Efficient estimation for ergodic SDE models sampled at high frequency

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Discretely observed diffusion

$$dX_t = b(X_t;\theta)dt + \sigma(X_t;\theta)dW_t \qquad \theta \in \Theta \subseteq \mathbb{R}^p$$

State space: $D = (\ell, r)$, $-\infty \le \ell < r \le \infty$

Data: $X_{t_0}, \cdots, X_{t_n}$ $\qquad \Delta_i = t_i - t_{i-1}$

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Review papers:

Helle Sørensen (2004) Int. Stat. Rev.

Bibby, Jacobsen and Sørensen (2004)

Sørensen (2008a,b)

Quadratic estimating functions

Approximate likelihood function

$$L_n(\theta) \doteq M_n(\theta) = \prod_{i=1}^{n} q(\Delta_i, X_{t_{i-1}}, X_{t_i}; \theta)$$

$$p(\Delta, x, y; \theta) \doteq q(\Delta, x, y; \theta) = \frac{1}{\sqrt{2\pi\Phi(\Delta, x; \theta)}} \exp\left[\frac{(y - F(\Delta, x; \theta))^2}{2\Phi(\Delta, x; \theta)}\right]$$

$$F(x;\theta) = E_\theta(X_\Delta | X_0 = x) \quad \text{and} \quad \Phi(x;\theta) = \text{Var}_\theta(X_\Delta | X_0 = x)$$

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Approximate score function

$$\partial_\theta \log M_n(\theta) = \sum_{i=1}^{n} \left\{ \frac{\partial_\theta F(\Delta_i, X_{t_{i-1}}; \theta)}{\Phi(\Delta_i, X_{t_{i-1}}; \theta)} [X_{t_i} - F(\Delta_i, X_{t_{i-1}}; \theta)] \right.$$

$$\left. + \frac{\partial_\theta \Phi(\Delta_i, X_{t_{i-1}}; \theta)}{2\Phi(\Delta_i, X_{t_{i-1}}; \theta)^2} [(X_{t_i} - F(\Delta_i, X_{t_{i-1}}; \theta))^2 - \Phi(\Delta_i, X_{t_{i-1}}; \theta)] \right\}$$

Martingale estimating functions

$$G_n(\theta) = \sum_{i=1}^{n} g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta),$$

$$g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta) \quad = \quad \sum_{j=1}^{N} a_j(X_{t_{i-1}}, \Delta; \theta)[f_j(X_{t_i}; \theta) - E_\theta(f_j(X_{t_i}; \theta) \mid X_{t_{i-1}})]$$

$\uparrow$ p-dimensional $\uparrow$ real valued

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Bibby and Sørensen (1995,1996)

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- Simple expression for Godambe-Heyde optimal estimating function
- Approximates the score function, which is a P_θ-martingale
- Particular and most efficient instance of GMM

Approximate martingale estimating functions

$$G_n(\theta) = \sum_{i=1}^{n} g(\Delta_i, X_{t_i}, X_{t_{i-1}}; \theta),$$

$$g(\Delta, y, x; \theta) = \sum_{j=1}^{N} a_j(x, \Delta; \theta)[f_j(y; \theta) - \pi_\theta^\Delta f_j(x; \theta)]$$

Transition operator: $\pi_\theta^\Delta f_j(x;\theta) = E_\theta(f_j(X_\Delta;\theta) \mid X_0 = x)$

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Dorogovcev (1976), Prakasa Rao (1988), Florens-Zmirou (1989),
Yoshida (1992), Chan et al. (1992), Kessler (1997),
Kelly, Platen and Sørensen (2004)

Jacobi diffusion

Larsen & Sørensen (2007):

$$dX_t = -\beta[X_t - (m + \gamma z)]dt + \sigma\sqrt{z^2 - (X_t - m)^2}dW_t$$

The eigenfunctions are given in terms of Jacobi polynomials

Asymptotic information at $(\beta, \gamma, \sigma^2) = (0.02, 0, 0.01)$:

Eigenfunction no.	1	2	1 & 2
Inf. for $\hat{\beta}$	47.4	44.8	49.2
Inf. for $\hat{\sigma}^2$	0	759	5016

For optimal estimating functions based on more than two eigenfunctions, the information is not increased by more than 1 - 3 per cent

High frequency asymptotics

$$dX_t = b(X_t;\alpha)dt + \sigma(X_t;\beta)dW_t$$

$\theta = (\alpha, \beta) \in \Theta \subseteq \mathbb{R}^2$ θ_0 is the true parameter value

State space: (ℓ, r)

Ergodic with invariant measure μ_θ.

