Estimation of pure-jump stable Cox-Ingersoll-Ross processes from high-frequency observations

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Plan

- Parametric estimation of pure-jump stable CIR process : overview
- Moments estimates
- Joint estimation
- Implementation
- Python simulations

Pure-jump stable CIR process

$$dX_t = (a - bX_t)dt + \delta X_{t^-}^{1/\alpha} dL_t^\alpha, \quad t \geq 0 \quad x_0 \geq 0$$

(L_t^α) non-symmetric pure-jump Lévy process, strictly α-stable, with triplet $(0, 0, F^\alpha)$ and representation

$$L_t^\alpha = \int_0^t \int z \tilde{\mu}(ds, dz),$$

where $\tilde{\mu} = \mu - \overline{\mu}$ is a compensated Poisson random measure with compensator

$$\overline{\mu}(dt, dz) = dt F^\alpha(dz)$$

where the Lévy measure is given by

$$F^\alpha(dz) = \frac{c_\alpha}{z^{1+\alpha}} 1_{\mathbb{R}_+^*}(z) dz, \quad 1 < \alpha < 2.$$

Self similarity : $L_t \overset{\mathcal{L}}{=} t^{1/\alpha} L_1$.

The characteristic function is

$$\mathbb{E}(e^{izL_1^\alpha}) = \exp\left(-|z|^\alpha (1 - i \tan \frac{\pi\alpha}{2} \mathrm{sgn}(z))\right).$$

$$dX_t = (a - bX_t)dt + \delta X_{t^-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0 \quad x_0 \geq 0$$

- Existence and uniqueness of a strong solution such that

$$\mathbb{P}(X_t \geq 0, \ \forall t \geq 0) = 1$$

if $a \geq 0, \ b \in \mathbb{R}, \ \delta \geq 0.$

- The solution satisfies

$$\mathbb{P}(X_t > 0, \ \forall t \geq 0) = 1$$

if $a > 0, \ b \in \mathbb{R}, \ \delta > 0.$

Kawazu-Watanabe (71), Fu-Li (10), Jiao-Ma-Scotti (18)

Parametric estimation : overview

- Small noise and pure-jump stable process

$$dX_t = (a - bX_t)dt + \varepsilon \delta X_{t^-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

$\alpha \in (1, 2)$ known

Estimation of (a, b, δ)

Observations : $(X_{i\Delta_n})_{i \in \{1,\dots,n\}}$, $\Delta_n \to 0$, $n\Delta_n = T$ fixed, $(T = 1)$

Maximising an approximation of the likelihood function (depends on α)
Consistency and asymptotic normality of estimators as $n \to \infty$ and $\varepsilon \to 0$ (with $\liminf \varepsilon n^{1-1/\alpha} > 0$)

Ma-Yang (14), Yang (17)

- Fixed step-size observations, long-time behavior

$$dX_t = (a - bX_t)dt + \delta X_{t^-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

Estimation of (a, b), assuming $b > 0$

Observations : $(X_{i\Delta})_{i \in \{1,\ldots,n\}}$, Δ fixed and $n \to \infty$

(X_t) is geometrically ergodic. Explicit least squares estimators of (a, b) (independent of α and δ) based on the equation satisfied by $e^{bt} X_t$

Consistency if $\alpha \in (1, 2)$ and rate of convergence $n^{(\alpha-1)/\alpha^2}$ if $\alpha \in (1, \frac{1+\sqrt{5}}{2})$

Li-Ma (15)

Moment estimates

We consider a pure-jump stable CIR process

$$dX_t = (a - bX_t)dt + \delta X_{t-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

with $a > 0$, $b \in \mathbb{R}$, $\delta > 0$, $\alpha \in (1, 2)$

Proposition (B. Clément (23))

We have

1. $\forall p \in (0, \alpha)$

$$\mathbb{E}_{|\mathcal{F}_s} \left(\sup_{s \leq u \leq t} X_u^p \right) \leq C_{p,t-s}(1 + X_s^p)$$

