A GMM Approach to Estimate the Roughness of Stochastic Volatility

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Joint work with Anine Bolko, Kim Christensen and Bezirgen Veliyev (Aarhus)

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- We derive asymptotic theory for the GMM estimator and assess its finite-sample properties in a simulation experiment.

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- Whilst the approach is fully generic, we focus on models that lend themselves to rough volatility.
- We take seriously the intrinsic sources of bias — integration effect and measurement error.
- We derive asymptotic theory for the GMM estimator and assess its finite-sample properties in a simulation experiment.
- Applying the method to equity index data, we study whether spot volatility is best described by a rough process.

Measuring the roughness of volatility

Generalised method of moments approach

Epilogue

Measuring roughness via scaling

Scaling of fractional Brownian motion

The q-th order variogram of fractional Brownian motion (fBm), for $q > 0$, scales as follows:

$$v(q, \Delta) := \mathbb{E}[|W^H_{t+\Delta} - W^H_t|^q] \propto \Delta^{qH}, \quad \Delta > 0$$

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Thus,

$$\frac{\log v(q, \Delta)}{q} = \text{const} + H \log \Delta.$$

Measuring roughness via scaling

Estimation of H

For daily realised variance RV_k on days $k = 0, 1, \ldots, N$, compute the empirical variogram

$$m(q, \Delta) = \frac{1}{N} \sum_{k=1}^{N} |\log(RV_{k\Delta}) - \log(RV_{(k-1)\Delta})|^q, \quad \Delta = 1, \ldots, \left\lfloor \frac{N}{\Delta} \right\rfloor.$$

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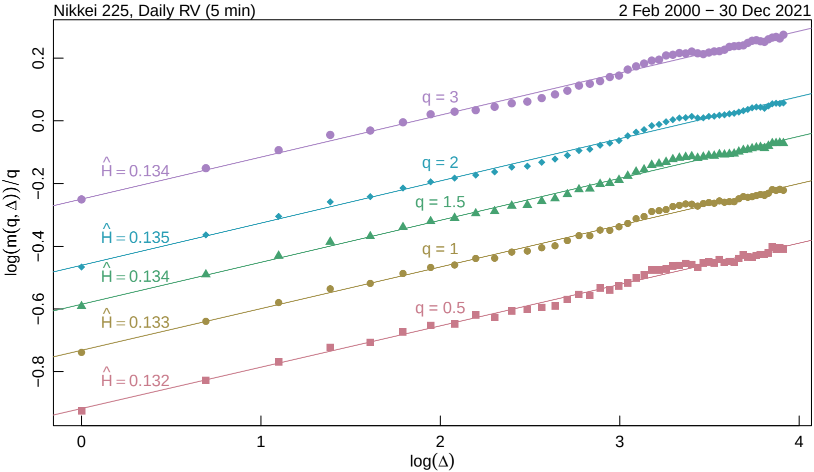
To estimate H, Gatheral, Jaisson and Rosenbaum (2018) suggest we...

regress $\frac{\log m(q,\Delta)}{q}$ on $\log \Delta$.

Roughness of Nikkei 225 volatility



Continuous-time model for asset prices

General semimartingale model

$$\frac{\mathrm{d}S_t}{S_t} = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t + J_t, \quad t \geq 0$$

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Jump part [omitted in what follows]

- Jump process (e.g., Lévy process) $(J_t)_{t\geq 0}$

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Roughness measured via proxy?

Is it okay to think that $RV_k^n \approx \sigma_k^2$ to infer that σ is rough too?

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- Gatheral, Jaisson and Rosenbaum (2018) compute an estimate of the bias.

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- Noise looks like roughness, "illusive scaling" (Fukasawa, Takabatake and Westphal, 2022).
- Mitigation via non-linear least squares (Bennedsen, Lunde and Pakkanen, 2022) or quasi-likelihood estimation (Fukasawa, Takabatake and Westphal, 2022).

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Generalised method of moments approach

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Log-normal spot volatility model

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3. Assume that Θ is compact with true θ_0 in its interior.

