ *there exists a random variable* $\eta = \eta_{\beta,\gamma}$ *with all finite moments such that* $\left|\hat{\theta}_n^{(3)} - \theta\right| \le \eta n^{\kappa+\gamma} 2^{-\tau n}$, *where* $\kappa = \gamma/\beta$, $\tau = (1 - H) \wedge (2\beta - 1)$.

Consider a stochastic differential equation

$$X_t = X_0 + \theta \int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s^H, \tag{12}$$

where X_0 is a non-random coefficient. In [Lyons (1998)], it is shown that this equation has a unique solution under the following assumptions: there exist constants $\delta \in (1/H - 1, 1]$, $K > 0$, $L > 0$ and for every $N > 0$ there exists $R_N > 0$ such that

(A) $|a(x)| + |b(x)| \le K \quad$ for all $x, y \in \mathbb{R}$,

(B) $|a(x) - a(y)| + |b(x) - b(y)| \le L\,|x - y| \quad$ for all $x, y \in \mathbb{R}$,

(C) $|b'(x) - b'(y)| \le R_N\,|x - y|^\delta \quad$ for all $x \in [-N, N], y \in [-N, N]$.

Our main problem is the following: to construct an estimator for θ based on discrete observations of X. Specifically, we will assume that for some $n \geq 1$ we observe values $X_{t_n^k}$ at the following uniform partition of $[0, 2^n]$: $t_k^n = k2^{-n}$, $k = 0, 1, \dots, 2^{2n}$.

In order to construct consistent estimators for θ, we need another technical assumption, in addition to conditions (A)–(C):

(D) there exists a constant $M > 0$ such that for all $x \in \mathbb{R}$

$$|a(x)| \geq M, \qquad |b(x)| \geq M.$$

We now define an estimator, which is a discretized version of a maximum likelihood estimator for $F(X)$, where $F(x) = \int_0^x b(y)^{-1} dy$:

$$\hat{\theta}_n^{(4)} = \frac{\sum_{k=1}^{2^{2n}} \left(t_k^n\right)^{-\alpha} \left(2^n - t_k^n\right)^{-\alpha} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\sum_{k=1}^{2^{2n}} \left(t_k^n\right)^{-\alpha} \left(2^n - t_k^n\right)^{-\alpha} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right) \frac{1}{2^n}},$$

where $\alpha = H - \frac{1}{2}$.

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where $\alpha = H - \frac{1}{2}$.

Theorem

With probability one, $\hat{\theta}_n^{(4)} \to \theta$, $n \to \infty$.
Moreover, for any $\beta \in (1/2, H)$ and $\gamma > 1/2$ there exists a random variable $\eta = \eta_{\beta,\gamma}$ with all finite moments such that $\left|\hat{\theta}_n^{(4)} - \theta\right| \le \eta n^{\kappa+\gamma} 2^{-\tau n}$, where $\kappa = \gamma/\beta$, $\tau = (1 - H) \wedge (2\beta - 1)$.

Consider a simpler estimator:

$$\hat{\theta}_n^{(5)} = \frac{\sum_{k=1}^{2^{2n}} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\frac{1}{2^n}\sum_{k=1}^{2^{2n}} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right)}.$$

This is a discretized maximum likelihood estimator for θ in equation (13), where B^H is replaced by Wiener process. Nevertheless, this estimator is consistent as well. Namely, we have the following result.

Theorem

With probability one, $\hat{\theta}_n^{(5)} \to \theta$, $n \to \infty$.
Moreover, for any $\beta \in (1/2, H)$ and $\gamma > 1/2$ there exists a random variable $\eta = \eta_{\beta,\gamma}$ with all finite moments such that $\left|\hat{\theta}_n^{(5)} - \theta\right| \le \eta n^{\kappa+\gamma} 2^{-\tau n}$, where $\kappa = \gamma/\beta$, $\tau = (1 - H) \wedge (2\beta - 1)$.

In the paper [Kozachenko et al (2015)] the following non-standard estimator for θ in the equation (3) was considered:

$$\hat{\theta}_t^{(6)} = \frac{\int_0^t a(s, X_s) b^{-2}(s, X_s) dX_s}{\int_0^t a^2(s, X_s) b^{-2}(s, X_s) ds}.$$

According to [Kozachenko et al (2015), Theorem 4], if the assumptions (I)–(IV), (VI)–(VII) hold and there exist such $\beta > 1 - H$ and $p > 1$ that

$$\frac{T^{H+\beta-1} (\log T)^p \int_0^T \left| (D_{0+}^{\beta} \varphi)(s) \right| ds}{\int_0^T \varphi_s^2 ds} \to 0 \ \text{ a.s. as } T \to \infty,$$

then the estimator $\hat{\theta}_T^{(6)}$ is well-defined and strongly consistent as $T \to \infty$. We define a discretized version of $\hat{\theta}_T^{(6)}$ for the equation

$$X_t = X_0 + \theta \int_0^t a(X_s) ds + \int_0^t b(X_s) dB_s^H. \tag{13}$$

Put

$$\hat{\theta}_n^{(7)} := \frac{\sum_{k=1}^{2^{2n}} a\left(X_{t_{k-1}^n}\right) b^{-2}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\sum_{k=1}^{2^{2n}} a^2\left(X_{t_{k-1}^n}\right) b^{-2}\left(X_{t_{k-1}^n}\right) \frac{1}{2^n}}.$$

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Let

$$\widehat{\varphi}_n(t) := \sum_{k=0}^{2^{2n}-1} \varphi(t_k^n) I_{[t_k^n, t_{k+1}^n)}(t).$$

Theorem

Assume that there exist constants $\beta > 1 - H$ and $p > 1$ such that

$$\frac{2^{n(H+\beta)} n^p \int_0^{2^n} \left|\left(D_{0+}^{\beta} \widehat{\varphi}_n\right)(s)\right| ds}{\sum_{k=1}^{2^{2n}} \varphi^2(t_{k-1}^n)} \to 0 \quad a.s. \ as\ n \to \infty.$$

Then with probability one, $\hat{\theta}_n^{(7)} \to \theta, n \to \infty$.

