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On the estimation of analytic intensity densities of Poisson processes.

1. The Problem

We are observing a Poisson random measure $X_\varepsilon(G)$ in R^d (or a Poisson random set Π_ε)

$$X_\varepsilon(G) = \#(\Pi_\varepsilon \cap G)$$

$$P\{X_\varepsilon(G) = k\} = \frac{(\Lambda_\varepsilon(G))^k}{k!} e^{-\Lambda_\varepsilon(G)}$$

The intensity measure Λ_ε

is a. c. w. r. t. Lebesgue m

$$\frac{d\Lambda_\varepsilon}{dm}(x) = \frac{1}{\varepsilon}\,\theta(x)$$

The small parameter ε is supposed to be known.

The unknown $\theta \in \Theta$

where Θ is a known set of functions

The problem: <u>to estimate θ and to study the behaviour of risk functio</u>

$$\underline{E_\theta \left\| \hat{\theta}_\varepsilon - \theta \right\|_p}$$

Dependence on Θ

Example: Θ is 1-dim., θ - constants

MLE:

$$\hat{\theta}_\varepsilon = \frac{\varepsilon}{\operatorname{mes} G} X_\varepsilon(G)$$

$$E_\theta \left| \hat{\theta}_\varepsilon - \theta \right|^2 = \frac{\varepsilon \theta}{\operatorname{mes} G}.$$

Example: Π_ε in R^1

$$\Theta = \{\theta : \sup |\theta'(x)| \leq 1\}$$

$$\sup_\theta E_\theta \left| \hat{\theta}_\varepsilon - \theta(x_0) \right|^2 \geq c\, \varepsilon^{\frac{2}{3}}$$

In this talk it is supposed that Θ consits of analytic functions.

Why analytic?

two quatations:

D. Hilbert, MATHEMATISCH Problemen.

S. Bernstein.

Kutoyants Yu., Statistical Inference for Spatial Poisson Processes, Lecture Notes in statistics, v.134, 1998.

2. Classes of analytic functions

a) Π_ε is observed in $G \subseteq R^d$.

The set $\Theta \subseteq A(D, M)$, $D \subseteq \mathbb{C}^d$ is a bounded region.

$A(D, M)$ consists of functions analytic in $D \supseteq G$ and

$$\sup_{z \in D} |f(z)| \leq M, \quad z = (z_1, \dots z_d$$

Example: $d = 1$, $G = [a, b]$

b) $\Pi_2 \subseteq G \subseteq R^d$. $\Theta \subseteq \mathcal{E}(M, \sigma, \rho)$,

$\sigma = (\sigma_1, \dots \sigma_d)$, $\rho = (\rho_1, \dots \rho_d)$.

$\mathcal{E}(M, \sigma, \rho)$ - the class of entire functions f:

$$\sup_{|z_j| \le R_j} |f(z)| \le M \exp\left\{ \sum_1^d \sigma_j R_j^{\rho_j} \right\}$$

c) $\Pi_2 \subseteq G \subseteq R^d$; $\Theta \subseteq \mathcal{E}_K$

$\mathcal{E}_K$ consists of f:

$$f(x) = \frac{1}{(2\pi)^d} \int_K e^{-i(t,x)} \varphi(t)\, dt,$$

$K \subset R^d$ is bounded, $\varphi \in L_2(K)$

Evidently

$\mathcal{E}_K \subseteq \mathcal{E}(M, \sigma, \rho)$, $\rho = (1, \dots 1)$

Theorem 2. Let $\Theta \supseteq A(\mathcal{D}, M)$. Then for any θ_ε

$$\sup_\theta E_\theta \|\theta_\varepsilon - \theta\|_p \geq c_p \sqrt{\varepsilon}\left(\sqrt{\ln \tfrac{1}{\varepsilon}}\right)^d, \; p<4$$

$$\sup_\theta E_\theta \|\theta_\varepsilon - \theta\|_4 \geq c_4 \varepsilon \left(\sqrt{\ln \tfrac{1}{\varepsilon}} \sqrt[4]{\ln \ln \tfrac{1}{\varepsilon}}\right)^{\cdots}$$

$$\sup_\theta E_\theta \|\theta_\varepsilon - \theta\|_p \geq c_p \sqrt{\varepsilon}\left(\ln \tfrac{1}{\varepsilon}\right)^{\left(1-\frac{2}{p}\right)\cdots}.$$

The constants depend on $M, \mathcal{D}$.