Covariance structure of integrated variance

Proposition

For any $\theta \in \Theta$, $k \in \mathbb{Z}$ *and* $\ell = 0, 1, \ldots,$

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If additionally $\lim_{u\to\infty} \kappa(u) = 0$ *and some regularity conditions on* κ *hold, then:*

3. $\mathrm{Cov}_\theta[IV_k, IV_{k+\ell}] \sim \mathrm{const}_{\xi,\kappa}\kappa(\ell)$ *as* $\ell \to \infty$

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3. $\mathbb{E}_\theta[\varepsilon_k | \varepsilon_{k-1}, \varepsilon_{k-2}, \ldots, Y] = 0$ for any $k \in \mathbb{Z}$ and any $\theta \in \Theta$.

Stylised measurement error

Example (CLT approximation)

The central limit theorem (CLT) for realised variance (e.g., Fukasawa, 2010) says:

$$\sqrt{n} e_k^n = \sqrt{n}(RV_k^n - IV_k) \xrightarrow[n \to \infty]{d} \left(2 \int_{k-1}^{k} \sigma_t^4 \mathrm{d}t \right)^{1/2} X_k$$

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$$\varepsilon_k := \left(\frac{2}{n}\int_{k-1}^{k} \sigma_t^4 \mathrm{d}t\right)^{1/2} X_k, \quad \text{for which} \quad c(\theta) = \frac{2\xi^2}{n} e^{\kappa(0)}.$$

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Moment selection

For a finite subset $\mathcal{L} \subset \{1, 2, \ldots\}$ of lags,

$$\widehat{\mathbb{IV}}_k := \left(\widehat{IV}_k, \widehat{IV}_k^2, \widehat{IV}_k \widehat{IV}_{k-\ell} : \ell \in \mathcal{L}\right), \quad k \in \mathbb{Z}$$

$$G_c(\theta) := (g^1(\theta), g_0^2(\theta) + c(\theta), g_\ell^2(\theta) : \ell \in \mathcal{L}), \quad \theta \in \Theta$$

Generalised method of moments (GMM)

Estimating function

Define

$$\widehat{m}_N(\theta) := \frac{1}{N}\sum_{k=1}^{N} \widehat{\mathbb{IV}}_k - G_c(\theta), \quad \theta \in \Theta,$$

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GMM estimator

For a data-dependent weight matrix $W \in \mathbb{R}^{(2+|\mathcal{L}|)\times(2+|\mathcal{L}|)}$,

$$\widehat{\theta}_N := \arg\min_{\theta\in\Theta} \widehat{m}_N(\theta)' W \widehat{m}_N(\theta)$$

The estimator is consistent and asymptotically normal under suitable conditions.

Specifying Y

(Rough) fractional stochastic volatility model (FSV/RFSV)

Let Y be a fractional Ornstein–Uhlenbeck process with Hurst index $H \in (0,1)$,

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Introduced as the fractional stochastic volatility (FSV) model by Comte and Renault (1998) with $H \in (\frac{1}{2}, 1)$. Repurposed in the rough case $H \in (0, \frac{1}{2})$ by Gatheral, Jaisson and Rosenbaum (2018).

Application to simulated FSV/RFSV data

Setup

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- Use Gatheral, Jaisson and Rosenbaum's (2018) procedure to derive initial parameter guesses ("baseline").
- Following Fukasawa, Takabatake and Westphal (2022), consider $H \in \{0.05, 0.1, 0.3, 0.5, 0.7\}$.
- Other parameters set to keep the data realistic.
- 10 000 replications with $N = 4\,000$.
- $\mathcal{L} := \{1, 2, 3, 5, 20, 50\}$, weight matrix W derived from Newey and West (1987) long-run covariance estimates.

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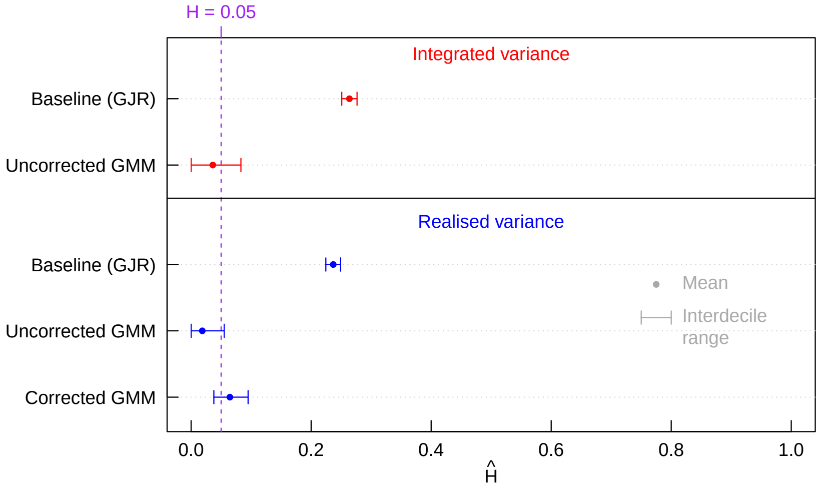

