Asymptotics for Student-Lévy regression *

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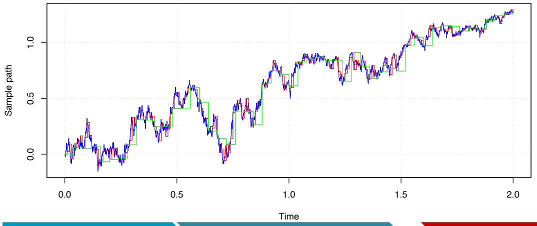
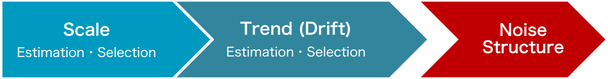
Objective in general

- Inference for **coefficients** and driving **non-Gaussian noise** process

$$X_{t+dt} \leftarrow X_t + (\text{Trend})dt + (\text{Scale}) \cdot d(\text{Noise})$$





Masuda, H., Mercuri, L., and Uehara, Y. (2022).
Noise inference for ergodic Lévy driven SDE.
Electron. J. Statist., 16(1):2432–2474.

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Quasi-likelihood analysis for Student-Lévy regression.
arXiv:2306.16790.

Objective in this talk: A non-trivial case study with t-noise

- Discrete-time sample $\{(X_{t_j}, Y_{t_j})\}_{j=0}^{[nT_n]}$ with $t_j = jh = j/n$ ($h = 1/n$) from

$$Y_t = X_t \cdot \mu + \sigma J_t, \qquad t \in [0, T_n].$$

  - $X = (X_t)$ is a càdlàg stochastic process in $\mathbb{R}^q$ satisfying some regularity conditions.
  - $J = (J_t)$ is a Lévy process such that the unit-time distribution $\mathcal{L}(J_1) = t_\nu$:

$$t_\nu \sim \text{pdf} \quad f(x;\nu) := \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}(1+x^2)^{-(\nu+1)/2}.$$

- Parameter of interest: $\theta := (\mu, \sigma, \nu) \in \Theta_\mu \times \Theta_\sigma \times \Theta_\nu = \Theta$
- **Our objective** is to estimate the true value $\theta_0 = (\mu_0, \sigma_0, \nu_0) \in \Theta$ from $\{(X_{t_j}, Y_{t_j})\}_{j=0}^{[nT_n]}$ when:
  - $T_n \to \infty$ as $n \to \infty$;
  - Θ is a bounded convex domain in $\mathbb{R}^q + 2$ such that $\overline{\Theta} \subset \mathbb{R}^q \times (0,\infty) \times (0,\infty)$.

- [Massing, 2019] studied the LAN property for Student-t Lévy process **when** $\nu > 1$ **is known**.

Sample $\{(X_{t_j}, Y_{t_j})\}_{j=0}^{[nT_n]}$ from $Y_t = X_t \cdot \mu + \sigma J_t$, where $t_j = jh = j/n$ and $\mathcal{L}(J_1) = t_\nu := t_\nu(0,1)$.

$$\sigma^{-1}\left(Y_{t_j} - Y_{t_{j-1}} - (X_{t_j} - X_{t_{j-1}}) \cdot \mu\right) \overset{\theta}{=} J_{t_j} - J_{t_{j-1}} \sim J_h$$

- The Student-t distribution $t_\nu(0,1)$ is **not closed under convolution**:

$$\varphi_\nu(u) := E[e^{iuJ_1}] = \frac{2^{1-\nu/2}}{\Gamma(\nu/2)}|u|^{\nu/2}K_{\nu/2}(|u|), \qquad u \in \mathbb{R},$$

$$\text{where} \qquad K_\nu(t) = \frac{1}{2}\int_0^\infty s^{\nu-1}\exp\left\{-\frac{t}{2}\left(s + \frac{1}{s}\right)\right\} ds.$$

