Detecting infinitesimal lead-lag effects from noisy high-frequency data

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Outline

- Introduction
- Model
- Formulation of the problem
- Main results
- Simulation study
- Conclusions

Introduction

- Lead-lag effect
  - One process (“leader”) is correlated with another process (“lagger”) at later times
- The investigation of such a relationship has a long history in economics
- Classically, it has been examined at moderate frequencies (day, week, month, quarter, . . .) using the statistics for discrete-time (stationary) processes
  - Ex. spectral analysis (cf. Granger & Hatanaka, 1964), distributed lags (cf. Griliches, 1967), cross-autocorrelations (cf. Campbell *et al.*, 1997), . . .

Introduction

- Recently, lead-lag effects at (ultra) high-frequencies have begun to attract notice (e.g. Huth & Abergel, 2014)
- For high-frequency data, discrete-time process modeling tends to be poor; a discretely observed continuous-time process is often more appropriate
- However, there are not many theoretical results on the statistical inference for lead-lag effects in such a setting
- The aim of this talk is to contribute to this area

Introduction

- There are a few approaches to express lead-lag effects
  - Hoffmann, Rosenbaum & Yoshida (2013)

    Model: continuous semimartingale

    Estimation: Hayashi-Yoshida estimator
  - Robert & Rosenbaum (2010)

    Model: continuous Gaussian martingale

    Estimation: random matrix theory
  - Bacry, Delattre, Hoffmann & Muzy (2013)

    Model: Hawkes process

    Estimation: parameter estimation

Introduction

- This talk focuses on the Hoffmann-Rosenbaum-Yoshida model and investigates
  - how to deal with observation noise
  - how to detect “small” lags
- In particular, we will provide a simple but effective hypothesis testing procedure to detect a small lead-lag effect
- We only consider a simple model; an extension to the general case would be possible (in progress)

Model

- $(X_t^1, X_t^2)_{t\in[0,1]}$: bivariate Brownian motion with a lead-lag effect

$$X_t^1 = \sigma_1 B_t^1, \qquad X_t^2 = \sigma_2 B_{t-\vartheta}^2,$$

  - $B_t = (B_t^1, B_t^2)$, $t \in \mathbb{R}$: two-sided bivariate standard Brownian motion with correlation $\rho \neq 0$ such that $B_0 = 0$
  - $\sigma_1, \sigma_2 > 0$; $\vartheta \in \mathbb{R}$ is the lag parameter

- We observe X at equidistant times with noise: $Y_0^p = 0$ and

$$Y_i^p = X_{t_i}^p + \epsilon_i^p, \quad t_i = i/n \qquad (i = 1, \dots, n) \tag{1}$$

  - $\epsilon_i^p \overset{i.i.d.}{\sim} N(0, \Upsilon_p)$ and ϵ^1 and ϵ^2 are mutually independent

Model

- We are interested in the inference for the parameter ϑ
- We restrict our attention to the situation where the lag is nearly zero

  $\Longrightarrow$ We consider the local asymptotics such that $\vartheta := \vartheta_n = c\eta_n$ for some $c \in [-\delta_c, \delta_c]$ and $\eta_n \to 0$ as $n \to \infty$
- Empirically, the sizes of lags are usually comparable with the sampling frequency, so such a setting is meaningful
- We assume $\eta_n = o(n^{-\frac{1}{2}})$; we show that this setting allows us to construct a simple, feasible and rate-optimal test for the absence of a lead-lag effect

Formulation of the problem

- We consider the following hypothesis testing problem:

$$H_0 : c = 0 \qquad \text{vs.} \qquad H_1 : c \neq 0 \tag{2}$$

- To discuss the rate optimality problem, we employ the minimax approach of Ingster (1993)
- Namely, we seek the fastest rate $r_n \to 0$ such that the hypothesis testing problem

$$H_0 : c = 0 \qquad \text{vs.} \qquad H_1(r_n) : c \in \mathcal{C}(r_n) \tag{3}$$

permits a uniformly consistent test, where

$$\mathcal{C}(r_n) = \{c : r_n \leq |c| \leq \delta_c\}$$

Formulation of the problem

- In terms of ϑ, (3) can be rewritten as

$$H_0 : \vartheta = 0 \qquad \text{vs.} \qquad H_1(r_n) : r_n \eta_n \le |\vartheta| \le \delta_c \eta_n$$

