Consistency of the drift parameter estimator for the discretized fractional Ornstein–Uhlenbeck process with Hurst index $H \in (0, \frac{1}{2})$

Yuliya Mishura

Taras Shevchenko National University of Kyiv

March 2015

Outline

Contents

Let $(\Omega, \mathfrak{F}, \mathbf{P})$ be a complete probability space. We consider fractional Brownian motion $B^H = \{B_t^H,\ t \geq 0\}$ on this probability space, that is, the centered Gaussian process with the covariance function

$$R(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

We restrict ourselves to the case $H \in \left(0, \frac{1}{2}\right)$ and consider the continuous (and even Hölder up to order H) modification that exists due to the Kolmogorov theorem. Introduce the Langevin equation,

$$X_t = x_0 + \theta \int_0^t X_s\, ds + B_t^H, \quad t \geq 0, \quad H \in (0, \tfrac{1}{2}). \tag{1}$$

According to Proposition A.1 from [Cheridito et al.(2003)], this equation has the unique solution that is named fractional Ornstein–Uhlenbeck process and can be presented as

$$X_t = x_0 e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} B_s^H\, ds + B_t^H,\ t \geq 0. \tag{2}$$

The goal of the paper is to construct consistent (strongly consistent) estimator of the unknown drift parameter θ by discrete observations of the process X.

The problem of the estimation of the drift parameter θ in the linear equation containing fBm and in the equation (1) when the Hurst index $H \geq \frac{1}{2}$ was investigated in many works. For linear models, mention only papers [Bertin et al.(2011)] and [Hu et al.(2011)]. Drift parameter estimators for fractional Ornstein–Uhlenbeck process with continuous time when the whole trajectory of X is observed, were studied in [Belfadli et al.(2011), Hu and Nualart(2010), Kleptsyna and Le Breton(2002)]. Kleptsyna and Le Breton [Kleptsyna and Le Breton(2002)] constructed the maximum likelihood estimator and proved its strong consistency for any $\theta \in \mathbb{R}$. They also investigated the asymptotic behaviour of the bias and mean square error of this estimator. The sequential maximum likelihood estimation was considered in [Prakasa Rao(2004)].

Hu and Nualart [Hu and Nualart(2010)] proved that in the ergodic case ($\theta < 0$) the least square estimator

$$\widehat{\theta}_T = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt}, \tag{3}$$

is strongly consistent for all $H \geq \frac{1}{2}$ and asymptotically normal for $H \in [\frac{1}{2}, \frac{3}{4})$. They also obtained the strong consistency and asymptotic normality of the estimator

$$\widehat{\theta}_T = \left(\frac{1}{H\Gamma(H)T} \int_0^T X_t^2 \, dt \right)^{-\frac{1}{2H}}. \tag{4}$$

In [Belfadli et al.(2011)] the corresponding non-ergodic case $\theta > 0$ was considered and the strong consistency of the least square estimator (3) was proved for $H > \frac{1}{2}$. It was obtained also that $e^{\theta t} \left(\widehat{\theta}_t - \theta \right)$ converges in law to $2\theta\mathcal{C}(1)$ as $t \to \infty$, where $\mathcal{C}(1)$ is the standard Cauchy distribution. Minimum contrast estimators in continuous and discrete case were studied in [Bishwal(2011)].

The distributional properties of maximum likelihood, minimum contrast and least square estimators were explored in [Tanaka(2013)]. For the two-parameter generalization see [Clarke De la Cerda and Tudor(2012)].

In [Cénac and Es-Sebaiy(2012), Es-Sebaiy(2013), Es-sebaiy and Ndiaye(2014)] the discretized version of (3) is considered, namely

$$\widehat{\theta}_n = \frac{\sum_{i=1}^n X_{t_{i-1}}\left(X_{t_i} - X_{t_{i-1}}\right)}{\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}, \tag{5}$$

where the process X was observed in the points $t_i = i\Delta_n$, $i = 0, \dots, n$, such that $\Delta_n \to 0$ and $n\Delta_n \to \infty$ as $n \to \infty$. In [Cénac and Es-Sebaiy(2012)] the ergodic case $\theta < 0$ was studied, the strong consistency of this estimator was proved for $H \geq \frac{1}{2}$ and the almost sure central limit theorem was obtained for $H \in \left(\frac{1}{2}, \frac{3}{4}\right)$.

