Parameter Estimation in Stochastic Hyperbolic Equations

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Joint work with Wei Liu

Stochastic wave equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial t} + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi;$$

zero initial and boundary conditions.

What is what:

- $u = u(t,x)$ — displacement of a string
- $a > 0$ — propagation speed
- $b \in \mathbb{R}$ — damping/amplification coefficient:

$$\frac{d}{dt}\int_0^\pi \big(u_t^2(t,x) + a^2 u_x^2(t,x)\big)dx = 2b\int_0^\pi u_t^2(t,x)dx$$

(amplification is $b > 0$; damping is $b < 0$).

- $\dot{W}(t,x)$ is space-time white noise.

Background

$u_{tt} = a^2 u_{xx} + b u_t + \dot{W}(t,x)$

Motivation:

- Guitar in the sand storm: Walsh (1984)
- Interest rate models: Santa-Clara and Sornette (2001)

Parameter Estimation:

- Huebner, Khasminskii, and Rozovskii (1992) — Heat equation
- Huebner and Rozovskii (1995) — Beyond the heat equation.

Our Objectives:

- Wave equation
  (1) Existence and uniqueness of solution.
  (2) Estimating a^2 and b.
- Beyond the wave equation.

The equation: Definition of solution

$u_{tt} = a^2 u_{xx} + b u_t + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi.$

Space-time white noise: $\dot{W}(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k \geq 1} \sin(kx) \dot{w}_k(t)$

$W_f(t) = \sum_{k \geq 1} f_k w_k(t),\ f \in L_2((0,\pi)).$

Sobolev spaces H^{γ}: $\|f\|_{\gamma}^2 = \sum_{k \geq 1} k^{2\gamma} f_k^2;$

Solution of the equation:

$u \in L_2\big(\Omega \times (0,T) \times (0,\pi)\big), \quad v \in L_2\big(\Omega; L_2((0,T); H^{-1})\big),$

$$(u(t,\cdot), f) = \int_0^t (v(t,\cdot), f)(s) ds,$$

$$(v(t,\cdot), f) = \int_0^t \big(a^2 (u(t,\cdot), f'') - 2b(v(t,\cdot), f)\big) ds + W_f(t)$$

The equation: Existence of solution

$u_{tt} = a^2 u_{xx} + b u_t + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi.$

Fundamental solution: $\varphi_k''(t) - b\varphi_k'(t) + k^2 a^2 \varphi(t) = 0$,
$\varphi_k(0) = 0,\ \varphi_k'(0) = 1.$

Fourier coefficients:

$$u_k(t) = \int_0^t \varphi_k(t-s)\,dw_k(s), \quad v_k(t) = \int_0^t \varphi_k'(t-s)\,dw_k(s).$$

Theorem.

$$u(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} u_k(t)\ \sin(kx), \quad v(t,x) = \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} v_k(t)\ \sin(kx),$$

$$u \in L_2\big(\Omega; L_2((0,T);H^{\gamma})\big);\ \ v \in L_2\big(\Omega; L_2((0,T);H^{\gamma-1})\big),\ \gamma < 1/2.$$

Note: If $4a^2k^2 > b^2$ and $\ell_k = \sqrt{4a^2k^2 - b^2}$, then

$$\varphi(t) = \frac{2}{\ell_k} \exp(bt/2)\,\sin(\ell_k t/2).$$

MLE, Part I

$u_{tt} = \theta_1 u_{xx} + \theta_2 u_t + \dot{W}(t,x),\ 0 < t < T,\ 0 < x < \pi.$

Observations: $\big(u_k(t),\ u_k'(t)\big),\ 0 < t < T,\ k = 1, \ldots, N.$

Notation: $v_k(t) = u_k'(t)$.

To get an MLE:

$$dv_k(t) = \left(-k^2\theta_1 \int_0^t v_k(s)ds + \theta_2 v_k(t) \right) dt + dw_k(t).$$

The result:

$$\hat{\theta}_{1,N} = \frac{B_{1,N}J_{2,N} + B_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}, \quad \hat{\theta}_{2,N} = \frac{B_{1,N}J_{12,N} + B_{2,N}J_{1,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}.$$

In case you want to know...

$$\hat{\theta}_{1,N} = \frac{B_{1,N}J_{2,N} + B_{2,N}J_{12,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}, \quad \hat{\theta}_{2,N} = \frac{B_{1,N}J_{12,N} + B_{2,N}J_{1,N}}{J_{1,N}J_{2,N} - J_{12,N}^2}.$$