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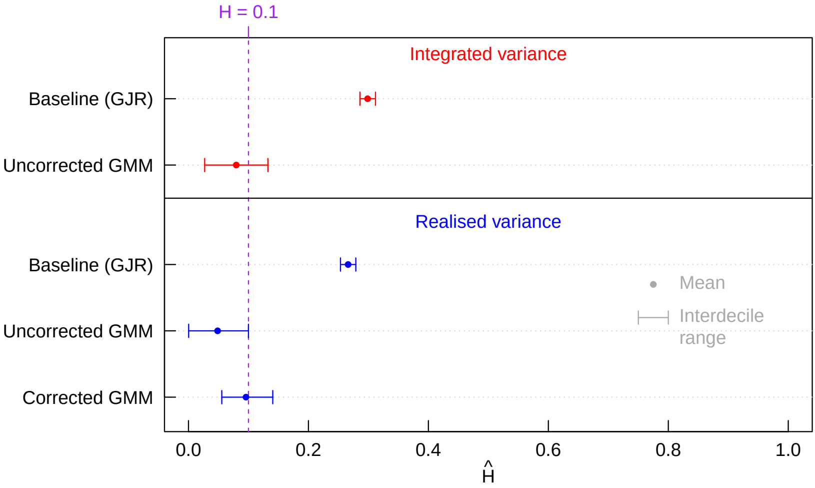

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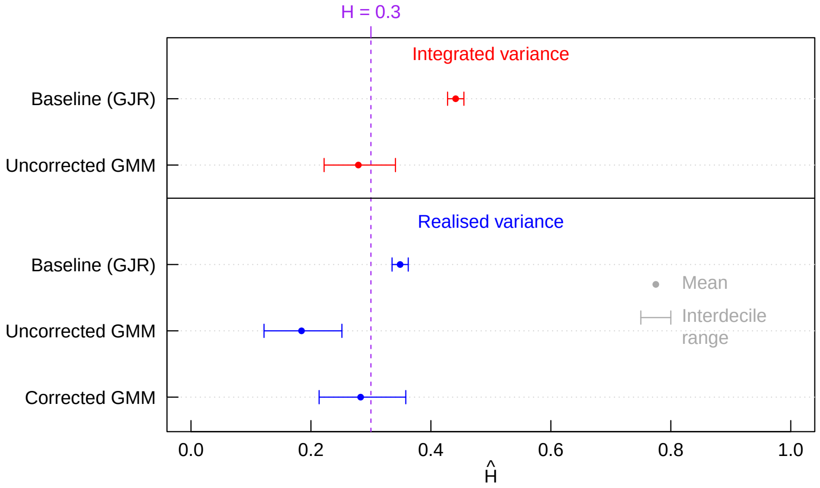

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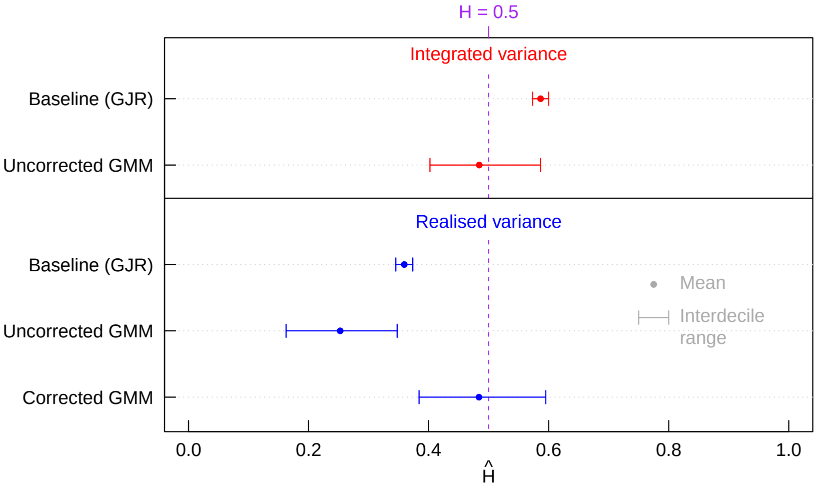