The non-ergodic case $\theta > 0$ was considered in Es-Sebaiy and Ndiaye [Es-sebaiy and Ndiaye(2014)]. They proved the strong consistency of the estimator (5) for $H \in \left(\frac{1}{2}, 1\right)$ assuming that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ as $n \to \infty$ for some $\alpha > 0$. The same result was obtained for the estimator

$$\widehat{\theta}_n = \frac{X_{t_n}^2}{2\Delta_n \sum_{i=1}^n X_{t_{i-1}}^2}.$$

In [Hu and Song(2013), Xiao et al.(2011)] the following discretized version of the estimator (4) was considered

$$\widehat{\theta}_n = -\left(\frac{1}{nH\Gamma(2H)} \sum_{k=1}^n X_{k\Delta}^2\right)^{-\frac{1}{2H}},$$

where $\theta < 0$ and the process X was observed in the points $\Delta, 2\Delta, \ldots, n\Delta$ for some fixed $\Delta > 0$. Hu and Song [Hu and Song(2013)] proved the strong consistency of the estimator for $H \geq \frac{1}{2}$ and the asymptotic normality for $\frac{1}{2} \leq H < \frac{3}{4}$.

In [Brouste and Iacus(2013), Zhang et al.(2014)] more general situation is studied, where the equation has the following form $dX_t = \theta X_t dt + \sigma dB_t^H$, $t > 0$, and $\vartheta = (\theta, \sigma, H)$ is the unknown parameter, $\theta < 0$. Consistent and asymptotically Gaussian estimators of the parameter θ are proposed by the discrete observations of the sample path $(X_{k\Delta_n}, k = 0, \dots, n)$ for $H \in (\frac{1}{2}, \frac{3}{4})$, where $n\Delta_n^p \to \infty$, $p > 1$, and $\Delta_n \to 0$ as $n \to \infty$. In [Zhang et al.(2014)] the strongly consistent estimator is constructed for the scheme, when $H > \frac{1}{2}$, the time interval $[0, T]$ is fixed and the process is observed at points $h_n, 2h_n, \dots, nh_n$, where $h_n = \frac{T}{n}$.

In [Diedhiou et al.(2011), Mendy(2013)] the so called sub-fractional Ornstein–Uhlenbeck process was studied, where the process B_t^H in (1) was replaced with a sub-fractional Brownian motion. In [Diedhiou et al.(2011)] the maximum likelihood estimator for such process was constructed, in [Mendy(2013)] the estimator (3) was investigated in the case $\theta > 0$. The maximum likelihood drift parameter estimators for fractional Ornstein–Uhlenbeck process and even more general processes involving fBm with Hurst index from the whole interval $(0, 1)$ were constructed and studied in [Tudor and Viens(2007)]. These estimators involve singular kernels therefore are more complicated to study and simulate. To the best of our knowledge, it is the only paper when discretized estimates of the drift parameter are constructed in the case $H < \frac{1}{2}$.

However, the observations of the real financial markets demonstrate that the Hurst index often falls below the level of $\frac{1}{2}$, taking values around 0.45–0.49 ([Bianchi et al.(2013)]). In order to consider the case of $H < \frac{1}{2}$ and to overcome the technical difficulties connected with singular kernels, we construct comparatively simple estimator that is similar in form to the maximum likelihood estimator for Langevin equation with standard Brownian motion. Observations are assumed to be discrete in time and we assume that the interval between observations is n^{-1}, i.e. tends to zero, so we consider high frequency data. At the same time, the number of observations increases to infinity with the speed n^m with $m > 1$. Let $n \geq 1$, $t_{k,n} = \frac{k}{n}$, $0 \leq k \leq n^m$, where $m \in \mathbb{N}$ be some fixed number.

Suppose that we observe X at the points $\{t_{k,n}, n \geq 1, 0 \leq k \leq n^m\}$. Consider the estimator

$$\widehat{\theta}_n(m) = \frac{\sum_{k=0}^{n^m-1} X_{k,n}\Delta X_{k,n}}{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2}, \tag{6}$$

where $X_{k,n} = X_{t_{k,n}}$, $\Delta X_{k,n} = X_{k+1,n} - X_{k,n}$.

By (1), estimator $\widehat{\theta}_n(m)$ from (6) can be represented in the following form, which is more convenient for evaluation:

$$\widehat{\theta}_n(m) = \theta + \frac{\theta \sum_{k=0}^{n^m-1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds + \sum_{k=0}^{n^m-1} X_{k,n}\Delta B^H_{k,n}}{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2}. \tag{7}$$

It is proved that for positive θ the estimator is strongly consistent for any $m > 1$ and for negative θ it is consistent for $m > \frac{1}{2H}$.