- Even if X is non-random, the exact log-likelihood is **highly intractable** ($\Delta_j \xi := \xi_{t_j} - \xi_{t_{j-1}}$):

$$\ell_n(\theta) = \sum_{j=1}^n \log\left(\sigma^{-1}\left(\frac{2^{1-\nu/2}}{\Gamma(\nu/2)}\right)^h \frac{1}{\pi}\int_0^\infty \cos\left(\frac{\Delta_j Y - \Delta_j X \cdot \mu}{\sigma}u\right) u^{\nu h/2}\left(K_{\nu/2}(u)\right)^h du\right)$$

  - The numerical integration $\int_0^\infty (\dots)du$ can be unstable, particularly for very small h.

$$Y_t = X_t \cdot \mu + \sigma J_t$$

$$\sigma^{-1}\left(Y_{t_j} - Y_{t_{j-1}} - (X_{t_j} - X_{t_{j-1}}) \cdot \mu\right) \overset{\theta}{=} J_{t_j} - J_{t_{j-1}} \sim J_h$$

Proposal for $\hat{\theta}_n = (\hat{\mu}_n, \hat{\sigma}_n, \hat{\nu}_n)$: Adaptive (two-stage) quasi-likelihood inference

1. Estimate (μ_0, σ_0) based on the **Cauchy quasi-likelihood**
2. Estimate ν_0 based on the **Student-t quasi-likelihood** through the "unit-time" residual sequence

$$\hat{\epsilon}_i := \hat{\sigma}_n^{-1}\left(Y_i - Y_{i-1} - \hat{\mu}_n \cdot (X_i - X_{i-1})\right), \qquad i = 1, \dots, [T_n],$$

Points in our study

- We can **asymptotically efficiently** estimate (μ_0, σ_0), with leaving ν_0 unknown.
- The two-stage procedure enables us to estimate ν_0 **as if we directly observe** $(J_i)_{i=1}^{[T_n]}$.
  - We will **thin the full data in the first stage** to obtain a "clean" result.

$$\{(X_{t_j}, Y_{t_j})\}_{j=0}^{[nT_n]} \quad \text{from} \quad \begin{cases} Y_t = X_t \cdot \mu + \sigma J_t \\ J_1 \sim t_\nu \end{cases} \quad \Rightarrow \quad \sqrt{??}\begin{pmatrix} \hat{\mu}_n - \mu_0 \\ \hat{\sigma}_n - \sigma_0 \end{pmatrix} = O_p(1)$$

- We have $h^{-1}J_h \xrightarrow{\mathcal{L}} t_1$ as $h \to 0$, whatever the degrees of freedom $\nu_0 > 0$ is:

$$\left(E_\theta\left[\exp(iuh^{-1}J_1)\right]\right)^h = (1+o(1))\left|u\nu^{-1/2}h^{-1}\right|^{\nu h/2} K_{\nu/2}\left(\left|uh^{-1}\right|\right)^h \to e^{-|u|}.$$

- We look at the **Cauchy quasi-(log-)likelihood** (QLF) in such a way:

$$Y_{t_j} \big| \{Y_{t_{j-1}} = y,\, X_{t_{j-1}},\, X_{t_j}\} \overset{\theta}{\approx} \text{Cauchy}\left(y + \mu\cdot(X_{t_j} - X_{t_{j-1}}),\, h\sigma\right)$$

  - We can make an inference for (μ, σ) without knowing the value ν_0;

Using the Cauchy QLF is not free:

- The information of ν_0 gets disappeared in the small-time limit.
- The Cauchy-approximation errors accumulate for $T_n \to \infty$.

