Martingale expansion in Finance

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Abstract:

- An application of Yoshida's formula to finance is given.
- More-or-less known regular and singular perturbation expansion formulas around the Black-Scholes model are validated in an unified way.
- As examples, asymptotic expansion formulas for the Black-Scholes implied volatility are given for several specific models, which are quite simple and explicit.

Typical financial practice:

1. Modeling theoretical prices of derivatives by several parameters.

2. Calibrating(=estimating) the parameters from traded call/put option prices.

3. Pricing other (exotic) derivatives by the model with calibrated parameters.

Complete matching between the theoretical call/put prices with the estimated parameters and the observed call/put option prices serves as a constraint to avoid an arbitrage opportunity. **The call/put prices should be recovered.**

Calibration and pricing procedure should be done fast and stably.

Example: the Black-Scholes framework

1. Theoretical European option price $P[h;T]$ with payoff function h, maturity T is determined by h, T and a volatility parameter σ; $P[h;T] = P_{h,T}(\sigma)$.

2. The Black-Scholes implied volatility $\hat{\sigma}$ is defined by $P_{h,T}(\hat{\sigma}) = \hat{P}[h;T]$, where $\hat{P}[h;T]$ is the observed price for $h(z) = (K - z)_+$, $h(z) = (z - K)_+$.

3. For any payoff h, the corresponding option price is given by $P_{h,T}(\hat{\sigma})$.

The function $P_{h,T}$ is explicit, so the procedure is fast. A well-known problem is the smile effect or the skew effect, which means that the implied volatility $\hat{\sigma}$ depends on the choice of K and T. **One parameter is not enough.**

Example: the Heston framework

$$dS_t = S_t(rdt + \sqrt{V_t}[\beta dW_t + \sqrt{1-\beta^2}dW'_t]),$$
$$dV_t = -(aV_t - b)dt + c\sqrt{V_t}dW_t, \quad V_0 = v,$$

where $\beta \in [-1, 1]$. The parameters are (β, a, b, c, v). A semi-analytic formula for theoretical call/put option prices is available.

- This is quite popular in practice, due to the fast calibration and the fact that the volatility smile and skew are reproduced.
- The calibrated parameters are not stable; **recalibration** is required.
- The Heston model is **rejected** by testing based on historical data of S_t

Fast mean reverting model: Fouque et.al. (2000)

$$dS_t = S_t(rdt + f(X_t)[\beta dW_t + \sqrt{1-\beta^2}dW'_t]),$$
$$dX_t = -r_n^{-2}(aX_t - b)dt + r_n^{-1}cdW_t, \quad X_0 = x,$$

where $\beta \in [-1, 1]$ and f is a bounded function. They obtained, by a singular perturbation argument of the corresponding partial differential equation,

$$P[h; T] = P_{h,T}(\sigma) + r_n P^1_{h,T}(\sigma, a) + O(r_n^2), \quad r_n \to 0,$$

for constants σ, a, where $P_{h,T}$ is the Black-Scholes price and $P^1_{h,T}$ is an explicit function of σ and a. In particular, the (theoretical) Black-Scholes implied volatility is expanded as

$$P^{-1}_{h,T}(P[h, T]) = a\frac{\log(K/S_0)}{T} + b + O(r_n^2), \quad h(z) = (K - z)_+.$$

The effective parameters are (σ, a). The volatility skew is reproduced.

Various perturbation approaches for stochastic volatility model:

$$dS_t = S_t(rdt + \varphi(X_t)[\beta dW_t + \sqrt{1-\beta^2}dW'_t]),$$

where $\beta \in [-1, 1]$ where (W, W') is a std. BM. $X = X^n$.

- slowly varying volatility: (Sircar, Lee, ...)

$$dX^n_t = (r_n b_1(X^n_t) + r_n^2 b_2(X^n_t))dt + r_n c(X^n_t)dW_t,$$

- small volatility of volatility: (Hull and White, Lewis, Sørensen et.al., ..)

$$dX^n_t = b(X^n_t)dt + r_n c(X^n_t)dW_t,$$

- fast mean-reverting: (Fouque et.al., Khasminskii et.al., Fukasawa)

$$dX^n_t = r_n^{-2} b(X^n_t)dt + r_n^{-1}\lambda(X^n_t)dt + r_n^{-1}c(X^n_t)dW_t,$$

- The above perturbations are all around the Black-Scholes model. Hence the Black-Scholes price appears as the leading term of the expansion.