Our paper is organized as follows. In Section 2 we consider auxiliary result, namely, bounds with probability 1 for the values and increments of fractional Brownian motion and fractional Ornstein–Uhlenbeck process. The bounds are factorized to the increasing non-random function and random variable not depending on time. In Section 3 we get the bounds for the numerator of the estimator, while in Section 4 we relate discretized integral sum in the denominator of the estimator to the corresponding integral $\int_0^t X_s^2\,ds$. This relation is convenient for some values of parameters because it is easier to apply L'Hôpital's rule to the integral $\int_0^t X_s^2\,ds$ than Stolz–Cesàro theorem to the sum $\sum_{k=0}^{n^m-1} X_{k,n}^2$ with terms depending on n. Section 5 contains two main theorems, Theorem 5.1 and Theorem 5.4 on strong consistency for $\theta > 0$ and consistency for $\theta < 0$. Section 6 contains some auxiliary results and Section 7 is devoted to numerics.

Contents

In what follows we shall use next auxiliary estimates for the rate of asymptotic growth with probability 1 of the fractional Brownian motion and its increments. Throughout the paper considering functions of the form $t^p \log t$, $p > 0$ we suppose that $0 \cdot \infty = 0$.

Proposition

(i) *For any $p > 1$ and any $H \in (0, 1)$ there exists nonnegative random variable $\xi(p, H)$ such that*

$$\sup_{0 \le s \le t} \left| B_s^H \right| \le \left(\left(t^H |\log t|^p \right) \vee 1 \right) \xi(p, H), \tag{8}$$

and there exists such number $c_\xi(p, H) > 0$ that for any $0 < y < c_\xi(p, H)$, $\mathbf{E} \exp\{y\xi^2(p, H)\} < \infty$.

Proposition

(ii) *For any* $q > \frac{1}{2}$ *and any* $H \in (0,1)$ *there exists nonnegative random variable* $\eta(q,H)$ *such that for any* $0 < t_1 < t_2 < \infty$

$$\left|B^H_{t_2} - B^H_{t_1}\right| \leq (t_2 - t_1)^H \left(|\log(t_2 - t_1)|^{1/2} + 1\right) (\log(t_2 + 2))^q \eta(q,H), \tag{9}$$

and there exists such number $c_\eta(q,H) > 0$ *that for any* $0 < y < c_\eta(q,H)$, $\mathbf{E}\exp\{y\eta^2(q,H)\} < \infty$.

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(ii) *For any* $q > \frac{1}{2}$ *and any* $H \in (0, 1)$ *there exists nonnegative random variable* $\eta(q, H)$ *such that for any* $0 < t_1 < t_2 < \infty$

$$\left| B^H_{t_2} - B^H_{t_1} \right| \le (t_2 - t_1)^H \left(\left|\log(t_2 - t_1)\right|^{1/2} + 1 \right) (\log(t_2 + 2))^q \eta(q, H), \tag{9}$$

and there exists such number $c_\eta(q, H) > 0$ *that for any* $0 < y < c_\eta(q, H)$, $\mathbf{E} \exp\{y\eta^2(q, H)\} < \infty$.

Now our goal is to estimate the numerator in (7) and compare it to the denominator. At first, we write the bounds for the values of X and its increments.

Lemma

We have the following bounds for fractional Ornstein-Uhlenbeck process X in terms of supporting fractional Brownian motion:

(*i*) *Let $\theta > 0$. Then for any $t > 0$*

$$\sup_{0\le s\le t} |X_s| \le |x_0| e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} \sup_{0\le u\le s} \left|B_u^H\right| ds + \sup_{0\le s\le t} \left|B_s^H\right| \quad (10)$$

and for any $s \in [\frac{k}{n}, \frac{k+1}{n})$

$$\sup_{\frac{k}{n}\le u\le s} |X_u - X_{k,n}| \le \int_{\frac{k}{n}}^s \left(e^{\theta u} \left(|x_0| + \theta \int_0^u e^{-\theta v} \sup_{0\le z\le v} \left|B_z^H\right| dv\right) + \sup_{0\le z\le u} \left|B_z^H\right|\right) du + \sup_{\frac{k}{n}\le u\le s} \left|B_u^H - B_{k,n}^H\right|. \quad (11)$$