where

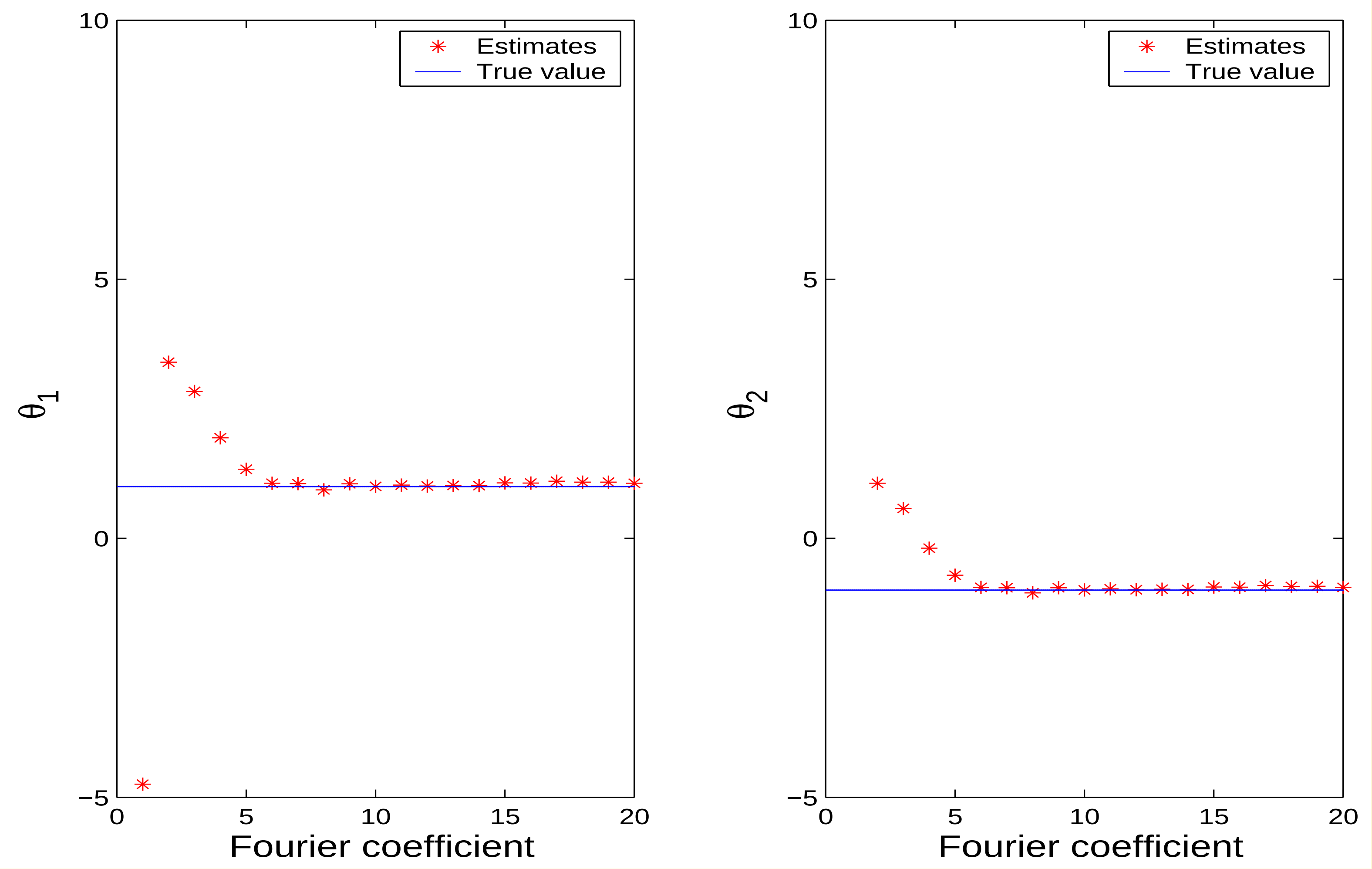
$$J_{1,N} = \sum_{k=1}^{N} k^4 \int_0^T u_k^2(t)dt, \quad J_{2,N} = \sum_{k=1}^{N} \int_0^T v_k^2(t)dt,$$

$$J_{12,N} = \sum_{k=1}^{N} k^2 \int_0^T u_k(t)v_k(t)dt;$$

$$B_{1,N} = -\sum_{k=1}^{N} k^2 \int_0^T u_k(t)dv_k(t), \; B_{2,N} = \sum_{k=1}^{N} \int_0^T v_k(t)dv_k(t).$$

Some pictures

$$u_{tt} = \mathbf{1} \cdot u_{xx} \, \mathbf{-1} \cdot u_t + \dot{W}$$



Theorem. We have

$$\lim_{N\to\infty} \hat{\theta}_{1,N} = \theta_1, \quad \lim_{N\to\infty} \hat{\theta}_{2,N} = \theta_2$$

with probability one and

$$\lim_{N\to\infty} N^{3/2}(\hat{\theta}_{1,N} - \theta_1) = \mathfrak{N}\left(0, \frac{3\theta_1}{C(\theta_2, T)}\right),$$

$$\lim_{N\to\infty} N^{1/2}(\hat{\theta}_{2,N} - \theta_2) = \mathfrak{N}\left(0, \frac{1}{C(\theta_2, T)}\right)$$

in distribution, where

$$C(\theta_2, T) = \begin{cases} \dfrac{e^{\theta_2 T} - \theta_2 T - 1}{2\theta_2^2}, & \text{if } \theta_2 \neq 0; \\ \dfrac{T^2}{4}, & \text{if } \theta_2 = 0. \end{cases}$$

$$\ddot{u} = (\mathcal{A}_0 + \theta_1 \mathcal{A}_1)u + (\mathcal{B}_0 + \theta_2 \mathcal{B}_1)\dot{u} + \dot{W}$$

or

$$\ddot{u}_k = (\kappa_k + \theta_1 \tau_k)u_k + (\rho_k + \theta_2 \nu_k)\dot{u}_k + \dot{w}_k.$$

