Sequential parameter estimators with guaranteed accuracy for delay differential equations

Uwe Küchler (Berlin), Vyacheslav A. Vasiliev (Tomsk)

Humboldt-Universität zu Berlin
www.math.hu-berlin.de/∼kuechler
kuechler@math.hu-berlin.de

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Outline

The problem

We start with a linear regression model

$$dX(t) = \vartheta' a(t)dt + dW(t),\ t \geq 0 \tag{1}$$

where

- $(W(t), t \geq 0)$ is an adapted one-dimensional standard Wiener process on a filtered probability space $(\Omega, F, (\mathfrak{F}_t)_{t\geq 0}, P)$,
- ϑ is an unknown parameter from some subset Θ of R^{p+1} and
- $(a(t), t \geq 0)$ forms an observable adapted $(p+1)$-dimensional cadlag process.

See e.g. Galtchouk, Konev (2001), Konev, Pergamenshchikov (1992)

The model (1) includes
Ornstein-Uhlenbeck-processes, linear stochastic differential equations (CARMA-processes) and linear stochastic delay differential equations.
Problem: Estimate ϑ based on observation of $X(t), t \geq 0$.

$$dX(t) = \vartheta' a(t)dt + dW(t),\ t \geq 0,$$

A natural candidate for estimating ϑ is the least squares estimator (LSE)

$$\vartheta(T) = (\int_0^T a(t)a'(t)dt)^{-1} \int_0^T a(t)dX(t), \quad T > 0.$$

In this talk, we restrict ourselves to the example of special delay differential equations

$$dX(t) = \vartheta_0 X(t)dt + \vartheta_1 X(t-1)dt + dW(t). \tag{2}$$

The considered candidate for estimating ϑ now equals (for fixed $T > 0$)

$$\vartheta(T) = [\int_0^T (X(t), X(t-1))'(X(t), X(t-1))dt]^{-1} \int_0^T (X(t), X(t-1))' dX(t).$$

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We use the notation

$$I_T := \begin{pmatrix} \int\limits_0^T X^2(t)dt & \int\limits_0^T X(t)X(t-1)dt \\ \bullet & \int\limits_0^T X^2(t-1)dt \end{pmatrix}$$

(Fishers information matrix), and

$$V_T := \begin{pmatrix} \int\limits_0^T X(t)dX(t) \\ \int\limits_0^T X(t-1)dX(t) \end{pmatrix},$$

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(Fishers information matrix), and

$$V_T^W := \begin{pmatrix} \int_0^T X(t)dW(t) \\ \int_0^T X(t-1)dW(t) \end{pmatrix},$$

to get

$$\vartheta(T) = (I_T)^{-1} V_T, \text{ and}$$

$$\vartheta(T) - \vartheta = (I_T)^{-1} V_T^{W} =$$

$$\begin{pmatrix} \int_o^T X^2(t)dt & \int_0^T X(t)X(t-1)dt \\ \bullet & \int_0^T X^2(t-1)dt \end{pmatrix}^{-1} \begin{pmatrix} \int_0^T X(t)dW(t) \\ \int_0^T X(t-1)dW(t) \end{pmatrix}.$$

We see, that

the asymptotic properties of $\vartheta(T)$ for $T \to \infty$ are determined by the asymptotic behavior of $X(T)$ for $T \to \infty$.

We define with $\mathrm{tr} I_T = \int_0^T (X^2(t) + X^2(t-1))dt$

$$\tau_\varepsilon := \inf\{T > 0 | \mathrm{tr}\, I_T \geq \varepsilon^{-1}\},$$
$$\vartheta_\varepsilon := I_{\tau_\varepsilon}^{-1} V_{\tau_\varepsilon}.$$

Fix any number $q \geq 2$. We are interested in the quantity

$$E_\vartheta ||\vartheta_\varepsilon - \vartheta||^q = E_\vartheta ||I_{\tau_\varepsilon}^{-1} V_{\tau_\varepsilon}^W||^q \leq E_\vartheta \left[||I_{\tau_\varepsilon}^{-1}||^q ||V_{\tau_\varepsilon}^W||^q \right].$$