- In particular, we have an expansion of the implied volatility

$$P_{h,T}^{-1}(P[h,T]) = a(T)\log(K/S_0) + b(T) + O(r_n^2), \quad h(z) = (K-z)_+. \qquad (1)$$

  Here the functions a and b of T depend on the perturbation model.

- Empirical studies showed that

  – (observed) implied volatility fits to (1) fairly good for fixed T,

  – $a(T) = a_1 + a_2/T$, $a(T) = aT^{-q}$ with $q \approx 1/2$ are more suitable than $a(T) = a/T$ (fast mean reverting) or $a(T) = aT$ (slowly varying).

- We shall see that the above perturbation approaches around the BS model are all validated in an extended form by Yoshida's formula.

- This is a probabilistic approach, so we don't need any smoothness condition on payoff h; the digital options are incorporated.

- We admit non-Markovian and jump models.

Let $(\Omega^n, \mathcal{F}^n, \{\mathcal{F}^n\}_{t\geq 0}, P^n)$ be a filtered probability space for each $n \in \mathbb{N}$ where an increasing adapted continuous process Λ^n with $\Lambda^n_0 = 0$, an adapted cag processes h^n and a continuous martingale X^n with $X^n_0 = 0$ are defined.

Let W be the canonical process $W_t(\omega) = \omega(t)$ of the Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$.

Let N be a standard Poisson process and U_j, $j = 1, 2, \dots$ be i.i.d. random variables independent to N defined on a probability space $(\Omega^0, \mathcal{F}^0, P^0)$.

Consider the product space $\Omega^0 \times \Omega^n \times \mathcal{W}$ with the product measure $P^0 \otimes P^n \otimes \mu$ and then define a time-changed compound Poisson process C^n as

$$C^n_t = r_n \sum_{j=1}^{N^n_t} U_j, \quad N^n_t = N_{\Lambda^n_t},$$

where $r_n > 0$ is a deterministic sequence with $r_n \to 0$ as $n \to \infty$.

Let ν be the distribution of U_j and assume that there exists $\epsilon > 0$ such that

$$\int_{\mathbb{R}} e^{\epsilon|z|}\nu(dz) < \infty.$$

Then put

$$\nu_k = \int_{\mathbb{R}} z^k \nu(dz), \quad k \in \mathbb{N}.$$

Let $\{\mathcal{G}_t^n\}_{t\geq 0}$ be the minimal filtration such that W, N^n, C^n and all $\{\mathcal{F}_t^n\}$-adapted processes are $\{\mathcal{G}_t^n\}$-adapted and that the usual conditions are satisfied.

The following argument is with respect to $\{\mathcal{G}_t^n\}$ without notice.

Let g^n be an adapted process which is independent to W.

We suppose that the log price process $Z = \log(S/S_0)$ is given by

$$Z = A^n + X^n + g^n \cdot W + h^n \cdot C^n$$

for a large n, where

$$A^n_t = rt - \frac{1}{2}\langle X^n \rangle_t - \frac{1}{2}\int_0^t |g^n_s|^2 ds - \int_0^t \int_{\mathbb{R}} (e^{h^n_s r_n z} - 1)\nu(dz)\Lambda^n(ds).$$

Note that $e^{-rt}S_t$ is a martingale for each n by Itô's formula due to the definition of A^n under suitable integrability conditions.

Negative relation between $(g^n, \langle X^n \rangle)$ and X^n induces the leverage effect.

The stochastic intensity Λ^n accounts for the clustering of jumps.

As is well-known, the European option price at time 0 with payoff function h and maturity T is given by

$$e^{-rT}E[h(S_T)] = E[H(Z_T)], \quad H(z) = e^{-rT}h(S_0 \exp(z)).$$

Our aim is to prove the validity of an asymptotic expansion of the distribution of Z_T for fixed $T > 0$ as $n \to \infty$.