- We use data on $[0, B_n]$, where

$$B_n \le T_n, \qquad B_n \to \infty, \qquad \exists \epsilon' \in (0,1),\ B_n \lesssim n^{1-\epsilon'}. \tag{2.1}$$

- Technically (Essentially?) we needed in the proof that $(\phi_1(y) := \pi^{-1}(1+y^2)^{-1})$

$$\sqrt{nB_n} \int |\zeta(y)|\, |f_h(y) - \phi_1(y)|\, dy \lesssim n^{r-1}\sqrt{nB_n} = \sqrt{n^{2r-1}B_n} \to 0,$$

for any $r > 0$ small enough and logarithmic-growth-order function ζ.

  - (2.1) restricts the speed of $T_n \to \infty$ as $T_n = O(n^{1-\epsilon'})$, when using the whole sample (i.e. $B_n = T_n$).

$$Y_t = X_t \cdot \mu + \sigma J_t$$

Assumption 2.1 (Covariate process X)

We can associate an $(\mathcal{F}_t)$-adapted process $t \mapsto X'_t$ having càdlàg sample paths s.t.:

1. *Some integrability conditions (omitted: [Masuda et al., 2023, Aasumption 2.1])*
2. *There exists a probability measure $\pi_X(dx)$ on the Borel space $(\mathbb{R}^q, \mathcal{B}^q)$ such that*

$$\forall f \in L^1(\pi_0), \quad \frac{1}{T}\int_0^T f(X'_t)dt \xrightarrow{p} \int f(x')\pi_0(dx'), \qquad T \to \infty.$$

3. *The integral $S_0 := \int x'^{\otimes 2}\pi_0(dx')$ is finite and positive-definite.*

- One may think of $X_t = \int_0^t X'_s ds$, where sample paths of X' may have certain discontinuity.

- Let $a := (\mu, \sigma)$, $a_0 := (\mu_0, \sigma_0)$, $N_n := [nB_n]$ with $T_n/N_n \to 0$, and $\Gamma_{a,0} := (2\sigma_0^2)^{-1}\mathrm{diag}(S_0, 1)$.
- **Cauchy quasi-MLE**: $\hat{a}_n := (\hat{\mu}_n, \hat{\sigma}_n) \in \mathrm{argmax}_{a \in \overline{\Theta_\mu \times \Theta_\sigma}} \mathbb{H}_{1,n}(a)$, where (apart from a constant)

$$\mathbb{H}_{1,n}(a) = -\sum_{j=1}^{N_n} \left\{ \log \sigma + \log \left(1 + \left(\frac{\Delta_j Y - \mu \cdot \Delta_j X}{h\sigma} \right)^2 \right) \right\}.$$

Theorem 2.2 (Cauchy quasi-likelihood estimation of (μ, σ))

1. *Asymptotic normality:* $\hat{u}_{a,n} := \sqrt{N_n}(\hat{a}_n - a_0) = \Gamma_{a,0}^{-1}\Delta_{a,n} + o_p(1) \xrightarrow{\mathcal{L}} N\left(0, \Gamma_{a,0}^{-1}\right)$
2. *Tail-probability estimate:* $\forall L > 0, \quad \sup_{r>0} \sup_n P\left[|\hat{u}_{a,n}| > r\right] r^L < \infty$

- Thanks to the LAN property [Ivanenko et al., 2015], we can conclude the **asymptotic efficiency**.

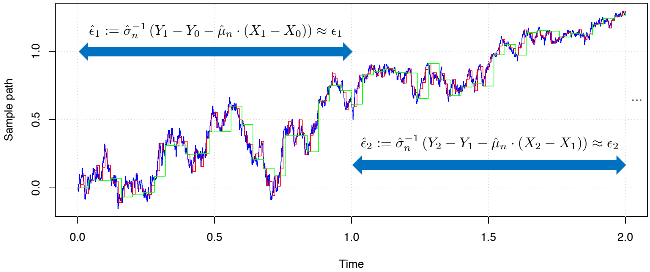
Estimation of the degrees of freedom ν from $\{(X_{t_j}, Y_{t_j})\}_{j=0}^{[nT_n]}$ under Assumption 2.1

$$\begin{cases} Y_t = X_t \cdot \mu + \sigma J_t \\ J_1 \sim t_\nu \end{cases} \Rightarrow \sqrt{N_n}\begin{pmatrix} \hat{\mu}_n - \mu_0 \\ \hat{\sigma}_n - \sigma_0 \end{pmatrix} \xrightarrow{\mathcal{L}} N_{q+1}(0, \Gamma_a^{-1}) \quad \Rightarrow \quad \sqrt{?}(\hat{\nu}_n - \nu_0) = O_p(1)$$