Lemma

(*ii*) *Let $\theta < 0$. Then for any $t > 0$*

$$\sup_{0\le s\le t} |X_s| \le |x_0| + 2 \sup_{0\le s\le t} \left|B_s^H\right| \tag{12}$$

and for any $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right)$

$$\begin{aligned} \sup_{\frac{k}{n}\le u\le s} |X_u - X_{k,n}| \le \frac{|\theta|\,|x_0|}{n} + \frac{2\,|\theta|}{n} \sup_{0\le u\le s} \left|B_u^H\right| \\ + \sup_{\frac{k}{n}\le u\le s} \left|B_u^H - B_{k,n}^H\right|. \end{aligned} \tag{13}$$

Remark

Plugging $p = 2$ *and* $q = 1$ *into the formulae* (8)–(9)*, we get the following bounds:*

$$\sup_{0 \le s \le t} \left| B_s^H \right| \le \left(t^H \log^2 t + 1 \right) \xi(2, H), \tag{14}$$

and for $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

$$\begin{aligned} \left| B_s^H - B_{\frac{k}{n}}^H \right| &\le \left(s - \tfrac{k}{n}\right)^H \left(\left| \log\left(s - \tfrac{k}{n}\right) \right|^{1/2} + 1 \right) \log(s+2) \eta(1, H) \\ &\le \left(\left(s - \tfrac{k}{n}\right)^H \left| \log\left(s - \tfrac{k}{n}\right) \right|^{1/2} + \left(s - \tfrac{k}{n}\right)^H \right) \log(s+2) \eta(1, H). \end{aligned} \tag{15}$$

Remark

Plugging $p = 2$ *and* $q = 1$ *into the formulae* (8)–(9), *we get the following bounds:*

$$\sup_{0 \le s \le t} \left| B_s^H \right| \le \left(t^H \log^2 t + 1 \right) \xi(2, H), \tag{14}$$

and for $s \in \left[\frac{k}{n}, \frac{k+1}{n} \right]$

$$\begin{aligned} \left| B_s^H - B_{\frac{k}{n}}^H \right| &\le \left(s - \tfrac{k}{n}\right)^H \left(\left| \log\left(s - \tfrac{k}{n}\right) \right|^{1/2} + 1 \right) \log(s+2) \eta(1, H) \\ &\le \left(\left(s - \tfrac{k}{n}\right)^H \left| \log\left(s - \tfrac{k}{n}\right) \right|^{1/2} + \left(s - \tfrac{k}{n}\right)^H \right) \log(s+2) \eta(1, H). \end{aligned} \tag{15}$$

Function $f(x) = x^r \left| \log x \right|^{\frac{1}{2}}$ *is bounded on the interval* $(0, 1]$ *for any* $r > 0$. *Therefore*

$$\left(s - \tfrac{k}{n}\right)^H \left| \log\left(s - \tfrac{k}{n}\right) \right|^{1/2} \le C \left(s - \tfrac{k}{n}\right)^{H-r}$$

for any $0 < r < H$. *Furthermore, for* $s \in \left[\frac{k}{n}, \frac{k+1}{n} \right]$ *we have that* $\left(s - \frac{k}{n}\right)^H \le \left(s - \frac{k}{n}\right)^{H-r}$.

Remark

Therefore, we get from (15) *that for any* $0 < r < H$ *and for* $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

$$\left|B_s^H - B_{\frac{k}{n}}^H\right| \leq C\left(s - \tfrac{k}{n}\right)^{H-r} \log(n^{m-1}+2)\eta(1,H). \qquad (16)$$

It follows immediately from (14) *that for* $\theta > 0$

$$\int_0^t e^{-\theta s} \sup_{0\leq u\leq s} \left|B_u^H\right| ds \leq \xi(2,H) \int_0^t e^{-\theta s}\left(s^H \log^2 s + 1\right) ds \leq C\xi(2,H), \qquad (17)$$

and therefore both integrals $\int_0^\infty e^{-\theta s} B_s^H ds$ *and* $\int_0^\infty e^{-\theta s} \sup_{0\leq u\leq s} \left|B_u^H\right| ds$ *exist with probability 1 and admit the same upper bound* $C\xi(2,H)$.