2. $\forall p > 0$

$$\sup_{t \in [0,1]} \mathbb{E} \left(\frac{1}{X_t^p} \right) < +\infty$$

2. Proof based on the expression of the Laplace transform of a CBI process.

Approximation results

Proposition (B. Clément (23))

1. $\forall p \in (0,\alpha)$

$$\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}}\left(\sup_{t\in[\frac{i-1}{n},\frac{i}{n}]}\left|\int_{\frac{i-1}{n}}^{t} X_{s-}^{1/\alpha}\,dL_s^{\alpha}\right|^{p}\right) \leq \frac{C_p}{n^{p/\alpha}}\left(1+X_{\frac{i-1}{n}}^{p/\alpha}\right)$$

2. $\forall p \in (0,\alpha)$

$$\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}}\left(\sup_{t\in[\frac{i-1}{n},\frac{i}{n}]}\left|X_t - X_{\frac{i-1}{n}}\right|^{p}\right) \leq \frac{C_p}{n^{p/\alpha}}\left(1+X_{\frac{i-1}{n}}^{p}\right)$$

3. $\forall p \in (0,\alpha)$

$$\mathbb{E}_{|\mathcal{F}_{\frac{i-1}{n}}}\left(\left|\int_{\frac{i-1}{n}}^{\frac{i}{n}}\left(X_{s-}^{1/\alpha} - X_{\frac{i-1}{n}}^{1/\alpha}\right)dL_s^{\alpha}\right|^{p}\right) \leq \frac{C_p}{n^{2p/\alpha}}\left(1+X_{\frac{i-1}{n}}^{p}+\frac{1}{X_{\frac{i-1}{n}}^{p}}\right)$$

Results of efficiency in high-frequency setting on a toy model

- *Masuda (06), Brouste-Masuda (18)* LAN property for (a, δ, α)

$$X_t = x_0 + at + \delta S_t^\alpha$$

S_t^α : symmetric α-stable process

- Diagonal rate in estimating $\theta = (a, \delta, \alpha)$

$$u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha - 1/2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{n}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{n}\log(n)} \end{pmatrix}$$

LAN property with singular information matrix (*Masuda (06)*)

$$I(a, \delta, \alpha) = \begin{pmatrix} \mathbb{E}h_\alpha^2(S_1^\alpha)/\delta^2 & 0 & 0 \\ 0 & \mathbb{E}k_\alpha^2(S_1^\alpha)/\delta^2 & \mathbb{E}k_\alpha^2(S_1^\alpha)/\delta\alpha^2 \\ 0 & \mathbb{E}k_\alpha^2(S_1^\alpha)/\delta\alpha^2 & \mathbb{E}k_\alpha^2(S_1^\alpha)/\alpha^4 \end{pmatrix}$$

with $h_\alpha = \varphi'_\alpha / \varphi_\alpha$ and $k_\alpha(z) = 1 + zh_\alpha(z)$.

(extension to L_t^α : $I_{2,1} = \mathbb{E}h_\alpha k_\alpha(L_1^\alpha)/\delta^2$, $I_{3,1} = \mathbb{E}h_\alpha k_\alpha(L_1^\alpha)/\delta\alpha^2$)

Brouste-Masuda (18)

- Non diagonal rate in estimating $\theta = (a, \delta, \alpha)$

$$u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha - 1/2}} & 0 \\ 0 & \frac{1}{\sqrt{n}} v_n \end{pmatrix}, \quad v_n = \begin{pmatrix} v_n^{1,1} & v_n^{1,2} \\ v_n^{2,1} & v_n^{2,2} \end{pmatrix}$$