$\Longrightarrow$ Therefore, our aim corresponds to seeking the fastest rate r_n such that the lag $\vartheta = r_n \eta_n$ is distinguishable from 0

- The formal formulation of the problem is given in the next slides:

☐ Notation

- $\mathcal{E}_n = (\mathcal{X}_n, \mathcal{A}_n, (P_{n,c})_{c\in[-\delta_c,\delta_c]})$: our statistical experiments
- Ψ_n: the set of all tests at the stage n, i.e.

$$\psi \in \Psi_n \quad \Leftrightarrow \quad \psi : \mathcal{X}_n \to \{0,1\} \text{ is } \mathcal{A}_n\text{-measurable}$$

  - ▽ $\psi = 0 \Rightarrow H_0$ is accepted
  - ▽ $\psi = 1 \Rightarrow H_0$ is rejected

- $\alpha_n(\psi) = P_{n,0}(\psi = 1)$: type I error probability for (3)
- $\beta_n(\psi, r_n) = \sup_{c\in\mathcal{C}(r_n)} P_{n,c}(\psi = 0)$: maximal type II error probability for (3)
- $\gamma_n(r_n) = \inf_{\psi\in\Psi_n} \{\alpha_n(\psi) + \beta_n(\psi, r_n)\}$: minimax total error probability for (3)

Definition 1 (Ingster, 1993; Spokoiny, 1996)

A sequence r_n^* is called the **minimax rate of testing** if $r_n^* \to 0$ and

(i) For any sequence r_n such that $r_n = o(r_n^*)$ we have $\gamma_n(r_n) \to 1$,

(ii) For any $\alpha, \beta > 0$, there is a constant $K > 0$ and a sequence $\psi_n \in \Psi_n$ of tests such that

$$\limsup_{n\to\infty} \alpha_n(\psi_n) \le \alpha, \qquad \limsup_{n\to\infty} \beta_n(\psi_n, K r_n^*) \le \beta$$

Warm-up: an idealized case

- As a warm-up, we consider the idealized situation such that $\sigma_1 = \sigma_2 = 1$ and ρ is known
- We start with the case that the noise is absent

Proposition 1

If $\Upsilon_1 = \Upsilon_2 = 0$ and $|\rho| < 1$, the minimax rate of testing for (2) is $r_n^* = n^{-\frac{3}{2}}\eta_n^{-1}$, provided that $r_n^* \to 0$

- If $\rho = 1$ (resp. $\rho = -1$), H_0 is equivalent to saying $X_t^1 = X_t^2$ (resp. $X_t^1 = -X_t^2$) for all t, so any lag is detectable

- A rate-optimal test is constructed based on the fact that lead-lag effects cause the Epps effects
- More formally, if $\vartheta \neq 0$ and it does not depend on n, the realized covariance $U_n = \sum_{i=1}^{n}(X^1_{t_i} - X^1_{t_{i-1}})(X^2_{t_i} - X^2_{t_{i-1}})$ tends to 0 as $n \to \infty$
- On the other hand, $\sqrt{n}(U_n - \rho) \xrightarrow{d} N(0, 1 + \rho^2)$ if $\vartheta = 0$
- This suggests the test rejecting H_0 if $|T_n| > z_{1-\alpha/2}$, where

$$T_n = \sqrt{n}\frac{U_n - \rho}{\sqrt{1 + \rho^2}}$$

and $z_{1-\alpha/2}$ is the $(1 - \alpha/2)$-quantile of $N(0, 1)$

- The following proposition due to Hoffmann *et al.* (2013) ensures that the above test indeed satisfies condition (ii) of Definition 1:

Proposition 2 (Hoffmann *et al.*, 2013, Proposition 1)

Assume $\Upsilon_1 = \Upsilon_2 = 0$. Then we have

$$U_n = \rho\varphi(n\vartheta) + n^{-\frac{1}{2}}\sqrt{1 + \rho^2\varphi(n\vartheta)}\xi_n$$

under $P_{n,c}$ for all n, c, where $\varphi(t) = (1 - |t|)1_{\{|t|\leq 1\}}$ and ξ_n is a random variable with zero mean and unit variance and converges in law to $N(0, 1)$ as $n \to \infty$ (under $P_{n,c}$ for all n, c).