Remark

Combining (10)–(13), (14) *and* (16), *we get that for* $\theta > 0$

$$\sup_{0 \le u \le s} |X_u| \le |x_0| e^{\theta s} + C\theta e^{\theta s} \xi(2, H) + \left(s^H \log^2 s + 1 \right) \xi(2, H),$$

and for $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

$$\sup_{\frac{k}{n} \le u \le s} |X_u - X_{k,n}| \le \theta \int_{\frac{k}{n}}^{s} \Big(e^{\theta u} \left(|x_0| + C\theta \xi(2, H) \right)$$

$$+ (u^H \log^2 u + 1) \xi(2, H) \Big) \, du + \left(n^{-H+r} \log n \right) \eta(1, H),$$

Remark

while for $\theta < 0$

$$\sup_{0\le u\le s} |X_u| \le |x_0| + 2\left(s^H \log^2 s + 1\right)\xi(2,H),$$

and for $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

$$\sup_{\frac{k}{n}\le u\le s} |X_u - X_{k,n}| \le \frac{|\theta|\,|x_0|}{n} + \frac{2\,|\theta|}{n}\sup_{0\le u\le s}\left|B_u^H\right|$$

$$+ \sup_{\frac{k}{n}\le u\le s}\left|B_u^H - B_{k,n}^H\right| \le \frac{|\theta|\,|x_0|}{n} + \frac{2\,|\theta|}{n}\left(s^H \log^2 s + 1\right)\xi(2,H)$$

$$+ \left(n^{-H+r}\log n\right)\eta(1,H).$$

Remark

To simplify the notations, we denote by C any constant whose value is not important for our bounds. Furthermore, we denote by $\mathfrak{Z}$ the class of nonnegative random variables with the property: there exists $C > 0$ not depending on n such that $\mathbf{E}\exp\{x\zeta^2\} < \infty$ for any $0 < x < C$. For example, $\xi(2, H) + C$ and $\eta(1, H) + C$, $C\xi(2, H)$ and $C\eta(1, H)$ for any constant C belong to $\mathfrak{Z}$. Also, note that for fixed $m > 1$ and $n > 3$ we have the upper bound $\log\left(n^{m-1} + 3\right) \leq C \log n$. Moreover, for any $\alpha > 0$ there exists such $n(\alpha)$ that for $n \geq n(\alpha)$ we have $\log n < n^{\alpha}$. Taking this into account and using the simplified notations, we get the bounds with the same $\zeta \in \mathfrak{Z}$: for $\theta > 0$ we have for any fixed $\alpha > 0$, starting with $n \geq n(\alpha)$:

$$\sup_{0 \leq u \leq s} |X_u| \leq \left(e^{\theta s} + s^H \log^2 s\right) \zeta \tag{18}$$

and for $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

$$\sup_{\frac{k}{n} \leq u \leq s} |X_u - X_{k,n}| \leq \left(\frac{1}{n} e^{\theta s} + \frac{1}{n} s^H \log^2 s + n^{-H+\alpha}\right) \zeta, \tag{19}$$

Remark

while for $\theta < 0$

$$\sup_{0 \le u \le s} |X_u| \le \left(1 + s^H \log^2 s\right) \zeta \tag{20}$$

and for $s \in \left[\frac{k}{n}, \frac{k+1}{n}\right]$

$$\sup_{\frac{k}{n} \le u \le s} |X_u - X_{k,n}| \le \left(\frac{1}{n} + \frac{1}{n} s^H \log^2 s + n^{-H+\alpha}\right) \zeta. \tag{21}$$

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Now we are in position to bound both terms in the numerator of the right-hand side of (7). At first, give the bound with probability 1 for the 1st term in the numerator of (7). All inequalities claimed in Lemma 3.1 hold for any $\alpha > 0$ starting with some nonrandom number $n(\alpha)$.

Lemma

(i) Let $\theta > 0$. Then for any $m > 1$ there exists such $\zeta \in \mathfrak{Z}$ that

$$\left| \sum_{k=0}^{n^m - 1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds \right| \le \zeta^2 n^{-1} e^{2\theta n^{m-1}}.$$

Lemma

(ii) Let $\theta < 0$. Then we have two cases.

