On the Invariant Measure for Branching Diffusions with Immigration

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Le Mans, March 18, 2009

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  Feller property: We assume that the semigroup $P_t f(x) = E_x(\xi_t)$ is strongly continuous on $\mathcal{C}_0(\mathbb{R}^d)$ (e.g., b and σ bounded Lipschitz).

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- Each particle branches [i.e. dies and produces a random number of offspring] at a random time according to a position-dependent branching rate $\kappa \in \mathcal{C}_b(\mathbb{R}^d)$, $\inf \kappa > 0$.

The model

- Particle dying at $x \in \mathbb{R}^d$: k offspring with probability $p_k(x)$, $k \in \mathbb{N}_0 \setminus \{1\}$, distributed on $(\mathbb{R}^d)^k$ according to

$$\bigotimes_{j=0}^{k} Q(x,\cdot), \qquad k \geq 1,$$

with a transition kernel $Q(\cdot,\cdot)$ on $\mathbb{R}^d \times \mathbb{R}^d$. Assume

$$\varrho \in \mathcal{C}_b(\mathbb{R}^d), \qquad \text{where } \varrho(x) := \sum_{k \in \mathbb{N} \setminus \{1\}}^{\infty} k p_k(x).$$

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- Immigration: at constant rate $c > 0$, new particles (one immigrant at a time) are added to the pre-existing configuration, distributed on $\mathbb{R}^d$ according to a probability measure π.

The model

$\rightsquigarrow$ Stochastic process $\eta = (\eta_t)_{t \geq 0}$ of (ordered) finite particle configurations, state space:

$$S := \biguplus_{\ell \geq 0} (R^d)^\ell, \qquad (\mathbb{R}^d)^0 := \{\Delta\} \text{ void configuration,}$$

càdlàg paths with jumps corresponding to either branching or immigration events, jump times $T_n \uparrow \infty$, Feller property.

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càdlàg paths with jumps corresponding to either branching or immigration events, jump times $T_n \uparrow \infty$, Feller property.
Generator:

$$Lf(x) = \mathscr{A} f(x) + \alpha(x) \int_S K(x, \mathrm{d}y)\, (f(y) - f(x))\,.$$

Here $\mathscr{A}$ describes a process $\tilde{\eta}$ of finitely many particles moving independently of each other on paths given by (1), without branching or immigration.

The model

'Kill' the process $\tilde{\eta}$ at rate

$$\alpha : S \to \mathbb{R}_+, \qquad \alpha(x) := c + \sum_{i=1}^{\ell(x)} \kappa(x^i),$$

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$\Rightarrow$ occupation time of $A \in \mathscr{B}(S)$ by the killed process given by the *generalized resolvent*

$$R_\alpha(x, A) := \int_0^\infty E_x \left(\mathbf{1}_A(\tilde{\eta}_t) \exp\left(- \int_0^t \alpha(\tilde{\eta}_s)\, ds \right) \right) dt$$

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Between jump times T_n and T_{n+1}, the process η evolves as $\tilde{\eta}$.

The model

Jumps are governed by the *jump kernel* $K : S \times \mathscr{B}(S) \to [0,1]$: at its death time, 'restart' $\tilde{\eta}$ with distribution $K(\tilde{\eta}_\tau, \cdot)$, where

$$K(x,A) := \sum_{i=1}^{\ell(x)} \frac{\kappa(x^i)}{\alpha(x)} \Bigg[p_0(x^i)\,\mathbf{1}_A(x^1,\dots,x^{i-1},x^{i+1},\dots,x^{\ell(x)}) + \sum_{k\in\mathbb{N}\setminus\{1\}} p_k(x^i) \int_{(\mathbb{R}^d)^k} \bigotimes_{j=1}^{k} Q(x^i;dv)\,\mathbf{1}_A(x^1,\dots,x^{i-1},v,x^{i+1},\dots,x^{\ell}) \Bigg] + \frac{c}{\alpha(x)} \int_{\mathbb{R}^d} \pi(\mathrm{d}v)\,\mathbf{1}_A(x^1,\dots,x^{\ell(x)},v), \qquad x=(x^1,\dots,x^\ell)\in(\mathbb{R}^d)^\ell.$$

