Continuity of stationary solutions of delay differential equations driven by Lévy processes

(joint work with Oleg Butkovsky)

Alexander Gushchin

[1]Steklov Mathematical Institute

[2]Higher School of Economics

Statistique Asymptotique des Processus Stochastiques X

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Main equation

Let λ be a finite signed measure on a finite interval $J = [-r, 0]$, $r \geqslant 0$. Consider the equation

$$\begin{aligned} X(t) &= X(0) + \int_0^t \int_J X(s+u)\,\lambda(du)\,ds + Z(t), && t \geqslant 0, \\ X(t) &= X_0(t), && t \in J. \end{aligned} \tag{1}$$

Here $Z = (Z(t),\ t \geqslant 0)$ is a real-valued process with independent stationary increments starting from 0 and having càdlàg trajectories, i.e. Z is a Lévy process, and $X_0 = (X_0(t),\ t \in J)$ is an initial process with càdlàg trajectories, independent of Z.

Notation

Let $\mathbb{D}_0 = \mathbb{D}([-r, 0], \mathbb{R})$ be the Skorokhod space of càdlàg real-valued functions on $[-r, 0]$ with the usual Skorokhod topology.
Denote by $\mathbb{M}$ the set of all finite signed measures on $[-r, 0]$.
Define as usual, $X_t := \{X(t+s), -r \leqslant s \leqslant 0\}$, $t \geqslant 0$.
Equation (1) has a unique strong solution and the process $X := (X_t)_{t\geqslant 0}$ is a homogeneous Markov process taking values in $(\mathbb{D}_0, \mathcal{B}(\mathbb{D}_0))$.
To emphasize that X is a solution to (1) with the delay measure $\lambda \in \mathbb{M}$ and the initial distribution $\alpha := \mathrm{Law}(X_0)$ we write $X = X^{(\lambda,\alpha)}$.
If the initial condition is deterministic, i.e., $\alpha = \delta_y$, $y \in \mathbb{D}$, then we simplify the notation and write $X = X^{(\lambda,y)}$.

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Assumption A1. The characteristic equation

$$h_\lambda(z) := z - \int_{[-r,0]} e^{zu} \lambda(du) = 0 \qquad (2)$$

has no complex solutions z with $\mathrm{Re}\, z \geqslant 0$.

Assumption A2. We have

$$\int_{|y|>1} \log |y| \, F(dy) < \infty,$$

where F is the Lévy measure of Z.

It was shown by G. and Küchler (2000) that the process X has an invariant measure if and only if **A1** and **A2** are satisfied. Moreover, in this case the invariant measure is unique. We assume that these conditions hold and denote the invariant measure of $X^{(\lambda)}$ by π^{λ}. We denote by $\mathbb{M}^{-}$ the set of all finite signed measures on $[-r, 0]$ that satisfy **A1**.

Additional assumption

We study the relationship between invariant measures of this SDDE under an additional condition on the Lévy process Z.

Assumption A3. The Lévy process Z has a non-zero Gaussian component.

This assumption implies that a Gaussian component X^c (which is also the continuous martingale component of X and Z) has mean zero and variance $\sigma^2 t$, $\sigma > 0$.

Formula for the likelihood ratio

For any interval $I \subset \mathbb{R}$ we denote
$\mathsf{P}_I^{(\lambda,\alpha)} := \mathrm{Law}(\{X^{(\lambda,\alpha)}(s),\ s \in I\})$, where $\lambda \in \mathbb{M}$.
Let $X = X^{(\lambda,\alpha)}$.