Minimax optimality in the noisy case

- We turn to the noisy case

Theorem 1

The minimax rate of testing for (2) is $r_n^* = n^{-\frac{3}{4}}\eta_n^{-1}$, provided that $r_n^* \to 0$

- This result is “canonical” in the sense that the smallest detectable lag $r_n^*\eta_n = n^{-\frac{3}{4}}(= (\sqrt{n})^{-\frac{3}{2}})$ coincides with the one in the non-noisy case with the sample size $\sqrt{n}$ (cf. Gloter & Jacod, 2001)

Construction of a rate-optimal test

- A natural idea is to consider a pre-averaged version of T_n (cf. Podolskij & Vetter, 2009; Vetter & Dette, 2012)
- Namely, we replace U_n with $\overline{U}_n = \frac{1}{\psi k_n} \sum_{i=0}^{n-k_n+1} \overline{Y}_i^1 \overline{Y}_i^2$, where $\psi = 1/12$ and

$$\overline{Y}_i = \frac{1}{k_n} \left(\sum_{p=0}^{k_n/2-1} Y_{i+p+k_n/2} - \sum_{p=0}^{k_n/2-1} Y_{i+p} \right)$$

with k_n being a positive even integer s.t. $k_n = \theta\sqrt{n} + o(n^{1/4})$ for some $\theta > 0$

Construction of a rate-optimal test

- According to Theorem 2 of Christensen, Kinnebrock & Podolskij (2010), $n^{1/4}(\overline{U}_n - \rho) \xrightarrow{d} N(0, \Gamma)$ for some constant $\Gamma > 0$ if $\vartheta = 0$
- Indeed, $\overline{U}_n$ is “too stable” for our purpose:

Proposition 3

If $\vartheta = o(n^{-5/8})$, $n^{1/4}(\overline{U}_n - \rho) \xrightarrow{d} N(0, \Gamma)$ as $n \to \infty$

- The proposition implies that the tests based on the statistic $n^{1/4}(\overline{U}_n - \rho)/\sqrt{\Gamma}$ cannot detect lags smaller than $n^{-5/8}$

Construction of a rate-optimal test

- We suppose $\vartheta \in \{k/n : k \in \mathbb{Z}\}$ for simplicity
- Fourier coefficients of $\mathrm{d}X$ (cf. Malliavin & Mancino, 2009):

$$c_f(\mathrm{d}X) = \sum_{i=1}^{n} \exp(-2\pi f \sqrt{-1} t_i)(X_{t_i} - X_{t_{i-1}})$$

- Since $E[(X^1_{t_i} - X^1_{t_{i-1}})(X^2_{t_i} - X^2_{t_{i-1}})] = \rho/n$ if $t_j - t_i = \vartheta$ and it vanishes otherwise, we have

$$E[c_f(\mathrm{d}X^1)c_{-f}(\mathrm{d}X^2)] = \exp(2\pi f \sqrt{-1}\vartheta)\rho,$$

ignoring the end effects

- This suggests that (a functional of) ϑ would be estimated by smoothing $c_f(\mathrm{d}X^1)c_{-f}(\mathrm{d}X^2)$ in the frequency domain
- However, this is not a good idea in the presence of noise:
  - The variance of $c_f(\mathrm{d}X^1)c_{-f}(\mathrm{d}X^2)$ due to the noise increases as f increases (cf. Mancino & Sanfelici, 2008)
  - The end effect due to the noise is crucial
- For these reasons
  - We consider “localized” Fourier coefficients and smooth them in the time domain
  - We only use Fourier sine coefficients; they do not suffer from the end effect because $\sin(0) = \sin(2\pi) = 0$

Construction of a rate-optimal test

- This results in considering *spectral statistics* of Bibinger, Hautsch, Malec & Reiß (2014) (with the lowest frequency):
  - Split $[0, 1]$ into blocks $[kh_n, (k+1)h_n)$ $(k = 0, 1, \dots, h_n^{-1} - 1)$
    - h_n is the width of the blocks and chosen so that $h_n^{-1} \in \mathbb{N}$ and $h_n \asymp n^{-\frac{1}{2}}$
  - Define

$$S_k = \sum_{i=1}^{n} (Y_i - Y_{i-1}) \, \Phi_k(\bar{t}_i), \qquad \bar{t}_i = \frac{t_{i-1} + t_i}{2},$$

where $\Phi_k(t) = \sin\left(\pi h_n^{-1}(t - kh_n)\right) 1_{[kh_n,(k+1)h_n)}(t)$