Using the Burkholder-Gundy inequality we obtain for some constant b_q:

$$E_\vartheta ||V_{\tau_\varepsilon}^W||^q \leq b_q \varepsilon^{-\frac{q}{2}}.$$

Otherwise we do not have a bound for

$$E_\vartheta ||I_{\tau_\varepsilon}^{-1}||^q.$$

Note: In the O-U-case $\vartheta_1 = 0$ we have

$$I_T = \int_0^T X^2(t)dt \text{ and thus } \operatorname{tr} I_T = I_T,$$

thus

$$I^{-1}_{\tau(\varepsilon)} = \left[\int_0^{\tau_\varepsilon} X^2(t)dt\right]^{-1} = \varepsilon.$$

Moreover it holds

$$E_\vartheta[\vartheta(\tau_\varepsilon) - \vartheta]^2 = E_\vartheta\left[\left(\int_0^{\tau_\varepsilon} X^2(t)dt\right)^{-1} \int_0^{\tau_\varepsilon} X(t)dW(t)\right]^2 =$$

$$\varepsilon^2 E_\vartheta\left[\int_0^{\tau_\varepsilon} X(t)dW(t)\right]^2 = \varepsilon^2 \cdot E_\vartheta \int_0^{\tau_\varepsilon} X^2(t)dt = \varepsilon^2 \cdot \frac{1}{\varepsilon} = \varepsilon$$

But for the delay equation we have more than one parameter and thus

$$\text{tr } I_T \neq I_T.$$

Exit: Choose an increasing sequence $(c_n > 0, n \geq 1)$ tending to infinity and define

$$\tau_\varepsilon(n) := \inf\{T > 0 | \text{tr } I_T \geq \varepsilon^{-1} c_n\},$$

then it holds $\tau_\varepsilon(n) \uparrow \infty$ for $n \to \infty$ and for $\varepsilon \downarrow 0$, .

Note that $\text{tr } I_{\tau_\varepsilon(n)} = \varepsilon^{-1} c_n$ and put

$$\vartheta_\varepsilon(n) := I^{-1}_{\tau_\varepsilon(n)} V_{\tau_\varepsilon(n)}.$$

To control the behaviour of

$$E_\vartheta || I^{-1}_{\tau_\varepsilon(n)} ||^q$$

we randomize the argument n as follows:

- Choose $(c_n > 0, n \geq 1)$ such that $\sum_{n\geq 1} c_n^{-\frac{q}{2}} < \infty$,
- define the random variables $\beta_\varepsilon(n) := \varepsilon c_n^{-1} || I_{\tau_\varepsilon(n)}^{-1} ||^{-1} > 0$,
- introduce $S_\varepsilon(N) := \sum_{n=1}^{N} \beta_\varepsilon^q(n)$,(increasing with N).
- put $\rho := b_q \sum_{n\geq 1} c_n^{-\frac{q}{2}}$,
- finally $\nu_\varepsilon := \inf\{N \geq 1 : S_\varepsilon(N) \geq \rho\}$.

Now the sequential estimation plan $(T_\varepsilon, \vartheta_\varepsilon)$ is constructed as

$$T_\varepsilon := \tau_\varepsilon(\nu_\varepsilon),$$

$$\vartheta_\varepsilon := S_\varepsilon^{-1}(\nu_\varepsilon) \sum_{n=1}^{\nu_\varepsilon} \beta_\varepsilon^q(n) \vartheta_\varepsilon(n), \quad \varepsilon > 0.$$

(Random convex linear combination, remember $\vartheta_\varepsilon(n) = I_{\tau_\varepsilon(n)}^{-1} V_{\tau_\varepsilon(n)}$)

Theorem

Assume Θ is the set of parameters $\vartheta = (\vartheta_0, \vartheta_1)'$, under which the equation

$$dX(t) = \vartheta_0 X(t)dt + \vartheta_1 X(t-1)dt + dW(t)$$

admits a stationary solution. Then it holds:

1. *For any $\varepsilon > 0$ and every $\vartheta \in \Theta$ the sequential estimation plan $(T_\varepsilon, \vartheta_\varepsilon)$ of ϑ is closed: $T_\varepsilon < \infty \;\; P_\vartheta - a.s.$,*
2. *for any $\varepsilon > 0$ it holds* $\sup_{\vartheta\in\Theta} \|\vartheta_\varepsilon - \vartheta\|_q^2 \le \varepsilon$,
3. $\limsup_{\varepsilon\to 0} \varepsilon \cdot T_\varepsilon < \infty \;\; P_\vartheta - a.s.$,
4. *the estimator ϑ_ε is strongly consistent:*

   $\lim_{\varepsilon\to 0} \vartheta_\varepsilon = \vartheta \;\; P_\vartheta - a.s.$.