- Define the **unit-time** residual sequence ($i = 1, \dots, [T_n]$):

$$\hat{\epsilon}_i := \hat{\sigma}_n^{-1}\left(Y_i - Y_{i-1} - \hat{\mu}_n \cdot (X_i - X_{i-1})\right) \overset{\theta}{\approx} J_i - J_{i-1} =: \epsilon_i \sim \text{i.i.d. } t_\nu$$

- We make use $\hat{\epsilon}_i \approx \epsilon_i$ to estimate ν as if $\hat{\epsilon}_1, \dots, \hat{\epsilon}_{[T_n]}$ are observed t_ν-i.i.d. samples.



- We consider the quasi-likelihood function

$$\widehat{M}_n(\nu) = \sum_{i=1}^{[T_n]} \rho(\hat{\epsilon}_i; \nu),$$

where ($f(\cdot;\nu)$ is the noise t_ν density)

$$\rho(\epsilon;\nu) := \log f(\epsilon;\nu) = \log\left(\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})} (1+x^2)^{-(\nu+1)/2} \right)$$

- We define the **Student quasi-MLE** of ν by any element $\hat{\nu}_n \in \mathrm{argmax}_{\nu \in \overline{\Theta}_\nu} \widehat{M}_n(\nu)$.
  - $\widehat{M}_n(\cdot)$ is a.s. convex.
  - Any other reasonable contrast function can be considered as well.

- We're left to investigate the asymptotic behavior of the random function $\widehat{M}_n(\cdot)$.

- Let $\Delta_{\nu,n} := \frac{1}{\sqrt{T_n}} \sum_{i=1}^{[T_n]} \partial_\nu \rho(\epsilon_i; \nu_0)$ and $\Gamma_\nu := \frac{1}{4}\left(\psi_1\left(\frac{\nu_0}{2}\right) - \psi_1\left(\frac{\nu_0+1}{2}\right)\right)$.
- Finer (High-frequency) sampling for estimating (μ, σ) makes the plugging-in effect ignorable:

Theorem 3.1 (Student-t quasi-likelihood estimation of ν)

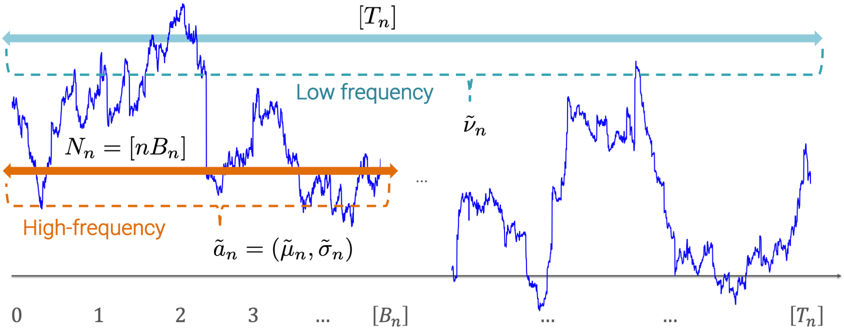
1. *Asymptotic normality:* $\hat{u}_{\nu,n} := \sqrt{T_n}(\hat{\nu}_n - \nu_0) = \Gamma_{\nu,0}^{-1}\Delta_{\nu,n} + o_p(1) \xrightarrow{\mathcal{L}} N\left(0, \Gamma_{\nu,0}^{-1}\right)$
2. *Tail-probability estimate:* $\forall L > 0, \quad \sup_{r>0} \sup_n P\left[|\hat{u}_{\nu,n}| > r\right] r^L < \infty$

- Question: **Joint** asymptotic distribution of $(\hat{u}_{a,n},\ \hat{u}_{\nu,n}) = \left(\sqrt{N_n}(\hat{a}_n - a_0),\ \sqrt{T_n}(\hat{\nu}_n - \nu_0)\right)$?