Construction of a rate-optimal test

- To make use of Fourier cosine coefficients, we rely on the same trick as in Bibinger & Winkelmann (2015)
  - We consider the spectral statistics on the shifted blocks $[(k-\frac{1}{2})h_n,(k+\frac{1}{2})h_n)$ as well, i.e. $S_{k-\frac{1}{2}}$ $(k=1,\dots,h_n^{-1}-1)$
  - Bibinger & Winkelmann (2015) use these statistics to handle jumps in their spectral covariance estimators
- The following formula plays a key role:

$$\Phi_{k-1}(t)-\Phi_k(t)=\cos\left(\pi h_n^{-1}\left(t-(k-1/2)\,h_n\right)\right)1_{[(k-1)h_n,(k+1)h_n)}(t)$$

- Therefore, noting that $|\vartheta| \le h_n/2$, we have

$$
\begin{aligned}
&E\left[\left(S^1_{k-1} - S^1_k\right) S^2_{k-\frac{1}{2}} - S^1_{k-\frac{1}{2}}\left(S^2_{k-1} - S^2_k\right)\right] \\
&= \frac{\rho}{n} \sum_{(k-\frac{1}{2})h_n \le \bar{t}_i < (k+\frac{1}{2})h_n} \Big\{\cos\left(\pi h_n^{-1}\left(\bar{t}_i - (k-1/2)\,h_n\right)\right)\sin\left(\pi h_n^{-1}\left(\bar{t}_i + \vartheta - (k-1/2)\,h_n\right)\right) \\
&\qquad - \sin\left(\pi h_n^{-1}\left(\bar{t}_i - (k-1/2)\,h_n\right)\right)\cos\left(\pi h_n^{-1}\left(\bar{t}_i + \vartheta - (k-1/2)\,h_n\right)\right)\Big\} \\
&= \rho h_n \sin\left(\pi h_n^{-1}\vartheta\right)
\end{aligned}
$$

due to the formula $\sin(y-x) = \cos(x)\sin(y) - \sin(x)\cos(y)$

- This motivates us to consider the following moment-type estimator:

$$
\Xi_n = \frac{1}{h_n^{-1}-1} \sum_{k=1}^{h_n^{-1}-1} \left\{\left(S^1_{k-1} - S^1_k\right) S^2_{k-\frac{1}{2}} - S^1_{k-\frac{1}{2}}\left(S^2_{k-1} - S^2_k\right)\right\}
$$

Theorem 2

Suppose that $\sqrt{n}h_n \to \kappa$ for some $\kappa > 0$. For model (1), we have

$$h_n^{-\frac{3}{2}} \left(\Xi_n - \rho h_n \sin(\pi h_n^{-1} \vartheta) \right) \xrightarrow{d} N(0, V)$$

as $n \to \infty$, where

$$\begin{aligned} V = & \left\{ \left(\sigma_1^2 + \pi^2 \kappa^{-2} \Upsilon_1 \right) \left(\sigma_2^2 + \pi^2 \kappa^{-2} \Upsilon_2 \right) - (\sigma_1 \sigma_2 \rho)^2 \right\} \\ & + \pi^{-2} \left\{ \left(\sigma_1^2 - \pi^2 \kappa^{-2} \Upsilon_1 \right) \left(\sigma_2^2 - \pi^2 \kappa^{-2} \Upsilon_2 \right) - (\sigma_1 \sigma_2 \rho)^2 \right\}. \end{aligned}$$

Construction of a rate-optimal test

- Theorem 2 suggests the test rejecting H_0 if $|T_n^{\mathrm{sp}}| > z_{1-\alpha/2}$, where $T_n^{\mathrm{sp}} = h_n^{-\frac{3}{2}} \Xi_n / \sqrt{V}$
- $h_n^{-\frac{3}{2}} \asymp n^{\frac{3}{4}}$ and $h_n \sin(\pi h_n^{-1} \vartheta) \asymp \vartheta$ (because $\vartheta = o(h_n)$) and we can directly check

$$\limsup_{n\to\infty} \sup_{|c|\le\delta_c} E_{n,c}\left[\left|h_n^{-\frac{3}{2}}\left(\Xi_n - \rho h_n \sin(\pi h_n^{-1}\vartheta)\right)\right|^r\right] < \infty$$

for all $r > 1$ (because Ξ_n is moment-type), so the test based on T_n^{sp} is indeed rate-optimal