To sketch the proof, we need some analytical properties of the considered delay equation.

$$X(t) = x_0(t)X_0(0) + \vartheta_1 \int_{-1}^{0} x_0(t-s-1)X_0(s)\,ds + \int_{0}^{t} x_0(t-s)dW(s), \quad t \geq 0. \quad (3)$$

where $x_0(.)$ is the *fundamental solution* of the deterministic delay equation

$$\dot{x}_0(t) = \vartheta_0 x_0(t) + \vartheta_1 x_0(t-1), \quad x_0(s) = \mathbf{1}_{\{0\}}(s),\ s \in [-1,0]. \quad (4)$$

Asymptotic properties of $X(t)$ for $T \to \infty$ are determined by the asymptotics of $x_0(t)$ for $t \to \infty$.

Let us study the asymptotic properties of $x_0(t)$ for $t \to \infty$.

Laplace transform of equation (4):

- $$\lambda \hat{x}_0(\lambda) = \vartheta_0 \hat{x}_0(\lambda) + \vartheta_1 \,\mathrm{e}^{-\lambda} \hat{x}_0(\lambda), \quad \mathrm{Re}\,\lambda > 0,$$
- $$\hat{x}_0(\lambda) = \frac{1}{\lambda - \vartheta_0 - \vartheta_1 \,\mathrm{e}^{-\lambda}}, \quad \mathrm{Re}\,\lambda > 0,$$
- Characteristic function
$$h(\lambda) := \lambda - \vartheta_0 - \vartheta_1 \,\mathrm{e}^{-\lambda}, \quad \lambda \in K,$$
- Set Λ of zeros of the characteristic function
$$\Lambda := \{\lambda \in K | h(\lambda) = 0\}$$
- $$v_0 := \max\{\mathrm{Re}\,\lambda \mid \lambda \in \Lambda\} < \infty, v_k := \max\{\mathrm{Re}\,\lambda \mid \lambda \in \Lambda,\ \mathrm{Re}\,\lambda < v_{k-1}\}.$$

Applying the inverse Laplace transform and Cauchy's residual theorem it can be shown

Lemma

For $c \in (v_2, v_1)$ the fundamental solution $x_0(\cdot)$ of (4) can be represented in the form

$$x_0(t) = \psi_0(t)\exp[v_0 t] + \psi_1(t)\exp[v_1 t] + o(\exp(\gamma t)) \tag{5}$$

for some $\gamma < c$.

Here $\psi_0(t)$ equals

1. $\dfrac{1}{v_0 - \vartheta_0 + 1}$ if $v_0 \in \Lambda, m(v_0) = 1$,
2. $2t + \dfrac{2}{3}$ if $v_0 \in \Lambda, m(v_0) = 2$,
3. $\dfrac{2(v_0 - \vartheta_0 + 1)}{(v_0 - \vartheta_0 + 1)^2 + \xi_0^2} \cos(\xi_0 t) + \dfrac{2\xi_0}{(v_0 - \vartheta_0 + 1)^2 + \xi_0^2} \sin(\xi_0 t)$

   if $v_0 \notin \Lambda$.

   (ξ_0 is the imaginary part of the $\lambda \in \Lambda$ with $Re(\lambda) = v_0$).

Asymptotics of $x_0(T)$ for $T \to \infty$ is determined by the element λ from the set Λ with greatest real part v_0.

Conclusion: As we will see later, the asymptotic properties of $(\vartheta(T) - \vartheta)$ for $T \to \infty$ depend on the two values v_0 and v_1.