Data: $X_{t_0^n}, \ldots, X_{t_n^n}$ $t_i^n = i\Delta_n$, $i = 0, \ldots, n$.

High frequency asymptotic scenario:

$$n \to \infty \qquad \Delta_n \to 0 \qquad n\Delta_n \to \infty$$

Condition 1: the process

- $\int_{x^{\#}}^{r} s(x;\theta)dx = \int_{\ell}^{x^{\#}} s(x;\theta)dx = \infty$ and $\int_{\ell}^{r} x^k \tilde{\mu}_\theta(x)dx < \infty$

for all $k \in \mathbb{N}$, where $x^{\#}$ is an arbitrary point in (ℓ, r),

$s(x;\theta) = \exp\left(-2\int_{x^{\#}}^{x} \frac{b(y;\alpha)}{\sigma^2(y;\beta)}dy\right)$ and $\tilde{\mu}_\theta(x) = [s(x;\theta)\sigma^2(x;\beta)]^{-1}$

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- $b, \sigma \in C_{p,4,1}((\ell, r) \times \Theta)$

Technical condition

$C_{p,k_1,k_2,k_3}(\mathbb{R}_+ \times (\ell, r)^2 \times \Theta)$ is the class of real functions $f(t, y, x; \theta)$ satisfying that

- $f(t, y, x; \theta)$ is k_1 times continuously differentiable with respect t, k_2 times continuously differentiable with respect y, and k_3 times continuously differentiable with respect α and with respect to β

- f and all partial derivatives $\partial_t^{i_1} \partial_y^{i_2} \partial_\alpha^{i_3} \partial_\beta^{i_4} f$, $i_j = 1, \ldots k_j$, $j = 1, 2$, $i_3 + i_4 \leq k_3$, are of polynomial growth in x and y uniformly for θ in a compact set (for fixed t)

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$C_{p,k_1,k_2}((\ell, r) \times \Theta)$ for $f(y; \theta)$ and $C_{p,k_1,k_2}((\ell, r)^2 \times \Theta)$ for $f(y, x; \theta)$

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$$|R(\Delta, y, x; \theta)| \leq F(y, x; \theta)$$

F is of polynomial growth in y and x uniformly for θ in a compact set

Condition 2: the estimating function

$$G_n(\theta) = \sum_{i=1}^{n} g(\Delta_n, X_{t_i^n}, X_{t_{i-1}^n}; \theta) \qquad 2-\text{dimensional}$$

- For some $\kappa \geq 2$

$$E_\theta(g(\Delta_n, X_{t_i^n}, X_{t_{i-1}^n}; \theta) \,|\, X_{t_{i-1}^n}) = \Delta_n^\kappa R(\Delta_n, X_{t_{i-1}^n}; \theta) \quad \text{for all } \theta \in \Theta$$

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- The function $g(\Delta, y, x; \theta)$ has an expansion in powers of Δ

$$g(\Delta, y, x; \theta) = g(0, y, x; \theta) + \Delta g^{(1)}(y, x; \theta) + \frac{1}{2}\Delta^2 g^{(2)}(y, x; \theta) + \Delta^3 R(\Delta, y, x; \theta)$$

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- The function $R(\Delta, y, x; \theta)$ in the expansion of g is differentiable with respect to θ, and

$$\begin{aligned} g(\Delta, y, x; \theta) &\in C_{p,6,2}((\ell, r)^2 \times \Theta) \text{ for fixed } \Delta, \\ g^{(1)}(y, x; \theta) &\in C_{p,4,2}((\ell, r)^2 \times \Theta), \\ g^{(2)}(y, x; \theta) &\in C_{p,2,2}((\ell, r)^2 \times \Theta) \end{aligned}$$

Theorem 1

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- Conditions 1 and 2

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- The identifiability condition that

$$\gamma(\theta,\theta_0) = \int_\ell^r [b(x,\alpha_0) - b(x,\alpha)]\partial_y g(0,x,x;\theta)\mu_{\theta_0}(x)dx + \frac{1}{2}\int_\ell^r [\sigma^2(x,\beta_0) - \sigma^2(x,\beta)]\partial_y^2 g(0,x,x;\theta)\mu_{\theta_0}(x)dx \neq 0$$

for all $\theta \neq \theta_0$

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- The matrix $S = \int_\ell^r A_{\theta_0}(x)\mu_{\theta_0}(x)dx$ is invertible, where

$$A_\theta(x) = \begin{pmatrix} \partial_\alpha b(x;\alpha)\partial_y g_1(0,x,x;\theta) & \frac{1}{2}\partial_\beta \sigma^2(x;\beta)\partial_y^2 g_1(0,x,x;\theta) \\ \\ \partial_\alpha b(x;\alpha)\partial_y g_2(0,x,x;\theta) & \frac{1}{2}\partial_\beta \sigma^2(x;\beta)\partial_y^2 g_2(0,x,x;\theta) \end{pmatrix}$$