Proposition

Assume **A3**. *Let α and β be two probability measures on $\mathbb{D}_0$. If $\beta \ll \alpha$ (resp. $\beta \sim \alpha$) then, for any $T > 0$ and any delay measures $\lambda, \mu \in \mathbb{M}$ we have $\mathsf{P}_{[-r,T]}^{(\mu,\beta)} \ll \mathsf{P}_{[-r,T]}^{(\lambda,\alpha)}$ (resp. $\mathsf{P}_{[-r,T]}^{(\mu,\beta)} \sim \mathsf{P}_{[-r,T]}^{(\lambda,\alpha)}$) and*

$$\frac{d\mathsf{P}_{[-r,T]}^{(\mu,\beta)}}{d\mathsf{P}_{[-r,T]}^{(\lambda,\alpha)}}(X) = \frac{d\beta}{d\alpha}(X_0) \exp\left\{ \frac{1}{\sigma^2} \int_0^T A(t)\, dX^c(t) - \frac{1}{2\sigma^2} \int_0^T A^2(t)\, dt \right\},$$

where

$$A(t) = \int_J X(t+u)\,(\mu - \lambda)(du), \quad \lambda,\ \mu \in \mathbb{M}.$$

Proposition

Assume **A2** *and* **A3**. *Then for any delay measures* $\lambda, \mu \in \mathbb{M}^-$, *the corresponding invariant measures* π^λ *and* π^μ *are equivalent.*

Proof: By the previous proposition,

$$\mathsf{P}_{[-r,r]}^{(\lambda,\pi(\lambda))} \sim \mathsf{P}_{[-r,r]}^{(-\delta_0,\pi(\lambda))},$$

therefore,

$$\mathsf{P}_{[0,r]}^{(\lambda,\pi(\lambda))} \sim \mathsf{P}_{[0,r]}^{(-\delta_0,\pi(\lambda))}.$$

However, $\mathsf{Law}\big(X^{(-\delta_0,\pi(\lambda))}(0)\big) = \mathsf{Law}\big(X^{(\lambda)}(0)\big)$ is a convolution of a nondegenerate Gaussian distribution and another one; hence it is equivalent to a nondegenerate Gaussian distribution. Therefore,

$$\mathsf{P}_{[0,r]}^{(-\delta_0,\pi(\lambda))} \sim \mathsf{P}_{[0,r]}^{(-\delta_0,\pi(-\delta_0))}.$$

Combining. we get

$$\pi(\lambda) = \mathsf{P}_{[0,r]}^{(\lambda,\pi(\lambda))} \sim \mathsf{P}_{[0,r]}^{(-\delta_0,\pi(-\delta_0))} = \pi(-\delta_0).$$

Our purpose

Our goal now is to show that invariant measures are close when delay measure are close. This was done in G. & Küchler (2003, 2011) in the case where Z is a Brownian motion.
$\| \cdot \|_{\mathrm{TV}}$ stands for the total variation norm.
We shall also need a Wasserstein-type distance between probability measures on $\mathbb{D}_0$. Namely,

$$W_d(\alpha, \beta) := \inf \mathsf{E} d(X, Y),$$

where

$$d(x, y) := \sup_{t \in J} |x(t) - y(t)| \wedge 1$$

and the infimum is taken over all random elements X and Y distributed as α and β, correspondingly.
It is known that $W_d(\alpha, \beta) \leqslant \|\beta - \alpha\|_{\mathrm{TV}}$.

Main result

Theorem (Butkovsky & G. (2014))

Let λ, λ_1, λ_2, ... be a sequence of finite signed measures on $[-r, 0]$, satisfying **A1**. *Suppose that the Lévy process Z satisfies* **A2–A3**. *If λ_n converges weakly to λ as $n \to \infty$ then*

$$\lim_{n\to\infty} d_{TV}(\pi^{\lambda_n}, \pi^{\lambda}) = 0.$$

Since the equation (1) involves no stochastic integrals and is treated pathwise, we will formulate a number of results for solutions of the equation (1) with deterministic Z and X_0.