The set Θ of parameters $\vartheta = (\vartheta_0, \vartheta_1)'$, under which the equation

$$dX(t) = \vartheta_0 X(t)dt + \vartheta_1 X(t-1)dt + dW(t)$$

admits a stationary solution, is the set where $v_0 < 0$ holds. This is the area N in the subsequent figure.
$X(.)$ is ergodic if $\vartheta \in \Theta$. We get in particular

$$\lim_{t\to\infty} t^{-1} \int_0^t X^2(s)ds = \int_0^\infty x_0^2(s)ds < \infty \;\; P_\vartheta - \text{a.s.} \tag{6}$$

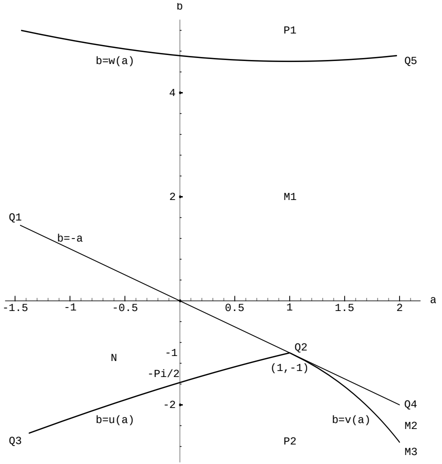
and therefore it follows

$$\lim_{T\to\infty} \frac{I_T}{T} = \lim_{T\to\infty} T^{-1} \begin{pmatrix} \int\limits_0^T X^2(s)ds & \int\limits_0^T X(s)X(s-1)ds \\ \int\limits_0^T X(s)X(s-1)ds & \int\limits_0^T X^2(s-1)ds \end{pmatrix} =$$

$$\begin{pmatrix} \int_0^\infty x_0^2(s)\,ds & \int_0^\infty x_0(s)x_0(s+1)\,ds \\ \int_0^\infty x_0(s)x_0(s+1)\,ds & \int_0^\infty x_0^2(s)\,ds \end{pmatrix} =: I_\infty. \tag{7}$$

(I_∞ is nonsingular)

$$lim_{T\to\infty} T^{-1} \text{ tr } I_T = 2\int_0^\infty x_0^2(s)\,ds = \text{ tr } I_\infty \tag{8}$$


b
P1
b=w(a)
Q5
4
2
M1
Q1
b=-a
-1.5
-1
-0.5
0.5
1
1.5
2
a
-1
Q2
N
(1,-1)
-Pi/2
-2
Q4
b=u(a)
b=v(a)
M2
Q3
P2
M3

1. Finiteness of the stopping time T_ε :

We have (note that tr $I_{\tau_\varepsilon(n)} = \varepsilon^{-1} c_n$)

- $$\lim_{n\to\infty} \frac{\tau_\varepsilon(n)}{\varepsilon^{-1} c_n} = \lim_{n\to\infty} \left[\mathrm{tr} \frac{I_{\tau_\varepsilon(n)}}{\tau_\varepsilon(n)} \right]^{-1} = [\lim_{T\to\infty} tr[\frac{I_T}{T}]]^{-1} = [\ tr\ I_\infty]^{-1} =: \alpha > 0$$
- $$\beta_\varepsilon(n) := (\varepsilon^{-1} c_n || I^{-1}_{\tau_\varepsilon(n)} ||)^{-1} =$$
$$\left[(\frac{\mathrm{tr}\ \ I_{\tau_\varepsilon(n)}}{\tau_\varepsilon(n)})(\tau_\varepsilon(n)) || I^{-1}_{\tau_\varepsilon(n)} || \right]^{-1} \to_{n\to\infty} [\mathrm{tr} I_\infty || I_\infty^{-1} ||]^{-1} =: \beta > 0$$
- $$\frac{\varepsilon^{-1} c_n}{\tau_\varepsilon(n)} \beta_\varepsilon^2(n) \to_{n\to\infty} \alpha^{-1} \beta^2 = [\mathrm{tr}\ I_\infty \, || I_\infty^{-1} ||^2]^{-1} =: C_* > 0$$

Because $\beta_\varepsilon(n) \to_{n\to\infty} [(\mathrm{tr} I_\infty)||I_\infty^{-1}||]^{-1} = \beta > 0$ we get

$$\sum_{n=1}^{\infty} \beta_\varepsilon^q(n) = \infty$$

Taking into account $S_\varepsilon(N) = \sum_{n=1}^{N} \beta_\varepsilon^q(n)$, it follows, that $P_\vartheta -$ a.s.

$$\nu_\varepsilon := \inf\{N \geq 1 : S_\varepsilon(N) \geq \rho\}$$

is finite and, consequently, that

$$T_\varepsilon := \tau_\varepsilon(\nu_\varepsilon)$$

is finite.