Assuming

$$\begin{pmatrix} \frac{1}{\delta} & \frac{\log n}{\alpha^2} \\ 0 & 1 \end{pmatrix} \times v_n \to \overline{v}$$

where $\overline{v}$ is non singular.
Then the LAN property holds with non-singular information (extension to L_t^α, the information is not bloc diagonal)

$$\overline{I}(a, \delta, \alpha) = \begin{pmatrix} \frac{1}{\delta^2} \mathbb{E} h_\alpha^2(S_1^\alpha) & 0 \\ 0 & \overline{v}^T \begin{pmatrix} \mathbb{E} k_\alpha^2(S_1^\alpha) & -\mathbb{E}(k_\alpha f_\alpha)(S_1^\alpha) \\ -\mathbb{E}(k_\alpha f_\alpha)(S_1^\alpha) & \mathbb{E} f_\alpha^2(S_1^\alpha) \end{pmatrix} \overline{v} \end{pmatrix}$$

with $f_\alpha = \partial_\alpha \varphi_\alpha / \varphi_\alpha$.
We lose $\log(n)$ in estimating simultaneously (δ, α) :
Rate for δ : $\sqrt{n}/\log(n)$, Rate for α : $\sqrt{n}$

Joint estimation of $\theta = (a, b, \delta, \alpha)$ for the pure-jump stable CIR process

- Observations : $(X_{\frac{i}{n}})_{i\in\{0,\dots,n\}}$ where $(X_t)_{t\in[0,1]}$ is a pure-jump stable CIR process for the parameter value $\theta_0 \in (0,\infty)\times\mathbb{R}\times(0,\infty)\times(1,2) = \Theta$

- Estimating functions method based on an approximation of the conditional distribution of $X_{\frac{i}{n}}$ given $X_{\frac{i-1}{n}}$

$$X_{\frac{i}{n}} \approx X_{\frac{i-1}{n}} + \frac{a_0}{n} - \frac{b_0}{n} X_{\frac{i-1}{n}} + \delta_0 X_{\frac{i-1}{n}}^{1/\alpha_0} (L^{\alpha_0}_{\frac{i}{n}} - L^{\alpha_0}_{\frac{i-1}{n}}).$$

$\hat{\theta}_n$ solution of $G_n(\theta) = -\nabla_\theta L_n(\theta) = 0$ on Θ

$$L_n(\theta) = \sum_{i=1}^{n} \log\left[\frac{n^{1/\alpha}}{\delta X_{\frac{i-1}{n}}^{1/\alpha}} \varphi_\alpha\left(n^{1/\alpha}\frac{X_{\frac{i}{n}} - X_{\frac{i-1}{n}} - \frac{a}{n} + \frac{b}{n} X_{\frac{i-1}{n}}}{\delta X_{\frac{i-1}{n}}^{1/\alpha}}\right)\right]$$

where φ_α is the density of L_1^α.

Clément-Gloter (19) (20) : SDE with Lipschitz coefficients, driven by symmetric locally stable process

Existence of the estimator

Theorem (B. Clément (23))

Let u_n be the previous non-diagonal rate (*Brouste-Masuda (18)*).
There exists a sequence $(\hat{\theta}_n)$, such that $\lim_n \mathbb{P}(G_n(\hat{\theta}_n) = 0) = 1$, that converges in probability to θ_0 .

Moreover we have the stable convergence in law with respect to $\sigma(L_s^{\alpha_0}, s \leq 1)$

$$u_n^{-1}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{\mathcal{L}_s} I(\theta_0)^{-1/2}\mathcal{N},$$

where $\mathcal{N}$ is a standard Gaussian variable independent of $I(\theta_0)$, where $I(\theta_0)$ is a symmetric non-negative and non-singular matrix.

- If α_0 is known (or δ_0 known), we obtain a similar result with the diagonal rate

$$u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha - 1/2}} & 0 \\ 0 & \frac{1}{\sqrt{n}} \end{pmatrix}, \quad u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha - 1/2}} & 0 \\ 0 & \frac{1}{\log n\sqrt{n}} \end{pmatrix}$$

and non-singular information.

The proof is based on the following convergences :

1. For $\eta > 0$, we set $W_n^{(\eta)} = \left\{ (\delta, \alpha);\ \left\| w_n^{-1} \begin{pmatrix} \delta - \delta_0 \\ \alpha - \alpha_0 \end{pmatrix} \right\| \leq \eta \right\}$ with $\forall q > 0,\ \ln(n)^q w_n \to 0$.