Examples

$$u_{tt} = \theta_1 \mathbf{\Delta} u + \theta_2 u_t + \dot{W}$$

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$$u_{tt} + \mathbf{\Delta}^2 u = \theta_1 \mathbf{\Delta} u - \theta_2 \mathbf{\Delta}^2 u_t + \dot{W}$$

The result

$$\ddot{u} = (\mathcal{A}_0 + \theta_1 \mathcal{A}_1)u + (\mathcal{B}_0 + \theta_2 \mathcal{B}_1)\dot{u} + \dot{W}$$

Assume:

$$\lambda_k = \lambda_k(\theta_1) = -(\kappa_k + \theta_1 \tau_k) \nearrow +\infty, \quad \mu_k = \mu_k(\theta_2) = -(\rho_k + \theta_2 \nu_k) \geq C$$

$$\Psi_{1,N} = \sum_{k=1}^{N} \mathbb{E} \int_0^T \tau_k^2 u_k^2(t)dt \sim \sum_{k=1}^{N} \frac{\tau_k^2}{\lambda_k \mu_k} \nearrow \infty,$$

$$\Psi_{2,N} = \sum_{k=1}^{N} \mathbb{E} \int_0^T \nu_k^2 v_k^2(t)dt \sim \sum_{k=1}^{N} \frac{\nu_k^2}{\mu_k} \nearrow \infty.$$

Then, as $N \nearrow \infty$,

$$\sqrt{\Psi_{1,N}}(\widehat{\theta}_1^N - \theta_1) \sim \mathfrak{N}(0,1), \; \sqrt{\Psi_{2,N}}(\widehat{\theta}_2^N - \theta_2) \sim \mathfrak{N}(0,1).$$

Examples

1. $u_{tt} = \theta_1 \mathbf{\Delta} u + \theta_2 \mathbf{\Delta} u_t + \dot{W}$ in $G \subset \mathbb{R}^2$
$\lambda_k = \tau_k = \mu_k = \nu_k \sim k$;

$$N^{1/2}(\widehat{\theta}_1^N - \theta_1) \sim \mathfrak{N}(0, \sigma_1^2), \; N(\widehat{\theta}_2^N - \theta_2) \sim \mathfrak{N}(0, \sigma_2^2).$$

Note: $\mathbf{\Delta} u$ is more regular that $\mathbf{\Delta} u_t$.

2. $u_{tt} + \mathbf{\Delta}^2 u = \theta_1 \mathbf{\Delta} u + \mathbf{\Delta} u_t + \theta_2 u_t + \dot{W}$ in $G \subset \mathbb{R}^2$
$\lambda_k \sim k^2, \; \tau_k \sim k, \; \mu_k \sim k, \; \nu_k = 1$;

$$(\ln N)^{1/2}(\widehat{\theta}_1^N - \theta_1) \sim \mathfrak{N}(0, \sigma_1^2), \; (\ln N)^{1/2}(\widehat{\theta}_2^N - \theta_2) \sim \mathfrak{N}(0, \sigma_2^2).$$

Note 1: $\mathbf{\Delta} u$ is as regular as u_t.
Note 2: In $\mathbb{R}^1$, no consistency for either $\widehat{\theta}_1^N$ or $\widehat{\theta}_2^N$.

Some technical issues

Parabolic equation $du_k = -\alpha_k u_k dt + dw_k,\ \alpha_k \nearrow +\infty$

$$u_k(t) = \int_0^t e^{-\alpha_k(t-s)} dw_k(s)$$

$$\mathbb{E}\int_0^T u_k^2(t)dt \sim \frac{1}{\alpha_k},\ \mathrm{Var}\int_0^T u_k^2(t)dt \sim \frac{1}{\alpha_k^3}$$

Hyperbolic equation $u_k'' + \beta_k u_k' + \alpha_k u_k = \dot{w}_k,$

$$\alpha_k \nearrow +\infty, \quad \beta_k > -C(T)\ln \alpha_k$$

- Many cases to consider, and harder computations.
- In general, cannot do better than

$$\mathrm{Var}\int_0^T u_k^2(t)dt \sim \left(\mathbb{E}\int_0^T u_k^2(t)dt\right)^2$$

- If $\beta_k \to +\infty$, then $\mathbb{E}\int_0^T u_k^2(t)dt \sim \dfrac{1}{\alpha_k \beta_k}$.
- *Multi-channel* observation model: not necessarily an SPDE.

Equations that are second-order time are *more than twice* the fun of the first-order equations!