Fundamental solution

Given a measure λ, we call a function $x_0\colon [-r,\infty[\to \mathbb{R}$ the fundamental solution of the homogeneous equation

$$\begin{aligned} X(t) &= X(0) + \int_0^t \int_J X(s+u)\,\lambda(du)\,ds, && t \geqslant 0, \\ X(t) &= X_0(t), && t \in J, \end{aligned} \qquad (3)$$

if it is the solution of (3) corresponding to the initial condition

$$X_0(t) = \begin{cases} 1, & t = 0, \\ 0, & -r \leqslant t < 0. \end{cases}$$

In other words, a function $x_0(t)$, $t \geqslant -r$, is the fundamental solution of (3) if it is absolutely continuous on $\mathbb{R}_+$, $x_0(t) = 0$ for $t < 0$, $x_0(0) = 1$, and

$$\dot{x}_0(t) = \int_J x_0(t+u)\,\lambda(du) \tag{4}$$

for Lebesgue-almost all $t > 0$.

The solution of (1) can be represented via the fundamental solution x_0 of (3):

$$X(t) = \begin{cases} x_0(t)X_0(0) + \int\limits_J \int\limits_u^0 X_0(s)x_0(t+u-s)\,ds\,\lambda(du) \\ \qquad + \int\limits_{[0,t]} Z(t-s)\,dx_0(s), & t \geqslant 0, \\ X_0(t), & t \in J. \end{cases} \tag{5}$$

Remark

The domain of integration in the last integral in (5) includes zero:

$$\int_{[0,t]} Z(t-s)\,dx_0(s) = Z(t) + \int_{]0,t]} Z(t-s)\,dx_0(s).$$

Characteristic equation

The asymptotic behaviour of solutions of the equations (1) and (3) for $t \to \infty$ is connected with the set of complex solutions of the characteristic equation

$$h(z) := z - \int_J e^{zu} \lambda(du) = 0. \tag{6}$$

Note that a complex number z solves (6) if and only if $(e^{zt},\ t \geqslant -r)$ solves (3) for the initial condition $X_0(t) = e^{zt}$, $t \in J$.

The set $\Lambda := \{z \in \mathbb{C} \mid h(z) = 0\}$ is not empty; moreover, it is infinite except the case where λ is concentrated at 0. Since $h(\cdot)$ is an entire function, Λ consists of isolated points only. It is easy to check that $z_n \in \Lambda$ and $|z_n| \to \infty$ imply $\mathrm{Re}\ z_n \to -\infty$, thus the set $\{z \in \Lambda \mid \mathrm{Re}\ z \geqslant c\}$ is finite for every $c \in \mathbb{R}$. In particular, it holds

$$v_0 = v_0(\lambda) := \max\{\,\mathrm{Re}\ z \mid z \in \Lambda\,\} < \infty. \tag{7}$$

It is easy to check from (4) that $1/h(z)$ is the Laplace transform of $(x_0(t),\ t \geqslant 0)$ at least if $\mathrm{Re}\ z$ is large enough. (In fact,

$$1/h(z) = \int_0^\infty e^{-zt} x_0(t)\, dt$$

if $\mathrm{Re}\ z > v_0$.) Applying a standard method based on the inverse Laplace transform and Cauchy's residue theorem, it is easy to show that for any $\gamma > v_0$

$$x_0(t) = o(e^{\gamma t}) \quad \text{as} \quad t \to \infty.$$

Formula for the solution of the equation (1)

Return to the case where Z is a Lévy process. Integration by parts gives

$$\int_{[0,t]} Z(t-s)\,dx_0(s) = \int_0^t x_0(t-s)\,dZ(s).$$

Thus, using (5), any solution of the equation (1) can be written in the form

$$X(t) = x_0(t)X_0(0) + \int_J \int_u^0 X_0(s)x_0(t+u-s)\,ds\,\lambda(du) + \int_0^t x_0(t-s)\,dZ(s), \tag{8}$$