   For smooth functions f and h_α and $K \subset \mathbb{R}_+^* \times \mathbb{R}$ a compact set, we have $\forall q > 0$

$$\sup_{A \times W_n^{(\eta)}} \ln(n)^q \left| \frac{1}{n} \sum_{i=1}^{n} f(X_{\frac{i-1}{n}}, \delta, \alpha) h_\alpha((z_n^i(\theta)) - \int_0^1 f(X_t, \delta_0, \alpha_0) dt \mathbb{E} h_{\alpha_0}(L_1^{\alpha_0}) \right| \xrightarrow{\mathbb{P}} 0$$

where
$$z_n^i(\theta) = n^{1/\alpha} \frac{X_{\frac{i}{n}} - X_{\frac{i-1}{n}} - \frac{a}{n} + \frac{b}{n} X_{\frac{i-1}{n}}}{\delta X_{\frac{i-1}{n}}^{1/\alpha}} \approx L_1^{\alpha_0}.$$

2. For $F = (f_{i,j})$ and $H = (h_i)$, we have the stable convergence in law

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} F(X_{\frac{i-1}{n}}) H(z_n^i(\theta_0)) \to \Sigma^{1/2} \mathcal{N}$$

   with $\mathcal{N}$ standard gaussian independent of Σ

$$\Sigma_{i,j} = \int_0^1 F(X_t) \tilde{\Sigma} F(X_t)^T ds, \quad \tilde{\Sigma}_{i,j} = \mathbb{E}(h_i h_j)(L_1^{\alpha_0}).$$

Sketch of proof

We define $J_n(\theta) = \nabla_\theta G_n(\theta)$ (G_n is an approximation of the score function).

We can prove uniform local convergence in probability and stable convergence in law

1. For all compact $A \subset (0, +\infty) \times (1, 2)$

$$\sup_{(a,b)\in A,\ (\delta,\alpha)\in W_n^{(\eta)}} ||u_n^T J_n(\theta) u_n - I(\theta_0)|| \to 0$$

2. $u_n^T G_n(\theta_0)$ stably converges in law to $I(\theta_0)^{1/2}\mathcal{N}$

   where $I(\theta_0)$ is a symmetric non-negative and non-singular matrix depending on $(X_t)_{t\in[0,1]}$ and $\mathcal{N}$ is a standard Gaussian variable independent of $I(\theta_0)$.

Information for the estimation of (a, δ, α)

$$I(a,\delta,\alpha) = \begin{pmatrix} \frac{1}{\delta^2}\mathbb{E}h_\alpha^2(L_1^\alpha)\int_0^1 \frac{ds}{X_s^{1/\alpha}} & Sym. \\ \overline{v}^T I^{2,1} & \overline{v}^T I^{2,2}\overline{v} \end{pmatrix}$$

$$I^{2,1} = \begin{pmatrix} \frac{1}{\delta}\mathbb{E}(h_\alpha k_\alpha)(L_1^\alpha)\int_0^1 \frac{ds}{X_s^{1/\alpha}} \\ -\frac{1}{\delta\alpha^2}\mathbb{E}(h_\alpha k_\alpha)(L_1^\alpha)\int_0^1 \frac{\log(X_s)}{X_s^{1/\alpha}}ds - \frac{1}{\delta}\mathbb{E}(f_\alpha h_\alpha)(L_1^\alpha)\int_0^1 \frac{ds}{X_s^{1/\alpha}} \end{pmatrix}$$

$$I^{2,2} = \begin{pmatrix} \mathbb{E}k_\alpha^2(L_1^\alpha) & -\frac{1}{\alpha^2}\mathbb{E}k_\alpha^2(L_1^\alpha)\int_0^1 \log(X_s)ds - \mathbb{E}(k_\alpha f_\alpha)(L_1^\alpha) \\ Sym. & \mathbb{E}f_\alpha^2(L_1^\alpha) + \frac{1}{\alpha^4}\mathbb{E}k_\alpha^2(L_1^\alpha)\int_0^1 \log(X_s)^2 ds + \frac{2}{\alpha^2}\mathbb{E}(f_\alpha k_\alpha)(L_1^\alpha)\int_0^1 \log(X_s)ds \end{pmatrix}$$

$$(x\ z\ t)I\begin{pmatrix} x \\ z \\ t \end{pmatrix} = \int_0^1 \int ds\phi(u)du\Big(\frac{1}{\delta}\frac{1}{X_s^{1/\alpha}}h_\alpha(u)x + k_\alpha(u)z - (f_\alpha(u) + \frac{\ln(X_s)}{\alpha}k_\alpha(u))t\Big)^2$$