Denote by M^n the local martingale part of Z:

$$M^n = X^n + g^n \cdot W + h^n \cdot C^n - r_n \nu_1 \int_0^{\cdot} h_s^n \Lambda^n(ds).$$

Further, put

$$D^n = r_n \sum_{j=1}^{N^n} |h_{\tau_j^n}^n U_j|^2 - r_n \nu_2 \int_0^{\cdot} |h_s^n|^2 \Lambda^n(ds),$$

where $\tau_j^n = \inf\{t \geq 0; N_t^n = j\}$, and

$$\Xi_n = \langle X^n \rangle_T + \int_0^T |g_s^n|^2 ds + r_n^2 \nu_2 \int_0^T |h_s^n|^2 \Lambda^n(ds),$$
$$\Xi_n^\alpha = r_n^2 \nu_3 \int_0^T \{h_s^n\}^3 \Lambda^n(ds).$$

Condition 1 *There exist a constant* $\alpha \in \mathbb{R}$ *and a deterministic sequence* Σ_n *with* $\liminf_n \Sigma_n > 0$ *such that*

$$\Sigma_n^{-1}\Xi_n \to 1, \ \ \Sigma_n^{-3/2}\Xi_n^{\alpha} \to \alpha,$$

in probability and that

$$(\Sigma_n^{-1/2}M_T^n, \Sigma_n^{-1}D_T^n, r_n^{-1}(\Sigma_n^{-1}\Xi_n - 1))$$

is uniformly bounded in L^p *for any* $p > 0$ *and converges in law to* (N_1, N_2, N_3)*.*

Condition 2 *For any* $p > 0$*,* $k \geq 2$*, the sequences*

$$r_n^2 \sum_{j=1}^{N_T^n} |h_{\tau_j^n}^n U_j|^4, \ \ r_n^2 \int_0^T |h_s^n|^k \Lambda^n(ds), \ \ \left\{\int_0^T |g_s^n|^2 ds\right\}^{-1}$$

are uniformly bounded in L^p*.*

Theorem 1 *Let H be a Borel function of polynomial growth. Under Conditions 1 and 2, it holds*

$$E[H(Z_T)] = \int_{\mathbb{R}} H(rT - \Sigma_n/2 + \sqrt{\Sigma_n}z)\phi_n(z)dz + o(r_n) \tag{2}$$

as $n \to \infty$, where

$$\phi_n(z) = \phi(z) + r_n\frac{1}{2}\partial_z^2(E[\xi|N_1 = z]\phi(z)) - r_n\partial_z(E[\eta|N_1 = z]\phi(z)),$$

and ϕ is the standard normal density,

$$\xi = \frac{1}{3}N_2 + N_3, \quad \eta = -\frac{\sqrt{\Sigma_n}}{2}\left(N_3 + \frac{\sqrt{\Sigma_n}}{3}\alpha\right).$$

Remark: Notice that

$$P_h(\sigma) := e^{-rT} \int h(S_0 \exp(rT - \sigma^2 T/2 + \sigma \sqrt{T} z))\phi(z)dz$$

is the Black-Scholes price for payoff function h with volatility σ, so that considering $H(z) = e^{-rT} h(S_0 \exp(z))$, the leading term of the expansion (2) corresponds to the Black-Scholes price with σ_n such that $\sigma_n^2 T = \Sigma_n$.

Theorem 2 *If, in addition, the distribution of* (N_1, N_2, N_3) *is normal with mean* δ *and covariance matrix* ρ*, then (2) holds with*

$$\phi_n(z) = \phi(z)\left\{1 + \frac{r_n}{2}\left\{\delta_3(z^2-1) + \left(\rho_{13} + \frac{1}{3}\alpha\right)(z^3 - 3z) - \sqrt{\Sigma_n}\left\{\left(\delta_3 + \frac{\sqrt{\Sigma_n}}{3}\alpha\right)z + \rho_{13}(z^2-1)\right\}\right\}\right\}.$$