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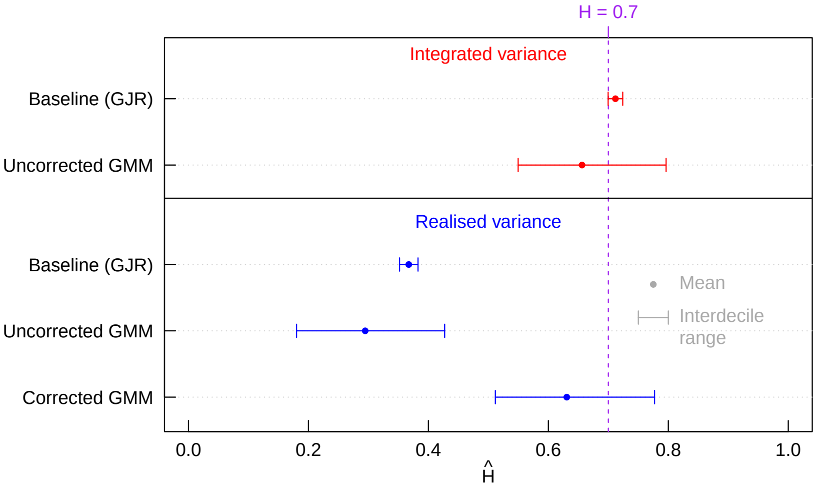

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- Apply GMM estimation to 5-min daily realised variance data on all indices in the database (except Karachi SE 100 Index and Straits Times Index, due to gaps in time series).

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- Inference (confidence intervals) using asymptotic normality.

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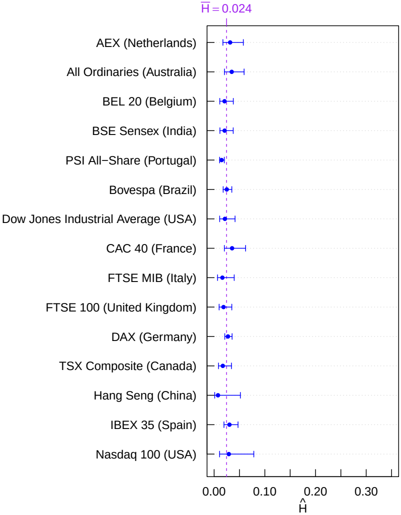

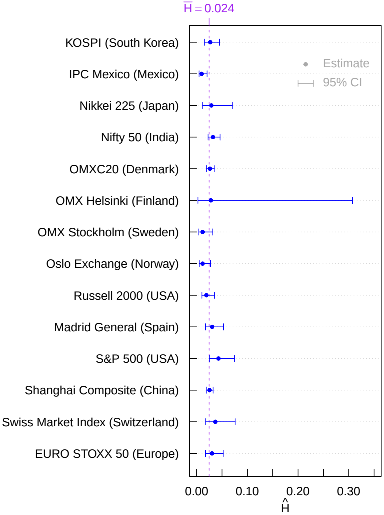

Measuring the roughness of volatility

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Concluding remarks

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- But within a log-normal spot volatility model using GMM, we can accommodate both of these effects.

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- But inferring that spot volatility is rough is harder.
- Integration (smoothes) and measurement error (roughens) become counteracting sources of bias.
- But within a log-normal spot volatility model using GMM, we can accommodate both of these effects.
- GMM estimation and inference suggest that spot volatility is under log-normality best described by a rough process.

Concluding remarks

Summary

- Volatility is rough, as far as realised variance is concerned.
- But inferring that spot volatility is rough is harder.
- Integration (smoothes) and measurement error (roughens) become counteracting sources of bias.
- But within a log-normal spot volatility model using GMM, we can accommodate both of these effects.
- GMM estimation and inference suggest that spot volatility is under log-normality best described by a rough process.
- Typically $\widehat{H} \approx 0.02$, consistent with implied volatility calibrations.

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