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Then a consistent estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ that solves the estimating equation $G_n(\theta) = 0$ exists and is unique in any compact subset of Θ containing θ_0 with a probability that goes to one as $n \to \infty$.

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For a martingale estimating function or more generally if $n\Delta^{2\kappa-1} \to 0$,

$$\sqrt{n\Delta_n}(\hat{\theta}_n - \theta_0) \xrightarrow{\mathcal{D}} N_2\left(0, S^{-1}V_0(S^T)^{-1}\right)$$

under P_{θ_0}, where

$$V_0 = \int_\ell^r \sigma^2(x, \beta_0)\partial_y g(0, x, x; \theta_0)\partial_y g(0, x, x; \theta_0)^T \mu_{\theta_0}(x)dx.$$

Optimal rate

Gobet (2002):

A discretely sampled diffusion is LAN in the high frequency asymptotics considered here, and the optimal rate of convergence is

For parameters in the drift coefficient: $\sqrt{n\Delta_n}$

For parameters in the diffusion coefficient: $\sqrt{n}$

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Jacobsen's condition:

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for all $x \in (\ell, r)$ and $\theta \in \Theta$

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Jacobsen (2001): small Δ-optimality

Theorem 2

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- The identifiability condition that

$$\int_\ell^r [b(x,\alpha_0) - b(x,\alpha)]\partial_y g_1(0,x,x;\theta)\mu_{\theta_0}(x)dx \quad \neq \quad 0 \qquad \text{when } \alpha \neq \alpha_0$$

$$\int_\ell^r [\sigma^2(x,\beta_0) - \sigma^2(x,\beta)]\partial_y^2 g_2(0,x,x;\theta)\mu_{\theta_0}(x)dx \quad \neq \quad 0 \qquad \text{when } \beta \neq \beta_0$$

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- $S_{11} \neq 0$ and $S_{22} \neq 0$

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- $S_{11} \neq 0$ and $S_{22} \neq 0$
- $\partial_y g_2(0,x,x;\theta) = 0$

HF-asymptotics: Theorem 2

Then a consistent estimator $\hat{\theta}_n = (\hat{\alpha}_n, \hat{\beta}_n)$ that solves the estimating equation $G_n(\theta) = 0$ exists and is unique in any compact subset of Θ containing θ_0 with a probability that goes to one as $n \to \infty$

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If, moreover,

$$\partial_\alpha \partial_y^2 g_2(0, x, x; \theta) = 0,$$

then for a martingale estimating function or more generally if $n\Delta^{2(\kappa-1)} \to 0$,

$$\begin{pmatrix} \sqrt{n\Delta_n}(\hat{\alpha}_n - \alpha_0) \\ \sqrt{n}(\hat{\beta}_n - \beta_0) \end{pmatrix} \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} W_1/S_{11}^2 & 0 \\ 0 & W_2/S_{22}^2 \end{pmatrix} \right)$$

where

$$W_1 = \int_\ell^r \sigma^2(x; \beta_0)[\partial_y g_1(0, x, x; \theta_0)]^2 \mu_{\theta_0}(x) dx$$

and

$$W_2 = \frac{1}{2} \int_\ell^r \sigma^4(x; \beta_0)[\partial_y^2 g_2(0, x, x; \theta_0)]^2 \mu_{\theta_0}(x) dx$$

Efficiency - 1

Gobet (2002):

A discretely sampled diffusion is LAN in the high frequency asymptotics considered here, and the Fisher information is

$$\begin{pmatrix} \int_\ell^r \frac{(\partial_\alpha b(x;\alpha_0))^2}{\sigma^2(x;\beta_0)} \mu_{\theta_0}(x)dx & 0 \\ 0 & \frac{1}{2}\int_\ell^r \left[\frac{\partial_\beta \sigma^2(x;\beta_0)}{\sigma^2(x;\beta_0)}\right]^2 \mu_{\theta_0}(x)dx \end{pmatrix}$$