Joint asymptotic normality?

- We now assume $\frac{B_n}{T_n} \to 0$: we thin the data for the 1st-stage inference of $a = (\mu, \sigma)$, then

$$\left(\sqrt{N_n}(\hat{a}_n - a_0),\ \sqrt{T_n}(\hat{\nu}_n - \nu_0)\right) \xrightarrow{\mathcal{L}} N\left(0,\ \mathrm{diag}\{\Gamma_{a,0}^{-1},\ \Gamma_{\nu,0}^{-1}\}\right).$$



A modified setting: Student-Lévy driven Markovian SDE

- Discrete-time sample $(Y_{t_j})_{j=0}^{[nT_n]}$ for $t_j = jh = j/n$ ($h := 1/n$) and $T_n \to \infty$ from

$$dY_t = \mu \cdot b(Y_t)dt + \sigma dJ_t, \qquad Y_0 = 0.$$

  - $J = (J_t)$ is a Lévy process s.t. $J_1 \sim$ pdf $f(x;\nu) := \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}(1+x^2)^{-(\nu+1)/2}$.
  - $\sup_{y\in\mathbb{R}}(|b(y)| \vee |\partial_y b(y)|) < \infty$ and $\limsup_{|y|\to\infty} \frac{\mu_0 \cdot b(y)}{y} < 0$.
- $\theta := (\mu, \sigma, \nu) =: (a, \nu) \in \Theta$, bounded convex domain s.t. $\overline{\Theta} \subset \mathbb{R}^q \times (0,\infty) \times (1,\infty)$.

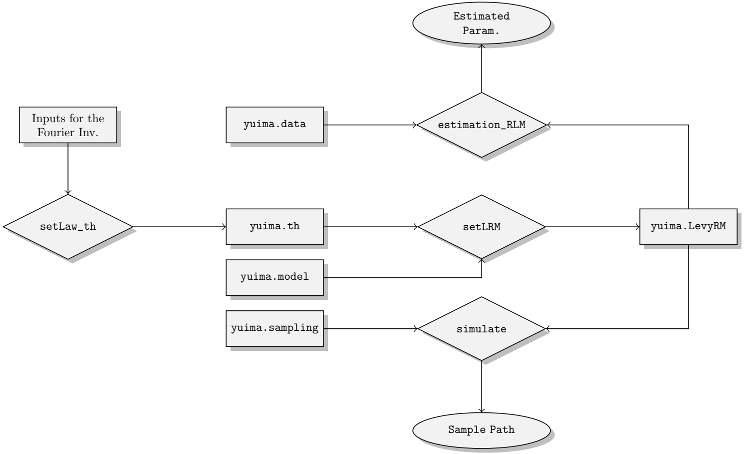
Theorem 3.2 (Joint asymptotic normality with orthogonality)

Under the above setup (including $B_n/T_n \to 0$), we have, with $\tilde{S}_n := N_n^{-1}\sum_{j=1}^{N_n} b_{j-1}^{\otimes 2}$,

$$\left(\mathrm{diag}\left(\frac{1}{2\tilde{\sigma}_n^2}\tilde{S}_n,\ \frac{1}{2\tilde{\sigma}_n^2}\right)^{1/2}\sqrt{N_n}(\tilde{a}_n - a_0),\ \left\{\frac{1}{4}\left(\psi_1\left(\frac{\tilde{\nu}_n}{2}\right) - \psi_1\left(\frac{\tilde{\nu}_n+1}{2}\right)\right)\right\}^{1/2}\sqrt{T_n}(\tilde{\nu}_n - \nu_0)\right) \xrightarrow{\mathcal{L}} N_{q+2}(0, I_{q+2}).$$

New classes/methods in yuima package [Brouste et al., 2014]

- New classes/methods in yuima package [Brouste et al., 2014] have been composed.