2. Accuracy of the estimator ϑ_ε :

$$\|\vartheta_\varepsilon - \vartheta\|_q^2 := \left(E_\vartheta(\|\vartheta_\varepsilon - \vartheta\|)^q\right) =$$

$$\left(E_\vartheta\left[S_\varepsilon^{-1}(\nu_\varepsilon)\left\|\sum_{n=1}^{\nu_\varepsilon}\beta_\varepsilon^q(n)(\vartheta_\varepsilon(n)-\vartheta)\right\|\right]^q\right)^{\frac{2}{q}} \le$$

$$\left(E_\vartheta\left[S_\varepsilon^{-1}(\nu_\varepsilon)\sum_{n=1}^{\nu_\varepsilon}\beta_\varepsilon^q(n)\,\|(\vartheta_\varepsilon(n)-\vartheta)\|\right]^q\right)^{\frac{2}{q}}$$

Applying Hölders inequality

$$\sum a_n b_n \le \left(\sum a_n^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}}\left(\sum b_n^q\right)^{\frac{1}{q}}$$

with

$$a_n = \beta_\varepsilon^{q-1}(n) \text{ and } b_n = \beta_\varepsilon(n)\|\vartheta_\varepsilon(n) - \vartheta\|,$$

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$$\left(E_\vartheta\left[S_\varepsilon^{-1}(\nu_\varepsilon)\left\|\sum_{n=1}^{\nu_\varepsilon}\beta_\varepsilon^q(n)(\vartheta_\varepsilon(n)-\vartheta)\right\|\right]^q\right)^{\frac{2}{q}} \le$$

$$\left(E_\vartheta\left[S_\varepsilon^{-1}(\nu_\varepsilon)\sum_{n=1}^{\nu_\varepsilon}\beta_\varepsilon^q(n)\,\|(\vartheta_\varepsilon(n)-\vartheta)\|\right]^q\right)^{\frac{2}{q}}$$

Applying Hölders inequality

we obtain by using $S_\varepsilon(N) = \sum_{n=1}^{N}\beta_\varepsilon^q(n)$ the inequalities

$$\|\vartheta_\varepsilon - \vartheta\|_q^2 \le \left(E_\vartheta S_\varepsilon^{-q}(\nu_\varepsilon)\left(\sum_{n\le\nu_\varepsilon}\beta_\varepsilon^q(n)\right)^{q-1}\sum_{n\le\nu_\varepsilon}\beta_\varepsilon^q(n)\,\|\vartheta_\varepsilon(n)-\vartheta\|^q\right)^{\frac{2}{q}} \le$$

$$\left(E_\vartheta S_\varepsilon^{-1}(\nu_\varepsilon)\sum_{n\ge1}\beta_\varepsilon^q(n)||\vartheta_\varepsilon(n)-\vartheta||^q\right)^{\frac{2}{q}}$$

Using the definitions

$$\rho := b_q \sum_{n\geq 1} c_n^{-\frac{q}{2}}, \qquad \nu_\varepsilon := \inf\{N \geq 1 : S_\varepsilon(N) \geq \rho\},$$

$$\beta_\varepsilon(n) := \varepsilon c_n^{-1} ||I_{\tau_\varepsilon}^{-1}||^{-1}, \qquad S_\varepsilon(N) := \sum_{n=1}^{N} \beta_\varepsilon^q(n)$$

and the inequalities $E_\vartheta ||V_{\tau_\varepsilon}^W||^q \leq b_q (c_n \varepsilon^{-1})^{\frac{q}{2}}$ and $S_\varepsilon^{-1}(\nu_\varepsilon) \leq \rho^{-1}$ we get

$$||\vartheta_\varepsilon - \vartheta||_q^2 \leq \left(E_\vartheta S_\varepsilon^{-1}(\nu_\varepsilon) \sum_{n\geq 1} \beta_\varepsilon^q(n) ||\vartheta_\varepsilon(n) - \vartheta||^q \right)^{\frac{2}{q}} \leq$$