Note also that

$$\int_0^t x_0(t-s)\,dZ(s) \overset{d}{=} \int_0^t x_0(s)\,dZ(s). \tag{9}$$

Let $Z = (Z(t),\ t \geqslant 0)$ and $\widetilde{Z} = (\widetilde{Z}(t),\ t \geqslant 0)$ be two independent Lévy processes with the same characteristics (b, c, F). We define a two-sided Lévy process $(Z(t),\ t \in \mathbb{R})$ by

$$Z(t) = \begin{cases} Z(t), & t \geqslant 0, \\ -\widetilde{Z}(-t-0), & t < 0, \end{cases}$$

and put

$$X(t) = \int_{-\infty}^{t} x_0(t-s)\, dZ(s).$$

The process $X = (X(t),\ t \geqslant -r)$ is well defined up to a modification under Assumptions **A1** and **A2** and is a stationary solution to (1).

Outline of the proof of the main theorem

The first part of the proof consists of a series of (more or less standard) lemmas having the aim to show that

$$\lim_{n\to\infty} W_d\big(\pi(\lambda_n), \pi(\lambda)\big) = 0.$$

The second part uses a coupling-type argument for stochastic functional equations due to Es-Sarhir, von Renesse, Scheutzow (2009).
Notation: $\pi_n := \pi(\lambda_n)$, $\pi := \pi(\lambda)$.

For $M \geqslant 0$ consider the "ball" in $\mathbb{D}_0$

$$B_M := \{x \in \mathbb{D}_0 \colon \sup_{t \in J} |x(t)| \leqslant M\}.$$

Lemma

Assume that a sequence of measures $\{\lambda_n\}$, $\lambda_n \in \mathbb{M}^-$, *converges weakly to a measure* $\lambda \in \mathbb{M}^-$ *as* $n \to \infty$. *Then for any* $\varepsilon > 0$ *there exists* $M > 0$ *such that*

$$\pi_n(\mathbb{D} \setminus B_M) < \varepsilon \quad \textit{for all } n.$$

Second lemma

Lemma

Under the assumptions of Theorem, we have, for any $M, N > 0$, $t \geqslant 0$,

$$\left\|\mathsf{P}^{(\lambda)}_{[-r,t]} - \mathsf{P}^{(\lambda_n,\pi)}_{[-r,t]}\right\|_{\mathrm{TV}} \leqslant C(n; N, t) + 2\pi(\mathbb{D} \setminus B_M) + 2 \sup_{x \in B_M} \mathsf{P}\Big(\sup_{s \in [-r,t]} \big|X^{(\lambda,x)}(s)\big| \geqslant N\Big),$$

where, for every N and t,

$$C(n; N, t) \to 0, \quad n \to \infty.$$

Third lemma

Lemma

Under the assumptions of Theorem, we have, for any $M > 0$ and $t \geqslant 2r$,

$$W_d(\mathsf{P}^{(\lambda)}_{[t-r,t]}, \mathsf{P}^{(\lambda,\lambda_n)}_{[t-r,t]}) \leqslant (1+\|\lambda\|_{\mathrm{TV}}r)(Mw^{(\lambda)}(t)+\pi(\mathbb{D}\setminus B_M)+\pi_n(\mathbb{D}\setminus B_M)),$$

where $w^{(\lambda)}(t) \to 0$ as $t \to \infty$.

Completing the first part of the proof

Combining the statements of previous lemmas and taking

$$\limsup_{M\to\infty} \limsup_{t\to\infty} \limsup_{N\to\infty} \limsup_{n\to\infty},$$

we get

$$\lim_{n\to\infty} W_d\big(\pi(\lambda_n), \pi(\lambda)\big) = 0.$$

Fourth lemma

Lemma

For any $x, y \in \mathbb{D}_0$ *and* $t > t_0 + r$, $t_0 > 0$ *being fixed,*

$$\|\mathsf{P}_t^{(\lambda,x)} - \mathsf{P}_t^{(\lambda,y)}\|_{\mathrm{TV}} \leqslant \frac{\sqrt{2}|x(0) - y(0)|}{\sigma t_0^{1/2}} + \sqrt{2}\|\lambda\|_{\mathrm{TV}}(t_0 + r)^{1/2} \sup_{u \in J} |x(u) - y(u)|\sigma^{-1}.$$