Case 1: Periodic deterministic regressors

- $Y_t = (\mu_1 \cos 5t + \mu_2 \sin t) + \sigma J_t$, $J_1 \sim t_3$; $(\mu_{1,0}, \mu_{2,0}, \sigma_0, \nu_0) = (5, -1, 3, 3)$
- $T_n = 50$, $B_n = 15$ $(T_n/N_n = 1/15)$, $h = 0.02$; $N_n = B_n/h = 750$; #MC $= 1000$

	Laguerre Method			COS Method			FFT Method		
h	0.01	0.005	1/365	0.01	0.005	1/365	0.01	0.005	1/365
$\hat{\mu}_1$	4.968	5.019	4.976	4.939	5.065	4.875	4.934	5.069	4.876
	(0.029)	(0.020)	(0.014)	(0.041)	(0.049)	(0.054)	(0.042)	(0.049)	(0.054)
$\hat{\mu}_2$	-1.065	-1.044	-0.914	-1.192	-1.195	-0.555	-1.179	-1.178	-0.528
	(0.146)	(0.104)	(0.074)	(0.210)	(0.247)	(0.277)	(0.213)	(0.248)	(0.277)
$\hat{\sigma}_0$	2.770	2.794	2.657	3.993	6.654	10.010	4.055	6.674	10.010
	(0.101)	(0.072)	(0.051)	(0.145)	(0.172)	(0.192)	(0.148)	(0.172)	(0.193)
$\hat{\nu}$	3.462	3.281	3.067	2.827	2.223	2.227	5.749	8.310	15.175
	(0.615)	(0.580)	(0.538)	(0.492)	(0.377)	(0.378)	(1.063)	(1.571)	(2.940)
sec.	2.07	2.02	2.07	1.10	1.24	1.34	0.15	0.24	0.39

Table: Estimated parameters for $h = 0.01, 0.005, 1/365$ and different integration methods. The number of points for the inversion of the characteristic function is $N = 180$. The parenthesis shows the asymptotic standard error. The last row reports the seconds necessary for the simulation of a sample path.