Efficiency - 1

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Condition for efficiency:

$$\partial_y g_1(0,x,x;\theta) = \partial_\alpha b(x;\alpha)/\sigma^2(x;\beta) \qquad \partial_y^2 g_2(0,x,x;\theta) = \partial_\beta \sigma^2(x;\beta)/\sigma^4(x;\beta)$$

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Jacobsen (2001): small Δ-optimality

Quadratic martingale estimating functions

$$\sum_{i=1}^{n} \begin{pmatrix} a_1(X_{t_{i-1}^n}, \Delta; \theta)(X_{t_i^n} - F(\Delta, X_{t_{i-1}^n}; \theta)) \\ \\ a_2(X_{t_{i-1}^n}, \Delta; \theta) \left[(X_{t_i^n} - F(\Delta, X_{t_{i-1}^n}; \theta))^2 - \phi(\Delta, X_{t_{i-1}^n}; \theta) \right] \end{pmatrix}$$

$$\begin{aligned} F(\Delta, x; \theta) &= E_\theta(X_\Delta | X_0 = x) = x + O(\Delta) \\ \phi(\Delta, x; \theta) &= \text{Var}_\theta(X_\Delta | X_0 = x) = O(\Delta) \end{aligned}$$

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$$\partial_y g_2(0, y, x; \theta) = 2a_2(x, \Delta; \theta)(y - x)$$ Jacobsen's condition satisfied

$$\partial_y g_1(0, x, x; \theta) = a_1(x, 0; \theta) = \partial_\alpha b(x; \alpha)/\sigma^2(x; \beta)$$ Approximately

$$\partial_y^2 g_2(0, x, x; \theta) = 2a_2(x, 0; \theta) = \partial_\beta \sigma^2(x; \beta)/\sigma^4(x; \beta)$$ optimal

Martingale estimating functions

$$g(\Delta, y, x; \theta) = \sum_{j=1}^{N} a_j(x, \Delta; \theta)[f_j(y;\theta) - \pi_\theta^\Delta f_j(x;\theta)]$$

$$= A(x, \Delta_n; \theta)[f(y;\theta) - \pi_\theta^{\Delta_n} f(x;\theta)]$$

$$G_n(\theta) = \sum_{i=1}^{n} A(X_{t_{i-1}^n}, \Delta_n; \theta)[f(X_{t_i^n};\theta) - \pi_\theta^{\Delta_n} f(X_{t_{i-1}^n};\theta)]$$

$f(y;\theta) = (f_1(y;\theta), \ldots, f_N(y;\theta))^T$

$A(x, \Delta; \theta)$ a $2 \times N$-matrix of weights

$\pi_\theta^\Delta f(x;\theta) = E_\theta(f(X_\Delta;\theta) \mid X_0 = x)$ is the transition operator

Efficiency - 2

Suppose Condition 1 is satisfied and that the functions f_j are twice continuously differentiable.

A sufficient condition that it is possible to find a specification of the weight matrix $A(x, \Delta; \theta)$ such that the estimating function $G_n(\theta)$ gives estimators that are rate optimal and efficient is that

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- $N \geq 2$
- and that the matrix

$$D(x) = \begin{pmatrix} f_1'(x) & f_1''(x) \\ \\ f_2'(x) & f_2''(x) \end{pmatrix}$$

is invertible for μ_θ-almost all x.

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Jacobsen (2002)

For a d-dimensional diffusion: $N \geq d(d+3)/2$

Efficiency - 3

Suppose Condition 1 is satisfied, that the functions f_j are six times continuously differentiable, that $N \geq 2$ and that $D(x)$ is invertible for μ_θ-almost all x.

If $A^*(x, \Delta; \theta)$ is the Godambe-Heyde optimal weight-matrix, then

$$g^*(\Delta, y, x; \theta) = \begin{pmatrix} 1 & 0 \\ & \\ 0 & 2\Delta \end{pmatrix} A^*(x, \Delta; \theta)[f(y) - \pi_\theta^\Delta f(x)]$$

satisfies that for all $x \in (\ell, r)$ and $\theta \in \Theta$

$$\partial_y g_2^*(0, x, x; \theta) = 0$$

and

$$\partial_y g_1^*(0, x, x; \theta) = \partial_\alpha b(x; \alpha)/\sigma^2(x; \beta) \qquad \partial_y^2 g_2^*(0, x, x; \theta) = \partial_\beta \sigma^2(x; \beta)/\sigma^4 \beta)$$