<u>Remark</u>. Observe iid

$$X_1 \ldots\ldots X_n$$

Poisson with int. density θ. The statistics $\sum_1^n X_j$

3. Results.

Th. 1. Let $\Theta \subseteq A(M,D)$.
Then $\exists\, \hat{\theta}_\varepsilon$

$$E_\theta \|\hat{\theta}_\varepsilon - \theta\|_p \leq c_p \sqrt{\varepsilon}\left(\sqrt{\ln \frac{1}{\varepsilon}}\right)^d, \quad 1 \leq p < 4$$

$$E_\theta \|\hat{\theta}_\varepsilon - \theta\|_4 \leq c_4 \sqrt{\varepsilon}\left(\sqrt{\ln \frac{1}{\varepsilon}}\, \sqrt[4]{\ln \ln \frac{1}{\varepsilon}}\right)^d,$$

$$E_\theta \|\hat{\theta}_\varepsilon - \theta\|_p \leq c_p \sqrt{\varepsilon}\left(\ln \frac{1}{\varepsilon}\right)^{\left(1-\frac{2}{p}\right)d},$$

$$4 < p \leq \infty.$$

The results are asymp. exact.

is sufficient. It is Poisson with $n \cdot \theta$

Thus in this case

$$\varepsilon = \frac{1}{n}$$

4. On proofs.

4.1. Construction of estimates.

$$G = \Gamma = [-1, 1]^d$$

$P_{\bar{n}}(x)$ - Legendre polynomials.

Any $f \in L_2(\Gamma)$

$$f(x) = \sum_{\bar{\nu}} f_{\bar{\nu}} P_{\bar{\nu}}(x)$$

$$f_{\bar{\nu}} = \int f(x) P_{\bar{\nu}}(x)\, dx$$

If

$$\theta(x) = \sum \theta_{\bar{v}} P_{\bar{v}},$$

Estimate $\theta_{\bar{v}}$ by

$$\hat{\theta}_{\bar{v}} = \varepsilon \int P_{\bar{v}}(t)\, dX_\varepsilon(t) =$$

$$= \varepsilon \sum_{x \in \Pi_\varepsilon} P_{\bar{v}}(x)$$

Set

$$\hat{\theta}_{\varepsilon N} = \sum_{|v_j| \le N} \hat{\theta}_{\bar{v}} P_v \,, \quad \bar{v} = (v_1, \dots v_d)$$

Then

$$E \|\hat{\theta}_{\varepsilon N} - \theta\|_2^2 = \sum_{|\theta_{v_j}| \le N} E|\hat{\theta}_{\bar{v}} - \theta_{\bar{v}}|^2 +$$

$$+ \sum \theta_v^2 .$$

The analyticity →

$$\rightarrow \quad |a_{\bar{j}}| \leq c_1 e^{-c_2|\bar{j}|}, \quad c_2 > 0.$$

Hence

$$\sup_{\theta} E_{\theta} \|\hat{\theta}_{\varepsilon} - \theta\|^2 \leq$$

$$\leq c_3 \left(\varepsilon N^d + e^{-c_2 N} \right).$$

Take

$$N \sim \frac{1}{c_2} \ln \frac{1}{\varepsilon}$$

4.2. Low bounds

$\Pi_\varepsilon = \{X\}$ - Poisson r. set i

$(\mathcal{X}, \mathfrak{A}, \mu)$ with $\theta_\varepsilon = \frac{\theta}{\varepsilon}$ (w.r.

Θ is equipped wit a metric ρ.

<u>Theorem</u>. Suppose that for any $\delta >$ there exist a set $\{\theta_{i\delta},\ i=1,2,\dots N(\delta)\} \subset \Theta$, $\underline{\rho(\theta_i,\theta_j)>\delta}$.