Remarks

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- Formal construction: cf. Ikeda/Nagasawa/Watanabe 1968
- Interactions between the particles in spatial motion and / or branching and reproduction mechanism possible (cf. Löcherbach 2002, 2004)
- One-particle motion need not be a diffusion: any Feller process on a LCCB space
- Replace S by the space of *unordered* finite particle configurations resp. finite point measures on $\mathbb{R}^d$ $\rightsquigarrow$ BDI as measure-valued process

Ergodicity: Positive Harris recurrence

We are interested in conditions ensuring that the particle process η is ergodic in the sense that it has the following properties:

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P1: Δ as a recurrent atom

The void configuration Δ is a recurrent atom for the process, i.e.

$$E_\Delta(R) < \infty, \qquad \text{where } R := \inf_{n \in \mathbb{N}} \{ T_n : T_n = \Delta \} .$$

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Under **P1**, $(\eta_t)_t$ is positive Harris recurrent with invariant probability measure

$$m(F) = \frac{1}{E_\Delta(R)} E_\Delta \left(\int_0^R \mathbf{1}_F(\eta_s)\, \mathrm{d}s \right).$$

Ergodicity: Finite invariant occupation measure

For $x \in S$, $B \in \mathscr{B}(\mathbb{R}^d)$ write

$$x(B) := \sum_{i=1}^{\ell(x)} \delta_{x^i}(B) = \sum_{i=1}^{\ell(x)} \mathbf{1}_B(x^i)$$

and define the *invariant occupation measure* $\overline{m}$ on $\mathbb{R}^d$:

$$\overline{m}(B) := \int_S x(B)\, m(dx) = \frac{1}{E_\Delta(R)} E_\Delta \left(\int_0^R \eta_s(B)\, \mathrm{d}s \right), \; B \in \mathscr{B}(\mathbb{R}^d).$$

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P2: Finite invariant occupation measure

$$\overline{m}(\mathbb{R}^d) = \int_S \ell(x)\, m(dx) = \sum_{\ell \geq 0} \ell m(\mathbb{R}^{d\ell}) < \infty.$$

Let A be the generator of the one-particle motion (1):

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\,\partial_{ij}^2 f + \sum_{i=1}^{d} b_i\,\partial_i f \qquad \text{for } f \in C_b^2(\mathbb{R}^d), \qquad a := \sigma\sigma^T,$$

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let $(\tilde{\xi}_t)_t$ be the Feller process with generator

$$Af(x) - \kappa(x)\varrho(x)\int Q(x,\mathrm{d}y)\big(f(y)-f(x)\big), \qquad f \in D(A), x \in \mathbb{R}^d$$

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and 'kill' $\tilde{\xi}$ at rate $\gamma := \kappa(1-\varrho)$ $\Rightarrow$ occupation time of the killed process given by

$$R_\gamma(x, B) = \int_0^\infty E_x\left(\mathbf{1}_B(\tilde{\xi}_t) e^{-\int_0^t \gamma(\tilde{\xi}_s)\,\mathrm{d}s}\right)\mathrm{d}t, \qquad B \in \mathscr{B}(\mathbb{R}^d).$$

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Assume that $\sup_{x\in\mathbb{R}^d} R_\gamma(x,\mathbb{R}^d) < \infty$. *Then:*

(i) *The process* η *has the properties* **P1** *and* **P2** *for every choice of an immigration law* π, *and* $\overline{m}$ *is given by*

$$\overline{m}(dy) = c\pi R_\gamma(dy) := c\int \pi(dx)R_\gamma(x,dy).$$

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$$\overline{m}(dy) = c\pi R_\gamma(dy) := c \int \pi(dx) R_\gamma(x, dy).$$

(ii) $\overline{m}$ *is the unique finite measure on* $\mathbb{R}^d$ *such that*

$$\overline{m}\left(Af - \kappa f + \kappa\varrho Q(f)\right) = -c\pi(f) \qquad \forall f \in D(A),$$

while m *is the unique probability measure on* S *such that*

$$m(Lf) = 0 \qquad \forall f \in D(L).$$

Statistical problems for BDIs

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- Non-parametric estimation of the diffusion coefficient σ from discrete-time observations (Brandt 2005)

$\rightsquigarrow$ Study of properties of m and $\overline{m}$:

- Finiteness of the q-th moments of the configuration length, $q \geq 1$:

$$\int_S \ell^q(x)\, m(\mathrm{d}x) = \sum_{\ell=0}^{\infty} \ell^q m(\mathbb{R}^{d\ell}) \overset{!}{<} \infty$$

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- Existence of continuous (or smooth) Lebesgue densities for $\overline{m}$ on $\mathbb{R}^d$ and m on S respectively.