In particular, the put option price $e^{-rT}E[(K - S_T)_+|S_0 = S]$ *with* $K > 0$ *is expanded as*

$$P_h(\sigma_n) + \frac{1}{2}r_n\sigma_n\sqrt{T}S\phi(d_1)\left\{\delta_3 - \rho_{13}d_2 - \frac{1}{3}\alpha(d_2 - \sigma_n\sqrt{T})\right\} + o(r_n),$$

where $\sigma_n^2 T = \Sigma_n$ *and* P_h *is the Black-Scholes price with* $h(s) = (K - s)_+$*, namely*

$$P_h(\sigma_n) = Ke^{-R(T)}\Phi(-d_2) - S\Phi(-d_1),$$
$$d_1 = \frac{\log(S/K) + R(T) + \Sigma_n/2}{\sqrt{\Sigma_n}}, \quad d_2 = d_1 - \sqrt{\Sigma_n}.$$

Recalling that the Black-Scholes put price P_h satisfies

$$\frac{\partial P_h}{\partial \sigma} = \sqrt{T} S \phi(d_1),$$

we obtain the following corollary.

Corollary 3 *Under the same conditions as Theorem 2, the Black-Scholes implied volatility*

$$P_h^{-1}(e^{-R(T)} E[h(S_T) | S_0 = S]), \quad h(s) = (K - s)_+$$

is expanded as

$$\sigma_n \left\{ 1 + \frac{r_n}{2} \left\{ \delta_3 - \rho_{13} d_2 - \frac{1}{3} \alpha (d_2 - \sigma_n \sqrt{T}) \right\} \right\} + o(r_n).$$

Pure jump effect: Let us analyze the effect of jump by considering a simple model of deterministic volatility. Suppose that

$$X^n \equiv 0, \;\; g_s^n \equiv \sigma_g > 0, \;\; h_s^n \equiv 1, \;\; \Lambda_s^n = r_n^{-2}\lambda s$$

for constants $r, \sigma_g, \lambda > 0$. Then we have a simple jump-diffusion model

$$Z_T = rT - \frac{\sigma_g^2 T}{2} - \lambda T \int (e^{r_n z} - 1)\nu(dz) + \sigma_g W_T + r_n \sum_{j=1}^{N_T^n} U_j$$

where N^n is a Poison process with $E[N_t^n] = r_n^{-2}\lambda t$. The larger n, the more and the smaller the jumps. If there exists $\epsilon > 0$ such that

$$\int e^{\epsilon z^2} \nu(dz) < \infty, \tag{3}$$

then Conditions 1, 2 are satisfied with

$$\Sigma_n \equiv \sigma_g^2 T + \nu_2 \lambda T, \;\; \alpha = \nu_3 \lambda T (\sigma_g^2 T + \nu_2 \lambda T)^{-3/2}.$$

The asymptotic distribution (N_1, N_2, N_3) is normal with $N_3 = 0$. By Theorem 2, (2) holds with

$$\phi_n(z) = \phi(z)\left\{1 + \frac{r_n\alpha}{6}\left\{(z^3 - 3z) - \sigma^2 T z\right\}\right\}$$

with $\sigma = \sqrt{\sigma_g^2 + \nu_2\lambda}$. By Corollary 3, the Black-Scholes implied volatility IV at time 0 is expanded as

$$\begin{aligned}\mathrm{IV} =& \sigma\left\{1 - \frac{r_n\alpha}{6}(d_2 - \sigma\sqrt{T})\right\} + o(r_n)\\ =& a\frac{\log(K/S)}{T} + b + o(r_n),\end{aligned}$$

where

$$a = \frac{\nu_3\lambda}{6\sigma^3}r_n, \quad b = \sigma - \frac{\nu_3\lambda}{6\sigma^3}\left(r - \frac{3}{2}\sigma^2\right)r_n.$$

Here the volatility skew results from not the stochastic volatility but the non-zero skewness $\nu_3\lambda$ of the marginal distribution of the compound Poisson process.

The leverage effect with jumps: Extending the preceding example, consider

$$X^n \equiv 0, \quad h_s^n \equiv 1, \quad \Lambda_s^n = r_n^{-2}\lambda s, \quad g_s^n = g(Y_s^n)$$

with

$$Y_s^n = y + r_n c W' + \beta \left\{ r_n^2 \sum_{j=1}^{N_s^n} U_j - \nu_1 \lambda s \right\}$$

for constants y and c, where W' is a standard Brownian motion independent to W. Assume that the function g has bounded derivatives up to order 2 and is positive, bounded away from 0.