Implementation

- The previous result states existence of a consistent estimator with optimal rate of convergence but there might be other sequences that solve the estimating equation that are not consistent.

- Uniqueness of the drift estimator is obtained if δ_0 and α_0 are known or if we have preliminary estimators $\tilde{\delta}_n$ and $\tilde{\alpha}_n$.

- We will give preliminary estimators $\tilde{\delta}_n$ and $\tilde{\alpha}_n$.

- Using these estimators, we obtain a preliminary estimator $(\tilde{a}_n, \tilde{b}_n, \tilde{\delta}_n, \tilde{\alpha}_n)$ that we will correct using a one-step procedure.

Drift estimator

Theorem (B. Clément (23))

Let $G_{n,d}$ be the approximation of the score function restricted to the drift parameters (a, b)

$$G_{n,d}(a, b) = -\nabla_{(a,b)} L_n(a, b, \tilde{\delta}_n, \tilde{\alpha}_n).$$

We assume that $(a_0, b_0) \in \mathrm{Int}(\Theta)$ for a compact set $\Theta \subset (0, +\infty) \times \mathbb{R}$ and that $\frac{\sqrt{n}}{\log(n)}(\tilde{\delta}_n - \delta_0)$ and $\frac{\sqrt{n}}{\log(n)}(\tilde{\alpha}_n - \alpha_0)$ are tight.
Then any sequence $(\tilde{a}_n, \tilde{b}_n)$ that solves $G_{n,d}(\tilde{a}_n, \tilde{b}_n) = 0$ converges in probability to (a_0, b_0) and this sequence is unique.

Moreover the sequence $\frac{n^{1/\alpha_0}}{\sqrt{n}\log(n)^2}(\tilde{a}_n - a_0, \tilde{b}_n - b_0)$ is tight.

Estimators $\tilde{\delta}_n$ and $\tilde{\alpha}_n$

Preliminary estimators $\tilde{\alpha}_n$

Non parametric methods for semimartingales based on p-order power variation (*Todorov-Tauchen (11), Todorov (13), Todorov (15)*)
The first and two order power variation are defined by

$$V_n^1(p, X) = \sum_{i=2}^{n} |\Delta_i^n X - \Delta_{i-1}^n X|^p \quad \text{where } \Delta_i^n X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}},$$

$$V_n^2(p, X) = \sum_{i=4}^{n} |\Delta_i^n X - \Delta_{i-1}^n X + \Delta_{i-2}^n X - \Delta_{i-3}^n X|^p.$$

Limit theorems (*Todorov (13)*)

1. For $p \in (0, \alpha)$, we have the convergences in probability

$$\frac{n^{p/\alpha}}{n} V_n^1(p, X) \to \int_0^1 \delta^p X_{t-}^{p/\alpha} dt \mathbb{E}|L_1 - \overline{L}_1|^p,$$

$$\frac{n^{p/\alpha}}{n} V_n^2(p, X) \to 2^{p/\alpha} \int_0^1 \delta^p X_{t-}^{p/\alpha} dt \mathbb{E}|L_1 - \overline{L}_1|^p$$

where $\overline{L}_1$ is an independent copy of L_1.