Sketch of the proof: coupling

Let $X = X^{(\lambda,x)}$,

$$R(t) = \begin{cases} x(t) - y(t), & \text{if } t \in J; \\ R(0)\chi(t), & \text{if } t \in [0, t_0]; \\ 0, & \text{if } t \geqslant t_0, \end{cases}$$

where $\chi(t)$, $0 \leqslant t \leqslant t_0$, is an absolutely continuous decreasing function, $\chi(0) = 1$, $\chi(t_0) = 0$, $Y(t) := X(t) - R(t)$, $Q(t) := R'(t) - \int_J R(t+u)\lambda(du)$. Note that $Q(t) = 0$ for $t \geqslant t_0 + r$. Then we can rewrite $\widetilde{Y}$ as the solution of

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t \int_J Y(s+u)\,\lambda(du)\,ds + \widetilde{Z}(t), && t \geqslant 0, \\ Y(t) &= y(t), && t \in J, \end{aligned}$$

where $\widetilde{Z}(t) = Z(t) - \int_0^t Q(s)\,ds$.

By Girsanov's theorem, $\mathrm{Law}(\widetilde{Z}|\mathrm{Q}) = \mathrm{Law}(Z|\mathrm{P})$, where Q is a new measure with the density process

$$\exp\Big(\sigma^{-2}\int_0^t Q(s)dZ^c(s) - (2\sigma^2)^{-1}\int_0^t Q^2(s)\,ds\Big)$$

with respect to P. This means that the law of Y under Q coincides with the law of $X^{(\lambda,y)}$.

Now it follows, for $t \geqslant t_0 + r$,

$$\begin{aligned}\|\mathrm{P}_t^{(\lambda,x)} - \mathrm{P}_t^{(\lambda,y)}\|_{\mathrm{TV}} &= 2\sup_A |\mathrm{P}(X_t \in A) - \mathrm{Q}(Y_t \in A)| \\ &\leqslant \|\mathrm{P}-\mathrm{Q}\|_{\mathrm{TV}} \\ &\leqslant \Big(2\sigma^{-2}\int_0^{t_0+r} Q^2(s)\,ds\Big)^{1/2} \\ &\leqslant \dots\end{aligned}$$

Rest of the proof of Theorem

It follows from the definition of W_d and Chebyshev's inequality that there exist random elements Y^n and Y distributed as π_n and π, respectively, such that

$$\mathrm{P}(\sup_{u \in J} |Y^n(u) - Y(u)| > \gamma) \leqslant W_d(\pi_n, \pi)/\gamma,$$

for any $\gamma \in (0, 1)$. The fourth lemma now yields

$$\|\mathrm{P}_t^{(\lambda,\lambda_n)} - \mathrm{P}_t^{(\lambda)}\|_{\mathrm{TV}} \leqslant \frac{W_d(\pi_n, \pi)}{\gamma} + \frac{\sqrt{2}\gamma}{\sigma(t-r)^{1/2}} + \sqrt{2}\|\lambda\|_{\mathrm{TV}} t^{1/2} \gamma \sigma^{-1}.$$

Using the first part of the proof and taking consequently limits in the inequality above as $n \to \infty$ and $\gamma \to 0$, we get

$$\lim_{n\to\infty} \|\mathrm{P}_t^{(\lambda,\lambda_n)} - \mathrm{P}_t^{(\lambda)}\|_{\mathrm{TV}} = 0.$$

Finally, the same argument as in the second lemma shows that

$$\lim_{n\to\infty} \|\mathrm{P}_t^{(\lambda,\lambda_n)} - \mathrm{P}_t^{(\lambda_n)}\|_{\mathrm{TV}} = 0.$$

Thank you for your attention!