$$\left(\rho^{-1} \sum_{n\geq 1} E_\vartheta \beta_\varepsilon^q(n) ||I_{\tau_\varepsilon(n)}^{-1}||^q ||V_{\tau_\varepsilon(n)}^W||^q \right)^{\frac{2}{q}} =$$

$$\varepsilon^2 \left(\rho^{-1} \sum_{n\geq 1} \frac{1}{c_n^q} E_\vartheta ||V_{\tau_\varepsilon(n)}^W||^q \right)^{\frac{2}{q}} \leq$$

$$\varepsilon \left(\rho^{-1} b_q \sum_{n\geq 1} c_n^{-\frac{q}{2}} \right)^{\frac{2}{q}} = \varepsilon.$$

3. The asymptotics of T_ε :

Analogously to the formulas above one can show that P_ϑ-a.s. it holds

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(n) = \lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n}||I^{-1}_{\tau_\varepsilon(n)}||^{-1} = [(\mathrm{tr} I_\infty)||I_\infty^{-1}||]^{-1} = \beta > 0.$$

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{-1}c_n}{\tau_\varepsilon(n)}\beta^2_\varepsilon(n) = \lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n\tau_\varepsilon(n)}||I^{-1}_{\tau_\varepsilon(n)}||^{-2} = [\mathrm{tr} I_\infty||I_\infty^{-1}||^2]^{-1} = \alpha^{-1}\beta^2 > 0,$$

If $\beta_* < \beta$, ε small enough and n sufficiently large, we get

$$S_\varepsilon(n) = \sum_{k=1}^{n}\beta^q_\varepsilon(k) \geq \beta_*^q n \geq \rho$$

Thus for all ε small enough and all n sufficiently large, we have

$$\nu(\varepsilon) = \inf\{m \geq 1 : S_\varepsilon(m) \geq \rho\} \leq n.$$

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Analogously to the formulas above one can show that P_ϑ-a.s. it holds

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(n)=\lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n}||I^{-1}_{\tau_\varepsilon(n)}||^{-1}=[(\mathrm{tr}I_\infty)||I_\infty^{-1}||]^{-1}=\beta>0.$$

$$\lim_{\varepsilon\to 0}\frac{\varepsilon^{-1}c_n}{\tau_\varepsilon(n)}\beta^2_\varepsilon(n)=\lim_{\varepsilon\to 0}\frac{\varepsilon}{c_n\tau_\varepsilon(n)}||I^{-1}_{\tau_\varepsilon(n)}||^{-2}=[\mathrm{tr}I_\infty||I_\infty^{-1}||^2]^{-1}=\alpha^{-1}\beta^2>0,$$

Because $\frac{\tau_\varepsilon(n)}{\varepsilon^{-1}c_n}\to_{\varepsilon\to 0}[\mathrm{tr}I_\infty]^{-1}$ for every $n\geq 1$, we get

$$\limsup_{\varepsilon\to 0}\varepsilon\tau_\varepsilon(\nu_\varepsilon)<\infty,$$

and therefore

$$\limsup_{\varepsilon\to 0}\varepsilon T_\varepsilon=\limsup_{\varepsilon\to 0}\varepsilon\tau_\varepsilon(\nu_\varepsilon)<\infty.$$

4. Consistency of the estimator $\vartheta(\varepsilon)$:

The Maximum-Likelihood-estimator is consistent:

$$\lim ||\vartheta(T) - \vartheta|| \leq \limsup_{T\to\infty} ||T \cdot I_T^{-1}|| \cdot ||\frac{V_T^W}{T}|| = 0$$

Because $\tau_\varepsilon(n) \to \infty$ as $\varepsilon \to 0$ the estimators $\vartheta_\varepsilon(n) = \vartheta(\tau_\varepsilon(n))$ and their weighted arithmetical mean ϑ_ε are also consistent.

What happens if the parameter ϑ is outside of N?