Decay of m_ℓ: A recursion formula

Proposition

Suppose that κ and $(p_k)_k$ are spatially constant. Then the following recursion formula for $m_\ell := m\left((\mathbb{R}^d)^\ell\right)$ holds:

$$m_1 = \frac{1}{\kappa p_0 E_\Delta(R)},$$

$$m_{\ell+1} = \frac{1}{(\ell+1)\kappa p_0}\Bigg((\ell\kappa + c)\cdot m_\ell - c\cdot m_{\ell-1} - \sum_{k=2}^{\ell}(\ell-k+1)\kappa p_k \cdot m_{\ell-k+1}\Bigg), \qquad \ell \geq 1.$$

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&\qquad - \sum_{k=2}^{\ell}(\ell - k + 1)\kappa p_k \cdot m_{\ell-k+1}\Bigg), \qquad \ell \geq 1.
\end{aligned}
$$

Proof.

Substitute $f := \mathbf{1}_{(\mathbb{R}^d)^\ell}$ in the equation $m(Lf) = 0$. □

Decay of m_ℓ: An explicit formula

Corollary

Suppose in addition that the branching is binary: $p_2 = 1 - p_0 < \frac{1}{2}$. Then the following explicit formula for m_ℓ holds:

$$m_\ell = \frac{1}{\ell \kappa p_0 E_\Delta(R)} \left(\frac{p_2}{p_0}\right)^{\ell-1} \cdot \prod_{j=1}^{\ell-1} \left(1 + \frac{c}{j\kappa p_2}\right), \qquad \ell \geq 1.$$

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In particular

$$\forall q \geq 1 : \int_S \ell^q(x)\, m(dx) = \sum_{\ell \geq 0} \ell^q m_\ell < \infty.$$

Decay of m_ℓ: Remark

Remark

- *The assumption that κ and p_k are spatially constant is not essential provided there are suitable bounds available (see the 'comparison results' in Höpfner/Hoffmann/Löcherbach 2002).*

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- **Conjecture**: *$m(\mathbb{R}^{d\ell})$ decays exponentially as long as the support of p_k is finite uniformly in $x \in \mathbb{R}^d$:*

$$p_k(x) = 0 \qquad \forall x \in \mathbb{R}^d, k \geq k_0.$$

Densities for $\overline{m}$

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When does $\overline{m}$ admit a (continuous) Lebesgue density?Recall the generator A of the one-particle motion (1):

$$Af(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\,\partial_{ij}^2 f + \sum_{i=1}^{d} b_i\,\partial_i f\,.$$

Assume $\pi(\mathrm{d}x) = p(x)\mathrm{d}x$. By the Theorem, $\overline{m}$ admits a Lebesgue density h iff

$$\langle h, Af - \kappa f + \kappa\varrho Qf\rangle = -c\langle p, f\rangle \qquad \forall f \in D(A)$$

has a nonnegative solution $h \in L^1(\mathbb{R}^d)$. Rewrite this problem as

$$A^* h - \kappa h + Q^*\kappa\varrho h = -cp.$$

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If $Q(x,\cdot) = \delta_x$, this becomes (see Höpfner and Löcherbach 2005)

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It turns out that

$$A^*h=A_0^*h-\frac{1}{2}\mathrm{div}(b-b^*)h,$$

where $b_i^*:=\sum_{j=1}^d\partial_j a_{ij}-b_i$ and A_0^* is the generator of

$$d\xi_t^*=b^*(\xi_t^*)dt+\sigma(\xi_t^*)dW_t. \tag{2}$$

$$\rightsquigarrow A_0^*h-\left(\frac{1}{2}\mathrm{div}(b-b^*)+\kappa\right)h+Q^*\kappa\varrho h=-cp.$$