Conditions 1, 2 are easily verified under (3) with

$$\Sigma_n \equiv g(y)^2 T + \nu_2 \lambda T, \quad \alpha = \nu_3 \lambda T (g(y)^2 T + \nu_2 \lambda T)^{-3/2}$$

and the asymptotic distribution (N_1, N_2, N_3) is normal with

$$\delta_3 = 0, \quad \rho_{13} = \beta g(y) g'(y) \nu_2 \lambda T^2 (g(y)^2 T + \nu_2 \lambda T)^{-3/2}.$$

By Corollary 3, we have

$$\begin{aligned}\mathrm{IV} =&\sigma\left\{1-\frac{r_n}{2}\left\{\rho_{13}d_2+\frac{1}{3}\alpha(d_2-\sigma\sqrt{T})\right\}\right\}+o(r_n)\\=&\left(\gamma_1+\frac{\gamma_2}{T}\right)\log(K/S)+\sigma-r\,(T\gamma_1+\gamma_2)+\frac{\sigma^2}{2}(T\gamma_1+3\gamma_2)+o(r_n),\end{aligned}$$

where

$$\sigma=\sqrt{g(y)^2+\nu_2\lambda},\quad \gamma_1=\frac{\beta g(y)g'(y)\nu_2\lambda}{2\sigma^3}r_n,\quad \gamma_2=\frac{\nu_3\lambda}{6\sigma^3}r_n.$$

Fractional Brownian motion: Consider a simple stochastic volatility model

$$Z_t = rt - \frac{1}{2}\int_0^t g(Y_s^n)^2 ds + \int_0^t g(Y_s^n)[\beta dW_s' + \sqrt{1-\beta^2} dW_s]$$

with

$$Y_s^n = y + r_n W_s^H, \quad W_t^H = \int_0^t K_H(t,s) dW_s',$$

where $\beta \in (-1,1), y \in \mathbb{R}$ are constants, (W, W') is a 2-dimensional Brownian motion and $H \in (0, 1/2)$,

$$K_H(t,s) = c_H\Big[\Big(\frac{t}{s}\Big)^{H-1/2}(t-s)^{H-1/2} \\ - (H-1/2)s^{1/2-H}\int_s^t u^{H-3/2}(u-s)^{H-1/2} du\Big],$$

Note that W^H is a fractional Brownian motion, so that it has stationary increments. We assume that g has bounded derivatives up to order 2 and is positive, bounded away from 0. Conditions 1, 2 are easily verified with

$$\Sigma_n \equiv g(y)^2 T, \quad \alpha = 0 \tag{4}$$

and the asymptotic distribution (N_1, N_2, N_3) is normal with $N_2 = 0$,

$$\delta_3 = 0, \quad \rho_{13} = 2\beta g'(y) c'_H T^H / g(y). \tag{5}$$

By Corollary 3, we have

$$\begin{aligned} \mathrm{IV} &= \sigma \left\{ 1 - \frac{r_n}{2} \rho_{13} d_2 \right\} + o(r_n) \\ &= a T^{H-1/2} \log(K/S) + \sigma + b T^{H+1/2} + o(r_n), \end{aligned} \tag{6}$$

where

$$\sigma = g(y), \quad a = \frac{\beta g'(y) c'_H}{\sigma} r_n, \quad b = -a \left(r - \frac{\sigma^2}{2} \right).$$

Regular perturbation: slowly varying and small vol-of-vol models

Suppose that

$$g_s^n = \sqrt{1-\beta^2} g(Y_s^n), \quad X^n = \beta g(Y^n) \cdot W', \quad h_s^n \equiv 0,$$

where (W, W') is a 2-dimensional standard Brownian motion, $\beta \in (-1, 1)$, and Y^n satisfies the stochastic differential equation

$$dY_s^n = (b_0(Y_s^n) + r_n b_1(Y_s^n) + r_n^2 b_2(Y_s^n))ds + r_n c(Y_s^n) dW_s', \quad Y_0^n = y.$$