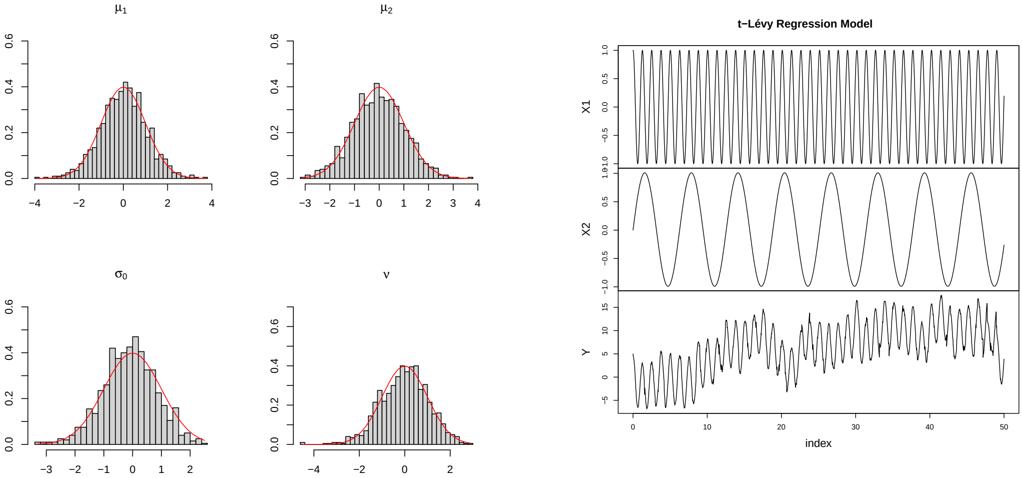


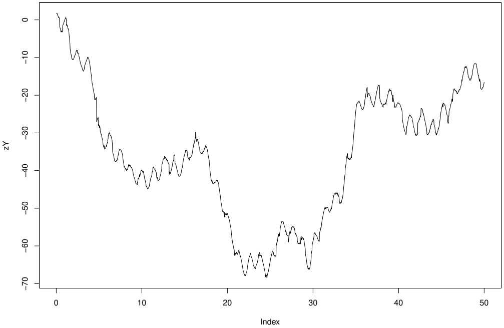
Figure: Studentized histograms ($\mu_1 = 5$, $\mu_2 = -1$, $\sigma_0 = 3$, $\nu = 2$) and path plots.

Case 2: Integrated Gaussian-OU regressor

$$Y_t = (\mu_1 \cos 5t + \mu_2 X_t) + \sigma J_t, \quad J_1 \sim t_3,$$

$$X_t = \int_0^t \int_0^s \exp\left(-\frac{1}{2}(s-u)\right) dw_u ds$$

- $T_n = 50, \quad B_n = 10, \quad h = 0.02,$
  $N_n = B_n/h = 500; \quad \#\mathrm{MC} = 1000$
    - $T_n/N_n = 1/10$
- $(\mu_{1,0}, \mu_{2,0}, \sigma_0, \nu_0) = (2, 5, 3, 3);$
  $\Theta_{\mu,\sigma} = [0.01, 10] \times [-10, 10] \times [0.1, 5]$



	$\hat{\mu}_{1,n}$	$\hat{\mu}_{2,n}$	$\hat{\sigma}_n$	$\hat{\eta}_n$
mean	2.00	5.00	2.67	2.73
sd	0.04	0.23	0.15	0.48

Summary and agenda

$$\{(X_{t_j}, Y_{t_j})\}_{j=0}^{[nT_n]} \text{ from } \begin{cases} Y_t = X_t \cdot \mu + \sigma J_t \\ J_1 \sim t_\nu \end{cases} \overset{\text{Cauchy QMLE}}{\Longrightarrow} \sqrt{N_n}\begin{pmatrix} \hat{\mu}_n - \mu_0 \\ \hat{\sigma}_n - \sigma_0 \end{pmatrix} \xrightarrow{\mathcal{L}} N_{q+1}(0, \Gamma_a^{-1})$$

$$\overset{\text{Student-}t\text{ QMLE}}{\Longrightarrow} \sqrt{T_n}(\hat{\nu}_n - \nu_0) \xrightarrow{\mathcal{L}} N_1(0, \Gamma_\nu^{-1})$$

- $N_n := nB_n$ with $B_n \le T_n$, $B_n \to \infty$, $\exists\epsilon' \in (0,1)$, $B_n \lesssim n^{1-\epsilon'}$, and $T_n \to \infty$, with $T_n/N_n \to 0$.

1. We do **not** know the asymptotic phenomenon in estimating ν from full data $\{(X_{t_j}, Y_{t_j})\}_{j=0}^{[nT_n]}$.
2. Implemented in the `yuima` package in CRAN, managing the **numerical** issues.

More general and systematic noise inference scheme through the β-**stable quasi-likelihood**:

$$dX_t = a(X_t, \alpha)dt + c(X_{t-}, \gamma)dJ_t^\beta, \qquad h^{-1/\beta} J_h^\beta \xrightarrow{\mathcal{L}} S_\beta \quad (h \to 0)$$

$$(X_{t_0}, X_{t_1}, \ldots, X_{t_{[nT_n]}}): \ t_j = jh; \ \ h = 1/n \to 0; \ \ T_n \to \infty.$$

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