(a) *Let $1 < m \leq \frac{1}{H}$. Then there exists such $\zeta \in \mathfrak{Z}$ that*

$$\left|\sum_{k=0}^{n^m-1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds\right| \leq \zeta^2 n^{mH+m-2H-1+\alpha}.$$

(b) *Let $m > \frac{1}{H}$. Then there exists such $\zeta \in \mathfrak{Z}$ that*

$$\left|\sum_{k=0}^{n^m-1} X_{k,n} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (X_s - X_{k,n})\, ds\right| \leq \zeta^2 n^{2Hm+m-2H-2+\alpha}.$$

Now we establish moment bounds for the 2nd term in the numerator of the right-hand side of (7). In order to do this, we apply well-known Isserlis' formula for calculation of higher moments of Gaussian distribution: let $\{\chi_1, \chi_2, \chi_3, \chi_4\}$ be a Gaussian vector, then

$$\mathbf{E}(\chi_1\chi_2\chi_3\chi_4) = \mathbf{E}(\chi_1\chi_2)\mathbf{E}(\chi_3\chi_4) + \mathbf{E}(\chi_1\chi_3)\mathbf{E}(\chi_2\chi_4) + \mathbf{E}(\chi_1\chi_4)\mathbf{E}(\chi_2\chi_3).$$

Therefore, we can calculate mathematical expectations $\mathbf{E}B_s^H B_u^H \Delta B_k^H \Delta B_j^H$ for $k \neq j$ as

$$\begin{aligned}\mathbf{E}B_s^H B_u^H \Delta B_k^H \Delta B_j^H &= \mathbf{E}B_u^H \Delta B_k^H \mathbf{E}B_s^H \Delta B_j^H + \mathbf{E}B_s^H \Delta B_k^H \mathbf{E}B_u^H \Delta B_j^H \\ &+ \mathbf{E}B_s^H B_u^H \mathbf{E}\Delta B_k^H \Delta B_j^H \leq \mathbf{E}B_u^H \Delta B_k^H \mathbf{E}B_s^H \Delta B_j^H \\ &+ \mathbf{E}B_s^H \Delta B_k^H \mathbf{E}B_u^H \Delta B_j^H \qquad (22)\end{aligned}$$

because for $H \in \left(0, \frac{1}{2}\right)$ the increments of fBm B^H are negatively correlated and so $\mathbf{E}\Delta B_k^H \Delta B_j^H < 0$. Similarly,

$$\mathbf{E}B_s^H B_u^H \left(\Delta B_{k,n}^H\right)^2 = 2\mathbf{E}B_u^H \Delta B_{k,n}^H \mathbf{E}B_s^H \Delta B_{k,n}^H + n^{-2H}\mathbf{E}B_s^H B_u^H. \qquad (23)$$

Lemma

(*i*) *Let* $\theta > 0$. *Then for any* $m > 1$ *we have the following moment bound*

$$\mathbf{E}\left(\sum_{k=0}^{n^m-1} X_{k,n}\Delta B^H_{k,n}\right)^2 \le Cn^{2-4H}e^{2\theta n^{m-1}}.$$

(*ii*) *Let* $\theta < 0$. *Then for any* $m > 1$ *we have the following moment bound*

$$\mathbf{E}\left(\sum_{k=0}^{n^m-1} X_{k,n}\Delta B^H_{k,n}\right)^2 \le Cn^{2m-4H}.$$

Corollary

(i) Let $\theta > 0$. Then

$$\mathbf{E}\left(n^{4H-2}e^{-2\theta n^{m-1}}\sum_{k=0}^{n^m-1} X_{k,n}\Delta B_{k,n}^H\right)^2 \le C.$$

If we denote $\xi_n = n^{2H-1}e^{-\theta n^{m-1}}\sum_{k=0}^{n^m-1} X_{k,n}\Delta B_{k,n}^H$ then $\sup_{n\ge 1}\mathbf{E}\xi_n^2 < \infty$. It means that for any $m > 1$ the numerator of (7) can be bounded by the sum

$$\zeta^2 n^{-1}e^{2\theta n^{m-1}} + n^{1-2H}e^{\theta n^{m-1}}\xi_n,$$

where $\sup_{n\ge 1}\mathbf{E}\xi_n^2 < \infty$.

Corollary

(ii) *Let* $\theta < 0$. *Then we have two cases.*

(a) *Let* $1 < m \leq \frac{1}{H}$. *Then for any* $\alpha > 0$ *the numerator of* (7) *can be bounded by the sum*

$$\zeta^2 n^{(m-2)H+m-1+\alpha} + n^{m-2H}\xi_n,$$

where $\sup_{n\geq 1} \mathbf{E}\xi_n^2 < \infty$.

(b) *Let* $m > \frac{1}{H}$. *Then for any* $\alpha > 0$ *the numerator of* (7) *can be bounded by the sum*

$$\zeta^2 n^{(2H+1)m-2H-2+\alpha} + n^{m-2H}\xi_n,$$

where $\sup_{n\geq 1} \mathbf{E}\xi_n^2 < \infty$.