Example

Consider the following model:

$$dX_t = \theta b(X_t)dt + b(X_t)dB_t^H.$$

In this case the estimator $\hat{\theta}_n^{(7)}$ has the form

$$\hat{\theta}_n^{(8)} = 2^{-n} \sum_{k=1}^{2^{2n}} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right), \tag{14}$$

$\widehat{\varphi}_n(t) = 1$. Then $\left(D_{0+}^{\beta}\widehat{\varphi}_n\right)(s) = \frac{1}{\Gamma(1-\beta)} \cdot s^{-\beta}$ and

$$\frac{2^{n(H+\beta)} n^p \int_0^{2^n} \left|\left(D_{0+}^{\beta}\widehat{\varphi}_n\right)(s)\right| ds}{\sum_{k=1}^{2^{2n}} \varphi^2(t_{k-1}^n)} = \frac{n^p}{\Gamma(2-\beta)\cdot 2^{n(1-H)}} \to 0, \quad n \to \infty.$$

Consequently the estimator (14) is strongly consistent.

The proof of the strong consistency is based on the following estimate for the fractional derivative of B^H.

Let some $\alpha \in (1 - H, 1/2)$ be fixed.

Denote for $t_1 < t_2$

$$
\begin{aligned}
Z(t_1, t_2) &= \left(D^{1-\alpha}_{t_2-} B^H_{t_2-}\right)(t_1) \\
&= \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{B^H_{t_1} - B^H_{t_2}}{(t_2 - t_1)^{1-\alpha}} + (1 - \alpha) \int_{t_1}^{t_2} \frac{B^H_{t_1} - B^H_u}{(u - t_1)^{2-\alpha}} du \right).
\end{aligned}
$$

The following proposition is a generalization of [Kozachenko et al (2015), Theorem 3].

Theorem

For any $\gamma > 1/2$,

$$\xi_{H,\alpha,\gamma} := \sup_{0 \le t_1 < t_2 \le t_1 + 1} \frac{|Z(t_1, t_2)|}{(t_2 - t_1)^{H+\alpha-1} \left(|\log(t_2 - t_1)|^{1/2} + 1 \right) (\log(t_2 + 3))^{\gamma}} \tag{15}$$

is finite almost surely.
Moreover, there exists $c_{H,\alpha,\gamma} > 0$ *such that* $\mathrm{E}\left[\exp\left\{ x \xi^2_{H,\alpha,\gamma} \right\} \right] < \infty$ *for* $x < c_{H,\alpha,\gamma}$.

Fix some $\beta \in (1/2, H)$.
Denote for $t_1 < t_2$

$$\Lambda_\beta(t_1, t_2) = 1 \vee \sup_{t_1 \le u < v \le t_2} \frac{|Z(u,v)|}{(v-u)^{\beta+\alpha-1}}.$$

Theorem

There exists a constant $M_{\alpha,\beta}$ depending on α, β, K, and L such that for any $t_1 \ge 0, t_2 \in (t_1, t_1+1]$

$$|X_{t_2} - X_{t_1}| \le M_{\alpha,\beta} \left(\Lambda_\beta(t_1,t_2)(t_2-t_1)^\beta + \Lambda_\beta(t_1,t_2)^{1/\beta}(t_2-t_1) \right).$$

Corollary

For any $\gamma > 1/2$, there exist random variables ξ and ζ such that for all $t_1 \geq 0$, $t_2 \in (t_1, t_1 + 1]$

$$|X_{t_2} - X_{t_1}| \leq \zeta (t_2 - t_1)^{\beta} (\log(t_2 + 3))^{\kappa}, \quad \Lambda_\beta(t_1, t_2) \leq \xi (\log(t_2 + 3))^{\kappa\beta},$$

where $\kappa = \gamma/\beta$. Moreover, there exists some $c > 0$ such that $\mathrm{E}\left[\exp\left\{x\xi^2\right\}\right] < \infty$ and $\mathrm{E}\left[\exp\left\{x\zeta^{2\beta}\right\}\right] < \infty$ for $x < c$. In particular, all moments of ξ and ζ are finite.

Lemma

For any $n \geq 1$ and any $t_1, t_2 \in [0, 2^{2n}]$ such that $t_1 < t_2 \leq t_1 + 1$

$$|X_{t_2} - X_{t_1}| \leq \zeta n^{\kappa} (t_2 - t_1)^{\beta}, \quad \Lambda_\beta(t_1, t_2) \leq \xi n^{\gamma}.$$

In order to construct a consistent estimator, we need a lemma concerning discrete approximation of integrals.

Lemma

For all $n \geq 1$ and $k = 1, 2, \ldots, 2^{2n}$

$$\left| \int_{t_{k-1}^n}^{t_k^n} \left(a(X_u) - a(X_{t_{k-1}^n}) \right) du \right| \leq C\zeta n^{\kappa} 2^{-n(\beta+1)}$$

and

$$\left| \int_{t_{k-1}^n}^{t_k^n} \left(b(X_u) - b(X_{t_{k-1}^n}) \right) dB_u^{H,\theta} \right| \leq C\xi\zeta n^{\gamma+\kappa} 2^{-2n\beta}.$$

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