Construction of a feasible test

- The test T_n^{sp} is infeasible in practice because V contains the parameters $\sigma_1, \sigma_2, \rho, \Upsilon_1, \Upsilon_2$ which are usually unknown
- However, a feasible test can be obtained once we construct a consistent estimator for V, and it is an easy task: Set

$$\widehat{\Upsilon}_p^n = -\frac{1}{n}\sum_{i=1}^{n-1}(Y_i^p - Y_{i-1}^p)(Y_{i+1}^p - Y_i^p),$$

$$\widehat{\Sigma}_{pq}^n = \sum_{k=1}^{h_n^{-1}-1}\left(S_k^p S_k^q + S_{k-1/2}^p S_{k-1/2}^q\right) - \frac{\pi^2}{nh_n^2}\widehat{\Upsilon}_p^n 1_{\{p=q\}}$$

for $p, q = 1, 2$

Construction of a feasible test

- We have $\widehat{\Upsilon}_p^n \to^p \Upsilon_p$, $\widehat{\Sigma}_{pp}^n \to^p \sigma_p^2$ and $\widehat{\Sigma}_{12}^n \to^p \sigma_1\sigma_2\rho$

  $\Longrightarrow$ Setting $\widehat{\Sigma}_{p,\pm}^n = \widehat{\Sigma}_{pp}^n \pm (\pi^2/nh_n^2)\widehat{\Upsilon}_p^n$ and

$$\widehat{V}^n = \widehat{\Sigma}_{1,+}^n \widehat{\Sigma}_{2,+}^n + \pi^{-2}\widehat{\Sigma}_{1,-}^n \widehat{\Sigma}_{2,-}^n - (1+\pi^{-2})\left(\widehat{\Sigma}_{12}^n\right)^2,$$

  we have $\widehat{V}^n \to^p V$

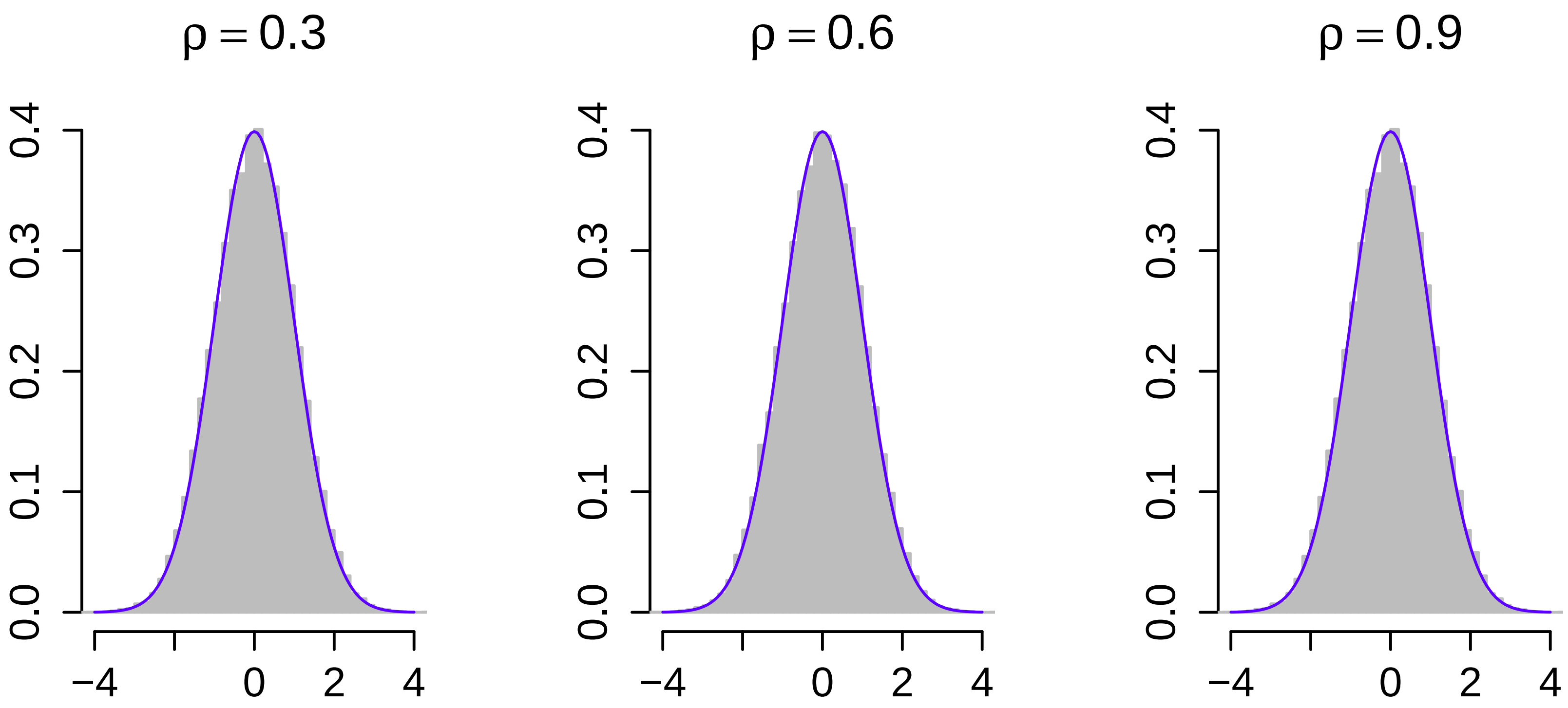
- Consequently, we obtain a feasible test statistic