2. For $p \in (\frac{\alpha-1}{2}, \frac{\alpha}{2})$

$$\sqrt{n} \begin{pmatrix} \frac{n^{p/\alpha}}{n} V_n^1(p, X) - \int_0^1 \delta^p X_{t-}^{p/\alpha} dt \mathbb{E}|L_1 - \overline{L}_1|^p \\ \frac{n^{p/\alpha}}{n} V_n^2(p, X) - 2^{p/\alpha} \int_0^1 \delta^p X_{t-}^{p/\alpha} dt \mathbb{E}|L_1 - \overline{L}_1|^p \end{pmatrix}$$

stably converges in law.

3. Considering $p = 1/2$

$$\tilde{\alpha}_n = \frac{\log 2}{2 \log(V_n^2(1/2, X)/V_n^1(1/2, X))} 1_{V_n^1(1/2,X) \neq V_n^2(1/2,X)}$$

$\sqrt{n}(\tilde{\alpha}_n - \alpha)$ stably converges in law.

Preliminary estimators $\tilde{\delta}_n$

We define a non parametric estimator of δ based on p-order power variation

$$\tilde{\delta}_n = \left(\frac{1}{m(\tilde{\alpha}_n)} \frac{n^{1/2\tilde{\alpha}_n}}{n} \sum_{i=2}^{n} \left| \frac{\Delta_i^n X - \Delta_{i-1}^n X}{X_{\frac{i-1}{n}}^{1/\tilde{\alpha}_n}} \right|^{1/2} \right)^2$$

where $m(\alpha) = \mathbb{E}|L_1^\alpha - \overline{L}_1^\alpha|^{1/2} = 2^{p/\alpha}\mathbb{E}|S_1^\alpha|^{1/2}$ where S_1^α is symmetric α-stable.

Theorem (B. Clément (23))

Then $\tilde{\delta}_n$ converges in probability to δ_0 and $\frac{\sqrt{n}}{\log(n)}(\tilde{\delta}_n - \delta_0)$ is tight.

One-step improvement

- There exists a joint estimator $\hat{\theta}_n = (\hat{a}_n, \hat{b}_n, \hat{\delta}_n, \hat{\alpha}_n)$ consistent, rate optimal and probably efficient that solves $G_n(\hat{\theta}_n) = 0$.
- We have explicit (easy to implement) preliminary estimators $\tilde{\delta}_n$ and $\tilde{\alpha}_n$. By plugging, we obtain consistent estimators of the drift $\tilde{a}_n$ and $\tilde{b}_n$ (not rate optimal).
  Finally we have a first step estimator $\tilde{\theta}_n = (\tilde{a}_n, \tilde{b}_n, \tilde{\delta}_n, \tilde{\alpha}_n)$.
- One-step improvement : define

$$\hat{\theta}'_n = \tilde{\theta}_n - J_n(\tilde{\theta}_n)^{-1} G_n(\tilde{\theta}_n)$$

We prove that $\hat{\theta}'_n$ inherits the asymptotic properties of $\hat{\theta}_n$ ie

$$u_n^{-1}\left(\hat{\theta}'_n - \theta_0\right) \xrightarrow{\mathcal{L}_s} I(\theta_0)^{-1/2}\mathcal{N} \text{ for } u_n = \begin{pmatrix} \frac{1}{n^{1/\alpha - 1/2}} Id_2 & 0 \\ 0 & \begin{pmatrix} \frac{\delta}{\sqrt{n}} & \frac{-\log n}{\alpha^2\sqrt{n}} \\ 0 & \frac{1}{\sqrt{n}} \end{pmatrix} \end{pmatrix}.$$

Simulating (X_t) solution of $dX_t = (a - bX_t)dt + \delta X_{t^-}^{1/\alpha} dL_t^\alpha$

We use the an implicit discretisation scheme which preserves the positivity proposed by *Li-Taguchi* (19) . For $h(x) = \min\{|x|^{1/\alpha_0}, H\}$, we set $X_0^{H,n} = x_0$ and for $i \geq 0$

$$X_{\frac{i+1}{n}}^{H,n} = \frac{|X_{\frac{i}{n}}^{H,n} + \frac{a}{n} + \delta h(X_{\frac{i}{n}}^{H,n})\Delta_i^n L^\alpha|}{(1 + \frac{b}{n})}.$$

We know that $\Delta_i^n L^\alpha = L_{\frac{i}{n}}^\alpha - L_{\frac{i-1}{n}}^\alpha \overset{\mathcal{L}}{=} L_{1/n}^\alpha \overset{\mathcal{L}}{=} n^{-1/\alpha} L_1^\alpha$.