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Assumption

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Assumption

Assume that

- σ *and* b *are smooth enough such that the diffusion* ξ^* *is Feller*
- $\pi(dx) = p(x)\,dx,\ Q(x,dy) = q(x-y)\,dy$ *with* $p, q \in \mathcal{C}_0(\mathbb{R}^d)$
- *Spatially constant rates* $\kappa > 0$, $0 < \varrho < 1$ *such that*

$$\inf \gamma^* := \inf \left(\frac{1}{2} div(b - b^*) + \kappa(1 - \varrho) \right) > 0.$$

Densities for $\overline{m}$

Proposition

Under the above assumptions, $\overline{m}$ has a Lebesgue density which is in $\mathcal{C}_0(\mathbb{R}^d)$ given by

$$\frac{d\overline{m}}{d\lambda}(x) = c \int_0^\infty dt\, E_x \left(p(\bar{\xi}_t) e^{-\int_0^t ds \gamma^*(\bar{\xi}_s)} \right) = c R^*_\gamma(p).$$

Here, $(\bar{\xi}_t)_t$ is the Feller process with generator

$$\bar{A}f(x) = A_0^* f(x) + \kappa\varrho \int_{\mathbb{R}^d} q(x-y)\Big(f(y) - f(x)\Big)\, dy.$$

For existence of a smooth density of $\overline{m}$ in the interactive case (under strong regularity assumptions) see Löcherbach 2004.

Calculating m

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When does m admit a Lebesgue density? By conditioning on η_{T_n}, we obtain

$$E_\Delta(R) \cdot m(F) = E_\Delta\left(\int_0^R \mathbf{1}_F(\eta_s)\,\mathrm{d}s\right)$$
$$= \sum_{n=0}^{\infty} E_\Delta\left(\mathbf{1}_{T_n<R}\int_{T_n}^{T_{n+1}} \mathbf{1}_F(\eta_s)\,\mathrm{d}s\right)$$
$$= \sum_{n=0}^{\infty} E_\Delta\left(\mathbf{1}_{T_n<R} \cdot R_\alpha(\eta_{T_n}, F)\right).$$

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&= \sum_{n=0}^{\infty} E_\Delta\left(\mathbf{1}_{T_n<R}\int_{T_n}^{T_{n+1}} \mathbf{1}_F(\eta_s)\,\mathrm{d}s\right) \\
&= \sum_{n=0}^{\infty} E_\Delta\left(\mathbf{1}_{T_n<R} \cdot R_\alpha(\eta_{T_n}, F)\right).
\end{aligned}
$$

Consequently, whenever the resolvent kernel $R_\alpha(x, \mathrm{d}y)$ admits a Lebesgue density, so does m.

Example (Höpfner 2004)

Let the one-particle motion (1) be a Brownian motion in $\mathbb{R}^d$, $d \geq 2$, $\kappa > 0$ and $\varrho \in (0,1)$ constants,

$$\pi(\mathrm{d}x) = p(x)\,\mathrm{d}x, \qquad \text{where } p \in \mathscr{S}(\mathbb{R}^d)$$

(here, $\mathscr{S}$ is the space of Schwartz functions) and

$$Q(x,\cdot) = \delta_x$$

(descendants of a branching particle start their diffusion at their parent's location). Then m is Lebesgue-absolutely continuous, but the density takes the value $+\infty$ on the set

$$\left\{x = (x^1,\dots,x^\ell) \in S : \ell \geq 2 \text{ and } \exists i \neq j : x^i = x^j\right\}.$$

Assumptions

Assumption (on the one-particle motion and branching rate)

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Assumption (on the one-particle motion and branching rate)

- *b and σ are smooth enough such that ξ and ξ^* are Feller;*
- $\frac{1}{2} div(b - b^*) + \kappa \geq 0$;
- *The diffusion ξ killed at rate κ with generator $A - \kappa$ has a continuous transition density $p_t(x, y)$ such that*

$$p_t(x, y) \leq c_1 t^{-d/2} \exp\left(-\frac{c_2 \|x - y\|^2}{2t} \right)$$

(e.g., b, σ, κ bounded Hölder + uniform ellipticity).