$$\widehat{T}_n^{\mathrm{sp}} = h_n^{-\frac{3}{2}} \frac{\Xi_n}{\left(\widehat{V}^n\right)^{1/2}}$$

Simulation study

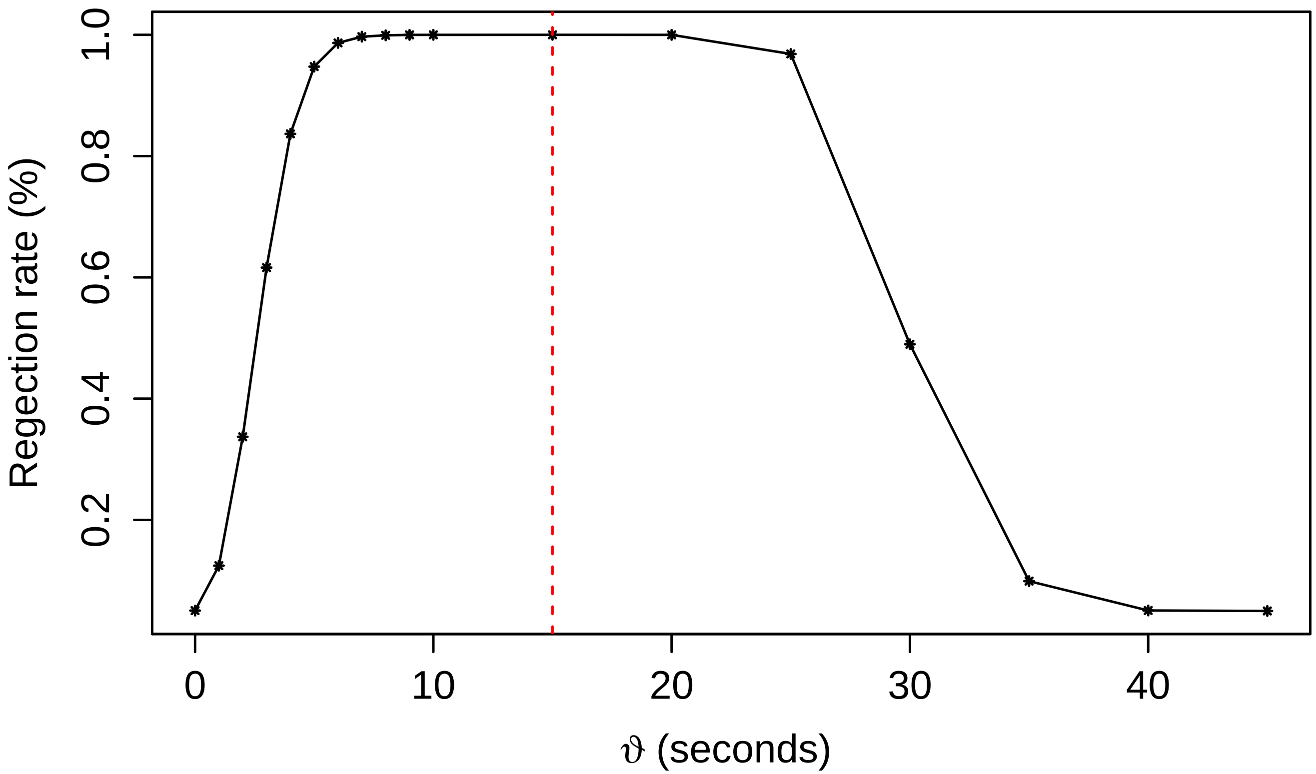
- We set $\sigma_p = 1$, $\Upsilon_p = 0.001$ for $p = 1, 2$ and $\rho \in \{0.3, 0.6, 0.9\}$
  - The noise variance is 0.1% of the quadratic variation, reflecting the empirical finding of Hansen & Lunde (2006)
- $n = 3,600$
- We regard $\frac{1}{n}$ as 1 second, so $[0, 1]$ corresponds to 1 hour
- $\vartheta = l/n$ and $l = 0, 1, \ldots, 10, 15, 20, \ldots, 45$
- $h_n = 30/n$; note that the consistency of the test is not ensured at the lags higher than $\vartheta = h_n/2 = 15/n$