For brevity, we assume that g has bounded derivatives up to order 2 and is positive, bounded away from 0. Let y_t be the solution of the ordinary differential equation

$$\frac{dy_t}{dt} = b_0(y_t), \quad y_0 = y.$$

Under suitable conditions on the smoothness of the coefficients b_0, b_1, b_2, c, Conditions 1, 2 are satisfied with

$$\Sigma_n \equiv \int_0^T g(y_t)^2 dt, \quad \alpha = 0.$$

We have

$$N_1 = \Sigma_n^{-1/2} \int_0^T g(y_t)[\beta dW_t' + \sqrt{1-\beta^2} dW_t],$$
$$N_2 = 0,$$
$$N_3 = 2\Sigma_n^{-1} \int_0^T g(y_t) g'(y_t) D_t dt,$$

which is normal with

$$\delta_3 = 2\Sigma_n^{-1} \int_0^T g(y_t) g'(y_t) \int_0^t \exp\left\{\int_s^t b_0'(y_u) du\right\} b_1(y_s) ds dt,$$
$$\rho_{13} = 2\Sigma_n^{-3/2} \beta \int_0^T g(y_t) g'(y_t) \int_0^t \exp\left\{\int_s^t b_0'(y_u) du\right\} c(y_s) g(y_s) ds dt.$$

Singular perturbation expansion: fast mean reverting model

Let $\bar{W} = (\bar{W}^1, \dots, \bar{W}^d)$ be a d-dimensional standard Brownian motion and consider a stochastic differential equation

$$dY_t^n = (r_n^{-2} b(Y_t^n) + r_n^{-1} b_1(Y_t^n))dt + r_n^{-1} c(Y_t^n) d\bar{W}_t, \quad Y_0^n = y,$$

where $b = (b^j), b_1 = (b_1^j) : \mathbb{R}^m \to \mathbb{R}^m$, $c = (c_i^j) : \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^d$. We assume that Y^n is ergodic for each n and denote by π^n the ergodic distribution.

Let $\psi = (\psi_1, \dots, \psi_m) : \mathbb{R}^m \to \mathbb{R}^m$ be a Borel function and put

$$\psi_0 = \sum_{i,j=1}^{m} \psi_i \psi_j a^{ij}, \quad a^{ij} = \sum_{k=1}^{d} c_k^i c_k^j.$$

Let us set

$$dX_s^n = \beta\psi(Y_s^n)c(Y_s^n)d\bar{W}_s = \beta\sum_{k,j}\psi_k(Y_s^n)c_j^k(Y_s^n)d\bar{W}_s^j,$$

$$g_s^n = \sqrt{(1-\beta^2)\psi_0(Y_s^n)}, \quad h_s^n \equiv 0,$$

where $\beta \in (-1, 1)$. Here we define g^n so that

$$\langle M^n\rangle_T = \int_0^T \psi_0(Y_s^n)ds, \quad M^n = X^n + g^n \cdot W.$$

Suppose that the Poisson equation $\mathcal{L}^n F^n = \psi_0 - \pi^n[\psi_0]$ has a smooth solution F^n on $\mathbb{R}^m$ for each $n \in \mathbb{N}$, where

$$\mathcal{L}^n = \sum_{j=1}^m (b^j + r_n b_1^j)\partial_j + \frac{1}{2}\sum_{i,j=1}^m a^{ij}\partial_i\partial_j.$$

Putting $\Sigma_n = T\pi^n[\psi_0]$, under suitable conditions on the uniform integrability, we have Conditions 1,2 with $\alpha = 0$ and that the asymptotic distribution (N_1, N_2, N_3) is normal with

$$N_2 = 0, \quad \delta_3 = 0, \quad \rho_{13} = -\frac{\beta}{\sqrt{T}\pi[\psi_0]^{3/2}} \sum_{i,j} \pi[f_i \psi_j a^{ij}]$$

by martingale central limit theorem. Notice that f appears in the coefficients of the expansion, which might cause a problem in practice because no analytic formula is available in general for the solution of the Poisson equations except one-dimensional case. However, if Y is a symmetric diffusion and there exists a function $\Psi : \mathbb{R}^m \to \mathbb{R}$ such that $\partial_j \Psi = \psi_j$, we have

$$\sum_{i,j} \pi[f_i \psi_j a^{ij}] = \sum_{i,j} \pi[\partial_i F \partial_j \Psi a^{ij}] = -2\pi[\mathcal{L} F \Psi] = 2\pi[(\pi[\psi_0] - \psi_0)\Psi]$$

by IBP formula, or equivalently, a well-known identity for the Dirichlet form.