Contents

Now our goal is to present the denominator of (7) in more convenient form. At first we compare the sum $\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2$ to the corresponding integral $\int_0^{n^{m-1}} X_s^2\,ds$. The reason to replace the sum with the corresponding integral is that for some values of H and m we can prove the consistency with the help of some kind of L'Hôpital's rule, however, the application of L'Hôpital's rule or Stolz-Cesàro theorem to the sum $\sum_{k=0}^{n^m-1} X_{k,n}^2$ is problematic because not only the upper bound but the terms in the sum depend on n.

Lemma

(*i*) *Let* $\theta > 0$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that*

$$\left| \int_0^{n^{m-1}} X_s^2 \, ds - \frac{1}{n} \sum_{k=0}^{n^m-1} X_{k,n}^2 \right| \leq \frac{\zeta_1^2}{n} e^{2\theta n^{m-1}}.$$

(*ii*) *Let* $\theta < 0$. *Then we have two cases.*

(*a*) *Let* $1 < m \leq \frac{1}{H}$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that for any* $\beta > 0$

$$\left| \int_0^{n^{m-1}} X_s^2 \, ds - \frac{1}{n} \sum_{k=0}^{n^m-1} X_{k,n}^2 \right| \leq \zeta_1^2 n^{mH+m-2H-1+\beta}.$$

(*b*) *Let* $m > \frac{1}{H}$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that for any* $\beta > 0$

$$\left| \int_0^{n^{m-1}} X_s^2 \, ds - \frac{1}{n} \sum_{k=0}^{n^m-1} X_{k,n}^2 \right| \leq \zeta_1^2 n^{2mH+m-2H-2+\beta}.$$

Corollary

(i) Let $\theta > 0$. Then there exists such $\zeta_1 \in \mathfrak{Z}$ that

$$\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2 = \int_0^{n^{m-1}} X_s^2\,ds + \vartheta_n,$$

where

$$|\vartheta_n| \le \frac{\zeta_1^2}{n} e^{2\theta n^{m-1}}.$$

Corollary

(*ii*) *Let* $\theta < 0$. *Then we have two cases.*

(*a*) *Let* $1 < m \leq \frac{1}{H}$. *Then there exists such* $\zeta_1 \in \mathfrak{Z}$ *that for any* $\beta > 0$

$$\frac{1}{n} \sum_{k=0}^{n^m - 1} X_{k,n}^2 = \int_0^{n^{m-1}} X_s^2 \, ds + \vartheta_n(\beta), \tag{24}$$

where

$$|\vartheta_n(\beta)| \leq \zeta_1^2 n^{mH+m-2H-1+\beta}. \tag{25}$$

(*b*) *Let* $m > \frac{1}{H}$. *Then for any* $\beta > 0$ *representation* (24) *holds with*

$$|\vartheta_n(\beta)| \leq \zeta_1^2 n^{2mH+m-2H-2+\beta}.$$

Contents

Consider separately cases $\theta > 0$ and $\theta < 0$. The case $\theta > 0$ is more simple and additionally the estimator is strongly consistent. The case $\theta < 0$ needs some additional calculations and we prove only that the estimator is consistent.

Theorem

Let $\theta > 0$. Then for any $m > 1$ estimator $\widehat{\theta}_n(m)$ is strongly consistent.

In the case when $\theta < 0$ we can establish consistency, but not the strong consistency of (7) as $n \to \infty$. According to Corollaries 3.4 and 4.2, we need to bound the following random variables: for $1 < m \leq \frac{1}{H}$

$$K_n^1(m, \alpha, \beta) := \frac{n^{mH+m-2H-1+\alpha}}{\int_0^{n^{m-1}} X_s^2\, ds + \vartheta_n} \quad \text{and} \quad K_n^2 := \frac{n^{m-2H}\xi_n}{\frac{1}{n}\sum_{k=0}^{n^m-1} X_{k,n}^2}.$$

and for $m > \frac{1}{H}$

$$\widetilde{K}_n^1(m, \alpha, \beta) := \frac{n^{2Hm+m-2H-2+\alpha}}{\int_0^{n^{m-1}} X_s^2\, ds + \vartheta_n}$$

and the same K_n^2.

Lemma

Let $\theta < 0$.

(*i*) *For any* $1 < m < \frac{2H+1}{H+1} < \frac{1}{H}$ *there exist such* $\alpha > 0$ *and* $\beta > 0$ *that*

$$K_n^1(m, \alpha, \beta) \to 0$$

a.s. as $n \to \infty$.