The terms $T^{-1}I_T$ never converge for $T \to \infty$. But

1. in some cases $\vartheta \in \Theta_1(v_0 > 0, v_0 \notin \Lambda, \quad \text{area } P_2)$ we have

$$\exp[-2v_0T]I_T - I_\infty(T) \to 0,$$

   where $I_\infty(T)$ are some (explicite known)random, periodic nonsingular matrices,
2. in some cases $\vartheta \in \Theta_2(v_0 > 0, v_0 \in \Lambda, m(v_0) = 1, v_1 > 0,\ M_2 \cup P_1)$ we have

$$\begin{pmatrix} \exp[-2v_0T] & 0 \\ 0 & \exp[-2v_1T] \end{pmatrix} (VI_T - I_\infty(T)) \to 0,$$

   where

$$V := \begin{pmatrix} 1 & -\exp[-2v_0T] \\ 0 & 1 \end{pmatrix}$$

   and $I_\infty(T)$ are some (explicite known) random, possibly periodic, nonsingular matrices.

3. in some cases $\vartheta \in \Theta_3(v_0 > 0, v_0 \in \Lambda, m(v_0) = 1, v_1 < 0$ area $M_1)$ we have

$$\begin{pmatrix} e^{-2v_0 T} & 0 \\ 0 & T^{-1} \end{pmatrix} (VI_T - I_\infty(T)) \to 0,$$

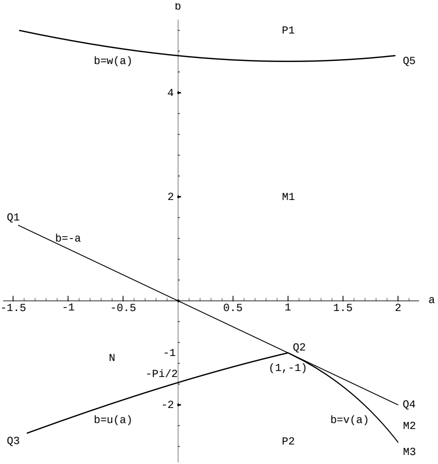
where I_∞ is a random (explicite known) nonsingular matrix.

In every of these cases, a sequential plan $(T_i(\varepsilon), \vartheta_i(\varepsilon))$ can be constructed with similar properties as $(T_\varepsilon, \vartheta_\varepsilon)$, taking into account the specific kind of convergence of the normalized (I_T) to a limit matrix. Put
$\Theta^* := \Theta \cup \Theta_1 \cup \Theta_2 \cup \Theta_3$.
The remaining cases $\vartheta \in \mathbb{R}^2 \setminus \Theta^*$ form a set of Lebesgue measure zero.
Because the true value of ϑ is not known before, we define

$$j^* := \arg\min T_j(\varepsilon), \ \ j = 0, 1, 2, 3,$$

$$\vartheta^*(\varepsilon) := \vartheta_{j^*}(\varepsilon)$$

where $T_0(\varepsilon) := T_\varepsilon$ and $\vartheta_0(\varepsilon) := \vartheta_\varepsilon$. Finally we get


b
P1
b=w(a)
Q5
4
2
M1
Q1
b=-a
-1.5
-1
-0.5
0.5
1
1.5
2
a
N
-1
Q2
-Pi/2
(1,-1)
-2
Q4
b=u(a)
b=v(a)
M2
Q3
P2
M3

Theorem

1. *For every $\varepsilon > 0$ and every $\vartheta \in \Theta^*$ the sequential estimation plans $(T^*(\varepsilon),\ \vartheta^*(\varepsilon))$ of ϑ are closed:*

$$(T^*(\varepsilon) < \infty \ \ P_\vartheta - a.s.).$$

2. *For any $\varepsilon > 0$ it holds*

$$\sup_{\vartheta \in \Theta^*} \|\vartheta^*(\varepsilon) - \vartheta\|_q^2 \leq \varepsilon;$$

3. *For every $\vartheta \in \Theta^*$ the estimator $\vartheta^*(\varepsilon)$ is strongly consistent:*

$$\lim_{\varepsilon \to 0} \vartheta^*(\varepsilon) = \vartheta \quad P_\vartheta - \mathrm{a.s.}$$

There are different asymptotic properties of $T^*(\varepsilon)$ for $\varepsilon \to 0$ in every of the cases $\vartheta \in \Theta_i, i = 0, 1, 2, 3$

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Thank you for your attention!