We simulate L_1^α according to *Weron-Weron* (05) with V a uniform random variable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and an independent exponential variable W with mean 1.

$$L_1^\alpha = S_\alpha \times \frac{\sin(\alpha(V + B_\alpha))}{(\cos(V))^{1/\alpha}} \times \left(\frac{\cos(V - \alpha(V + B_\alpha))}{W}\right)^{(1-\alpha)/\alpha},$$

where

$$B_\alpha = \frac{\arctan(\tan\frac{\alpha\pi}{2})}{\alpha}, \quad S_\alpha = \left(1 + \tan^2\frac{\alpha\pi}{2}\right)^{1/(2\alpha)}.$$

Estimating (a,b) for δ_0 and α_0 known

Theoretical limit :

$$u_n^{-1}(\hat{\theta}_n - \theta_0) \xrightarrow[n\to\infty]{\mathcal{L}-s} \Sigma^{1/2}\mathcal{N}$$

where $\mathcal{N}$ is a standard Gaussian variable independent of $\Sigma = I^{11}(\theta_0)^{-1}$.

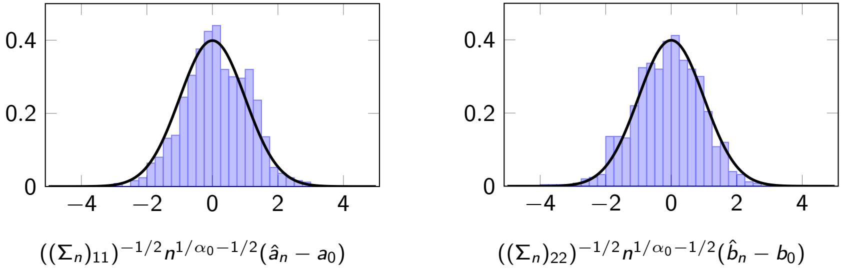


Figure - Distribution of the rescaled errors of estimation and comparison with $\mathcal{N}(0,1)$ ($a_0 = 3$, $b_0 = 5$, $\delta_0 = 1$, $\alpha_0 = 1.3$, n=4096, $n_{dis} = (1000n)^{-1}$, $n_{MC} = 1000$)

One-step improvement of a, δ, α

We correct our preliminary estimator $\tilde{\theta}_n = (\tilde{a}_n, \tilde{\delta}_n, \tilde{\alpha}_n)$ which is consistent but not rate optimal to the one-step estimator $\hat{\theta}'_n = (\hat{a}_{1,n}, \hat{\delta}_{1,n}, \hat{\alpha}_{1,n})$.