(*ii*) *For any* $\frac{2H+1}{H+1} \leq m \leq \frac{1}{H}$ *there exist such* $\alpha > 0$ *and* $\beta > 0$ *that*

$$K_n^1(m, \alpha, \beta) \to 0$$

in probability as $n \to \infty$.

(*iii*) *There exist such* $\alpha > 0$ *and* $\beta > 0$ *that for any* $m > \frac{1}{H}$

$$\widetilde{K}_n^1(m, \alpha, \beta) \to 0$$

in probability as $n \to \infty$.

Remark

We can prove more than it was mentioned in (i), namely, to establish that

$$\int_0^{n^{m-1}} X_s^2 \, ds \to \infty$$

a.s. as $n \to \infty$ (see Lemma 6.2 in Section 42).

Lemma

Let $m > \frac{1}{2H}$. Then $K_n^2 \to 0$ in probability as $n \to \infty$.

Theorem

Let $m > \frac{1}{2H}$. Then the estimator $\widehat{\theta}_n(m)$, introduced in (7), is consistent.

Contents

At first we establish an auxiliary result concerning the bounds for several sums of integral type that will participate in the bounds for the numerator of (7)

Lemma

For any $m > 1$ *and* $n \geq 2$ *there exists* $C > 0$ *not depending on* n *such that*

(i)

$$\sum_{k=0}^{n^m-1} \left(\frac{k+1}{n}\right)^H \log^2 \frac{k+1}{n} \leq C n^{(m-1)H+m} \log^2 n,$$

(ii)

$$\sum_{k=0}^{n^m-1} \left(\frac{k+1}{n}\right)^{2H} \log^4 \frac{k+1}{n} \leq C n^{2H(m-1)+m} \log^4 n.$$

Next auxiliary result establishes asymptotic behavior of integral $\int_0^T X_s^2\,ds$ as $T \to \infty$.

Lemma

Let process X satisfy equation (1). Then $\int_0^T X_s^2\,ds \to \infty$ with probability 1 as $T \to \infty$.

Contents

In this section, we present the results of simulation experiments. We simulate 20 trajectories of the fractional Ornstein–Uhlenbeck process (1) with $x_0 = 1$ for different values of θ and H. Then we compute values of $\widehat{\theta}_n(m)$. For each combination of θ, H, n and m the mean of the estimator is reported.

In Tables 1–3 the true value of the drift parameter θ equals 2. In this case the behavior of the estimators is almost the same for different values of H. Also we can see that the value of $\widehat{\theta}_n(m)$ is determined by n and does not depend on m. Further, we consider the case of negative θ. We simulate the process with $H = 0.45$, $\theta = -3$ and $m = 4, 5$. The results are reported in Tables 4–5. One can see that the method works but the rate of convergence to the true value of a parameter is not very high. There are two reasons for this: the estimator is only consistent not strongly consistent and moreover, the trajectories are so irregular that even the length of the interval is small we can not “catch” the trajectory.

Table: $\theta = 2,\ m = 2.$

n	5	10	50	100	500	1000
$H = 0.05$	2.45763	2.21281	2.0395	2.01911	2.00300	2.00100
$H = 0.25$	2.45766	2.21281	2.0395	2.01911	2.00300	2.00100
$H = 0.45$	2.45794	2.21281	2.0395	2.01911	2.00300	2.00100

Table: $\theta = 2,\ m = 3.$

n	5	10	20	25
$H = 0.05$	2.45763	2.21281	2.10231	2.08109
$H = 0.25$	2.45763	2.21281	2.10231	2.08109
$H = 0.45$	2.45763	2.21281	2.10231	2.08109

Table: $\theta = 2,\ m = 4.$

n	5	8	10	12	15
$H = 0.05$	2.45763	2.27092	2.21281	2.17240	2.13566
$H = 0.25$	2.45763	2.27092	2.21281	2.17240	2.13566
$H = 0.45$	2.45763	2.27092	2.21281	2.17240	2.13566

Table: $\theta = -3,\ H = 0.45,\ m = 4.$

n	2	4	6	8	10
$\widehat{\theta}_n(4)$	-1.50913	-2.41157	-2.71411	-2.9546	-3.12058

Table: $\theta = -3,\ H = 0.45,\ m = 5.$

n	2	3	4	5	6
$\widehat{\theta}_n(5)$	-1.63396	-2.04297	-2.38237	-2.5595	-2.72538

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