Set

$$\delta(\varepsilon,\Theta) = \sup_{\theta_0,\{\theta_{i\delta}\}} \Big\{ \delta :$$

$$\frac{1}{\ln N(\delta)} \max_i \left\| \frac{\theta_{i\delta}-\theta_0}{\sqrt{\theta_0}} \right\|_{L_2(d\mu)} \le$$

$$\le \frac{1}{2}\varepsilon \Big\}$$

Then

$$\sup_\theta E_\theta\, l\left(\frac{\rho(\theta_\varepsilon,\theta)}{\delta(\varepsilon,\Theta)}\right) \ge \frac{1}{2}\, l\left(\frac{1}{2}\right)$$

Θ is equipped wit a metric ρ.

<u>Theorem</u>. Suppose that for any $\delta >$ there exist a set $\{\theta_{i\delta}, i=1,2,\dots N(\delta)\} \subset \Theta$, $\rho(\theta_i, \theta_j) > \delta$.

Set

$$\delta(\varepsilon, \Theta) = \sup_{\theta_0, \{\theta_{i\delta}\}} \Bigg\{ \delta : \frac{1}{\ln N(\delta)} \max_i \left\| \frac{\theta_{i\delta} - \theta_0}{\sqrt{\theta_0}} \right\|_{L_2(d\mu)} \leq \frac{1}{2} \varepsilon \Bigg\}$$

Then

$$\sup_\theta E_\theta \, l\left(\frac{\rho(\theta_\varepsilon, \theta)}{\delta(\varepsilon, \Theta)} \right) \geq \frac{1}{2} \, l\left(\frac{1}{2}\right)$$

Application to the proof of Th. 2.

$$\theta_a = e^{-\gamma n} \sum_{0}^{n-1} a_j P_j$$

$$a = (a_0, \dots a_{n-1}), \quad a_j = 0, 1$$

$$\#\{a\} = 2^n$$

The distance

$$\| \theta_a - \theta_{a'} \|_2 = e^{-\gamma n} \left(\sum | a_j - a'_j | \right)^{1/2}$$

$$\geq c\sqrt{n}\, e^{-\gamma n}$$

and $N \geq 2^{cn}$, $c > 0$

$$c \frac{1}{\ln N} e^{-2\gamma n} \cdot n \leq \frac{\varepsilon}{2\Sigma}$$

if $e^{-\gamma n} \leq c_1 \varepsilon \Rightarrow$

$$c\sqrt{n}\, e^{-\gamma n} \sim c_2 \sqrt{\varepsilon} \sqrt{\ln \frac{1}{\varepsilon}}.$$

6. $\Theta \subseteq \mathcal{B}(M, \sigma, \rho)$, $\rho_j \geq 1$.

$$\left[\ |\theta(x+iy)| \leq M e^{\sum \sigma_j |y_j|^{\rho}}\ \right].$$

Th. 5.

$$\sup_{\theta} E_{\theta} \|\theta - \theta_{\varepsilon}\|_p \leq c_p \left(\frac{\left(\ln \frac{1}{\varepsilon}\right)^{\frac{\rho-1}{\rho}}}{} \cdot \varepsilon \right)^{\frac{p-1}{p}},$$

$$1 \leq p \leq 2$$

$$\sup_{\theta} E_{\theta} \|\theta - \theta_{\varepsilon}\|_p \leq c_p \left(\varepsilon \left(\ln \frac{1}{\varepsilon}\right)^{\frac{\rho-1}{\rho}} \right)^{\frac{1}{2}},$$

$$2 \leq p < \infty$$

$$\sup_{\theta} E_{\theta} \|\theta - \theta_{\varepsilon}\|_{\infty} \leq$$

$$\leq c_{\infty} \left(\varepsilon \left(\ln \frac{1}{\varepsilon}\right)^{\frac{\rho-1}{\rho}} \right)^{\frac{1}{2}} \left(\ln \ln \frac{1}{\varepsilon}\right.$$