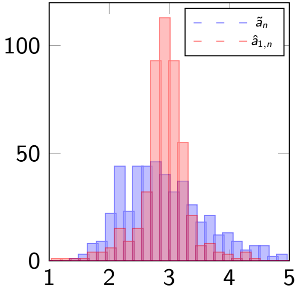


Figure – Estimation of a

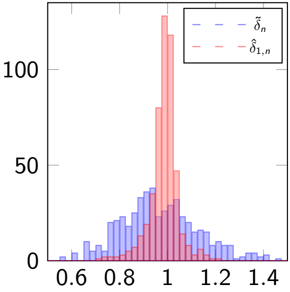


Figure – Estimation of δ

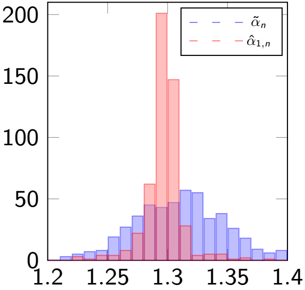


Figure – Estimation of α

Estimating (a,b) for δ_0 and α_0 known

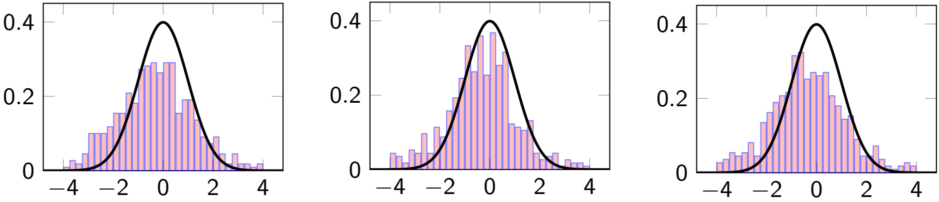
Theoretical limit :

$$((\overline{\Sigma}_n)_{11})^{-1/2} n^{1/\alpha_0 - 1/2}(\hat{a}_{1,n} - a_0) \xrightarrow[n\to\infty]{\mathcal{L}-s} \mathcal{N}$$

$$((\overline{\Sigma}_n)_{33})^{-1/2} \frac{\sqrt{n}}{\ln(n)} \frac{\alpha_0^2}{\delta_0}(\hat{\delta}_{1,n} - \delta_0) \xrightarrow[n\to\infty]{\mathcal{L}-s} \mathcal{N}$$

$$((\overline{\Sigma}_n)_{33})^{-1/2} \sqrt{n}(\hat{\alpha}_{1,n} - \alpha_0) \xrightarrow[n\to\infty]{\mathcal{L}-s} \mathcal{N}.$$

where $\mathcal{N}$ is a standard Gaussian variable independent of $\overline{\Sigma} = \overline{I}(\theta_0)^{-1}$.



$((\overline{\Sigma}_n)_{11})^{-\frac{1}{2}} n^{\frac{1}{\alpha_0} - \frac{1}{2}}(\hat{a}_{1,n} - a_0)$ $((\overline{\Sigma}_n)_{33})^{-\frac{1}{2}} \frac{\alpha_0^2}{\delta_0} \frac{\sqrt{n}}{\log(n)}(\hat{\delta}_{1,n} - \delta_0)$ $((\overline{\Sigma}_n)_{33})^{-\frac{1}{2}} \sqrt{n}(\hat{\alpha}_n - \alpha_0)$

Figure – Distribution of the rescaled errors of estimation and comparison with $\mathcal{N}(0,1)$ ($a_0 = 3$, $b_0 = 5$, $\delta_0 = 1$, $\alpha_0 = 1.3$, n=5000, $n_{dis} = (1000n)^{-1}$, $n_{MC} = 1000$)

Futur work : Stable CIR process

We will try to estimate all parameters in presence of a Brownian motion.

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t + \delta X_{t-}^{1/\alpha} dL_t^\alpha, \quad t \geq 0 \quad x_0 \geq 0$$

- (B_t) standard Brownian motion independent of (L_t^α)
- Existence and uniqueness of a strong solution such that

$$\mathbb{P}(X_t > 0, \; \forall t \geq 0) = 1$$

if $x_0 > 0, \; a \geq \frac{\sigma^2}{2} > 0, \; b \in \mathbb{R}, \; \delta \geq 0$

Kawazu-Watanabe (71), Fu-Li (10), Jiao-Ma-Scotti (18)

- Continuous time observations with $\sigma > 0$

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t + \delta X_{t^-}^{1/\alpha} dL_t^{\alpha}, \quad t \geq 0, \quad x_0 > 0$$

$a \geq 0$, $\sigma > 0$, $\delta > 0$ and $\alpha \in (1, 2)$ are known

Estimation of b

Observations : $(X_t)_{t\in[0,T]}$, $T \to \infty$

Explicit expression of the maximum likelihood estimator $\hat{b}$ from Girsanov's Theorem (depends on $a \geq 0$, $\delta > 0$ and α)

- $b > 0$: consistency and asymptotic normality with rate $\sqrt{T}$
- $b = 0$: consistency
- $b < 0$: consistency and asymptotic mixed normality with rate e^{-bT}

Barczy-Ben Alaya-Kebaier-Pap (19)

Thank you for your attention