Figure 1: Histograms of $\widehat{T}_n^{\mathrm{sp}}$ under H_0



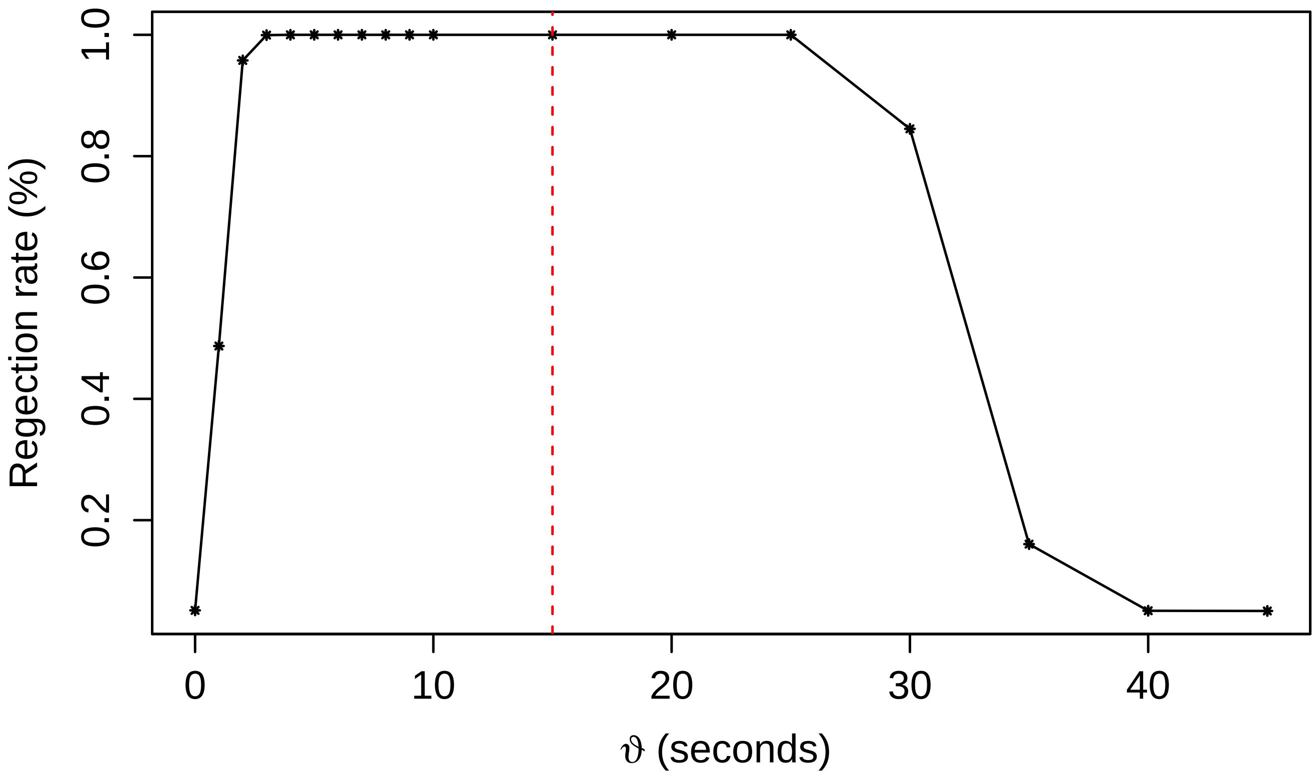
Note. Monte Carlo distribution of $\widehat{T}_n^{\mathrm{sp}}$ under H_0 based on 50,000 repetitions (grey). Blue solid lines denote the $N(0,1)$ density.