Assumptions

Assumption (on the reproduction and immigration mechanisms)

- *Strictly subcritical binary branching*

$$\forall x \in \mathbb{R}^d : p_2(x) = 1 - p_0(x), \qquad \varrho(x) = 2p_2(x) \leq \bar{\varrho} < 1.$$

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$$\forall x \in \mathbb{R}^d : p_2(x) = 1 - p_0(x), \qquad \varrho(x) = 2p_2(x) \leq \bar{\varrho} < 1.$$

- *Offspring and immigration distribution:*

$$\pi(dx) = p(x)\,dx, \quad Q(x,dy) = q(x-y)\,dy, \qquad p,q \in \mathcal{C}_0(\mathbb{R}^d).$$

Theorem

Under the above assumptions, the invariant measure m admits a continuous and bounded Lebesgue-density on S.

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Strategy of proof: Remember

$$m(\cdot) = \frac{1}{E_\Delta(R)} \sum_{n=0}^{\infty} E_\Delta \left(\mathbf{1}_{T_n < R} \cdot R_\alpha(\eta_{T_n}, \cdot) \right).$$

Under our assumptions, for every $n \in \mathbb{N}$ the measure $E_\Delta \left(\mathbf{1}_{T_n < R} \cdot R_\alpha(\eta_{T_n}, \cdot) \right)$ has a continuous and bounded density given by

$$R_\alpha^* (B R_\alpha^*)^{n-1} p.$$

Here, R_α^* is the adjoint of R_α while B involves the adjoint of K.

Strategy of proof

B is an operator given by

$$(Bf)^{\ell}(x)$$
$$= \sum_{i=1}^{\ell+1} \int_E \mathrm{d}v\,(\kappa p_0)(v) f^{(\ell+1)}(x^1, \dots, x^{i-1}, v, x^i, \dots, x^{\ell})$$
$$+ \sum_{i=1}^{\ell-1} \int_E \mathrm{d}v\,(\kappa p_2)(v) f^{(\ell-1)}(x^1, \dots, x^{i-1}, v, x^i, \dots, x^{\ell-2}) q(v, x^{\ell-1}) q(v, x$$
$$+ c \cdot f^{(\ell-1)}(x^1, \dots, x^{\ell-1}) \cdot p(x^{\ell}), \qquad x \in (R^d)^{\ell}, \ell \geq 2$$

for a function $f = (f^{(\ell)})_{\ell \geq 0} : S \to \mathbb{R}$.

Strategy of proof

The series $\sum_{n=0}^{\infty} R_\alpha^*(BR_\alpha^*)^n p$ converges uniformly on each layer $\mathbb{R}^{d\ell}$:

$$\forall \ell : \sum_{n=0}^{\infty} \|R_\alpha^*(BR_\alpha^*)^n p|_{\mathbb{R}^{d\ell}}\|_\infty < \infty.$$

However, there is no convergence of $\sum_n R_\alpha^*(BR_\alpha^*)^n$ in operator norm.

- C. Brandt: Partial reconstruction of the trajectories of a discretely observed branching diffusion with immigration and an application to inference, PhD Thesis, Mainz 2005
- R. Höpfner and E. Löcherbach: *Remarks on ergodicity and invariant occupation measure in branching diffusions with immigration*, Ann. I. H. Poincaré - PR 41 (2005), 1025 - 1047
- R. Höpfner, M. Hoffmann and E. Löcherbach: *Non-parametric estimation of the death rate in branching diffusions*, Scand. J. Stat. 29 (2002), 665 - 692
- R. Höpfner: *Strange shape of invariant density in branching diffusions with immigration*, Preprint, Mainz 2004
- N. Ikeda, M. Nagasawa and S. Watanabe: *Branching Markov Processes. I, II, III*, J. Math. Kyoto Univ. 8 (1968), 233 - 278, 365 - 410; 9 (1969), 95 - 160

E. LÖCHERBACH: *Smoothness of the intensity measure density for interacting branching diffusions with immigration*, J. Funct. Analysis 215 (2004), 133 - 177

E. LÖCHERBACH: *LAN and LAMN for Systems of Interacting Diffusions with Branching and Immigration*, Ann. I. H. Poincaré - PR 38 (2002), 59-90