Figure 2: Rejection rate of H_0 at the 5% level ($\rho = 0.6$)



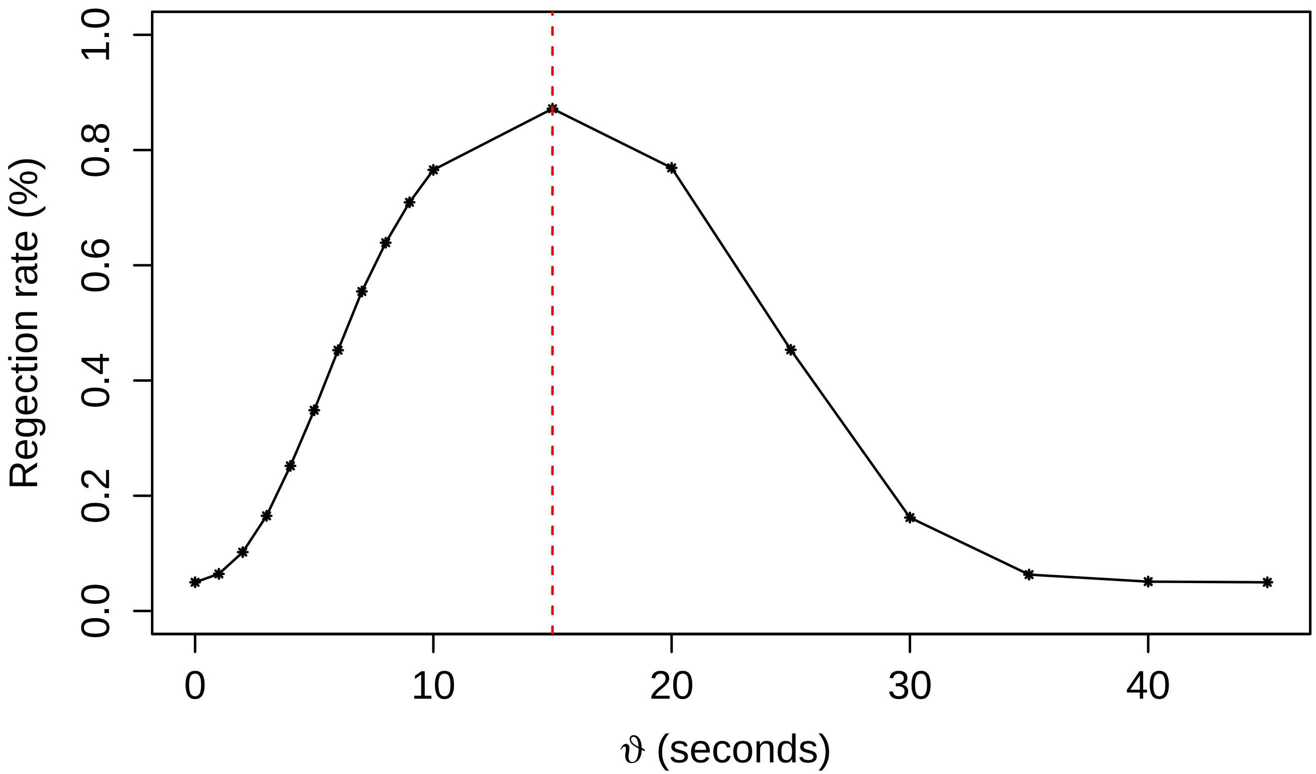
Note. Monte Carlo empirical rejection rate of H_0 at the 5% level based on 50,000 repetitions ($\rho = 0.6$). Red dash line denotes $\vartheta = h_n/2$.

Figure 3: Rejection rate of H_0 at the 5% level ($\rho = 0.9$)



Note. Monte Carlo empirical rejection rate of H_0 at the 5% level based on 50,000 repetitions ($\rho = 0.9$). Red dash line denotes $\vartheta = h_n/2$.

Figure 4: Rejection rate of H_0 at the 5% level ($\rho = 0.3$)



Note. Monte Carlo empirical rejection rate of H_0 at the 5% level based on 50,000 repetitions ($\rho = 0.3$). Red dash line denotes $\vartheta = h_n/2$.

Conclusions

- Contributions of this study
  - For the Hoffmann-Rosenbaum-Yoshida model of lead-lag effects, lower bounds of detectable lags' rate have been provided both in the non-noisy case and the noisy case
  - In the noisy case, a simple feasible test that attains the optimal rate is proposed
- Future works
  - Extension of the model: stochastic volatility and non-synchronous observations (probably routine)
  - More general model of lags (e.g. time varying one)

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