

Block bootstrap for Poisson sampled almost periodic processes

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I – Almost periodically correlated processes

II – Sampling

III – Estimation of the cyclic mean from Poisson sampling

IV – Bootstrap method

V – Some simulations

I – Almost periodically correlated processes (APC)

Examples .

$$(i) \quad X_1(t) = \sin(2\pi t/10) + Z(t), \quad t \in \mathbb{R},$$

$$(ii) \quad X_2(t) = \sin(2\pi t/10) + Z(t) \cos(2\pi t/10), \quad t \in \mathbb{R},$$

where $\{Z(t), t \in \mathbb{R}\}$, zero mean, stationary with continuous covariance.

$$(iii) \quad X_3(t) = U(t) \cos(t), \quad t \in \mathbb{R},$$

$$(iv) \quad X_4(t) = U(t) \cos(t) + U(t - 1) \cos(\pi t), \quad \forall t \in \mathbb{R},$$

where U is an Ornstein Uhlenbeck process (stationary Gaussian process) with $r(\tau) = e^{-|\tau|}$.

Almost periodically correlated processes (APC)

1 – Definitions

A function $f : \mathbb{R} \rightarrow \mathbb{C}$ is **almost periodic** when

$\forall \epsilon > 0, \exists L_\epsilon > 0$ s.t. $\forall I$ interval with $\text{length}(I) \geq L_\epsilon, \exists p_\epsilon \in I$ with

$$\sup_{s \in \mathbb{R}} |f(s + p_\epsilon) - f(s)| < \epsilon.$$

This is equivalent to the existence of trigonometric polynomials $P_n(\cdot)$, $n \in \mathbb{N}$ such that

$$\sup_{s \in \mathbb{R}} |P_n(s) - f(s)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Examples:

$$f(s) = \cos(s) + \cos(\pi s)$$

$$f(s) = \cos(s) + \frac{1}{2} \cos\left(\frac{s}{2}\right) + \frac{1}{4} \cos\left(\frac{s}{4}\right) + \dots + \frac{1}{2^n} \cos\left(\frac{s}{2^n}\right) + \dots$$

A process $X = \{X(s), s \in \mathbb{R}\}$ is **periodically correlated (PC)** (Gudzenko, 1959) (Hurd, 1969) (Dragan, Rozhkov, Javoskyj, 1987) or *wide sense almost cyclostationary* (Gardner, 1994) when

- $E[X(s)^2] < \infty$, for any $s \in \mathbb{R}$,
- $s \mapsto E[X(s)]$ and $s \mapsto B(s, \tau) = \text{cov}[X(s), X(s + \tau)]$ are periodic in s for each τ with the same period P .

A process $X = \{X(s), s \in \mathbb{R}\}$ is **almost periodically correlated (APC)** (Gladyshev, 1963) or *wide sense almost cyclostationary* (Gardner, 1994) when

- $E[X(s)^2] < \infty$, for any $s \in \mathbb{R}$,
- $s \mapsto E[X(s)]$ and $s \mapsto B(s, \tau) = \text{cov}[X(s), X(s + \tau)]$ are almost periodic in s for each τ , has some continuity property in τ .

Applications

Mechanical systems (Diagnostic and Vibro-Acoustic): Antoni (2009),

Communication and signal processing : Gardner et al. (1994), Napolitano (2013)

Economics : Lenart and Pipien (2013)

Hydrology, meteorology, ...

Bibliography in Gardner et al. (2006), Serpendin et al. (2005), Hurd and Miamee (2007)

2 – Spectrum

$X = \{X(s) : s \in \mathbb{R}\}$ APC process.

Cyclic mean : for $\lambda \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$m_\lambda := \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \mathbb{E}[X(s)] e^{-i\lambda s} ds.$$

Cyclic covariance : for $\lambda \in \mathbb{R}$, $t, \tau \in \mathbb{R}$

$$a(\lambda, \tau) := \lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{t+T} \text{cov}[X(s), X(s + \tau)] e^{-i\lambda s} ds.$$

Cycle frequency sets of X :

$$\text{first order} \quad \Lambda_1 := \{ \lambda : m_\lambda \neq 0 \}$$

$$\text{second order} \quad \Lambda_2 := \{ \lambda : a(\lambda, \tau) \neq 0 : \text{for some } \tau \}.$$

They are at most countable.

For a **PC process** with period $P > 0$ we have

$$\Lambda_1, \Lambda_2 \subset \frac{2k\pi}{P}\mathbb{Z} = \left\{ \frac{2k\pi}{P} : k \in \mathbb{Z} \right\}$$

$$m_\lambda = \frac{1}{P} \int_0^P \mathbb{E}[X(s)X(s+\tau)] e^{-i\lambda s} ds, \quad \forall \lambda \in \Lambda_1$$

$$a_\lambda(\tau) = \frac{1}{P} \int_0^P \text{cov}[X(s), X(s+\tau)] e^{-i\lambda s} ds, \quad \forall \lambda \in \Lambda_2$$

and

$$m_\lambda = 0, \quad a_\lambda(\tau) = 0 \quad \text{otherwise.}$$

For $\{X(t) : t \in \mathbb{R}\}$ a **stationary process** with mean m and covariance function $r(\tau) = E[X(s + \tau)X(s)]$, we have

$$\Lambda_1 \subset \{0\}, \quad \Lambda_2 = \{0\}$$

$$m_0 = m \in \mathbb{R},$$

$$a_0(\tau) = r(\tau)$$

$$m_\lambda = 0, \quad a_\lambda(\tau) = 0 \quad \forall \lambda \neq 0$$

3 – Examples

Example (i) $X_1(t) = \sin\left(\frac{2\pi t}{10}\right) + Z(t), t \in \mathbb{R},$

$\{Z(t), t \in \mathbb{R}\}$, zero mean, stationary with continuous covariance $r(\cdot)$.

Then X_1 is PC and $\Lambda_1 = \left\{\pm\frac{\pi}{5}\right\}, \quad \Lambda_2 = \{0\}$

$$m_{\pi/5} = \overline{m_{-\pi/5}} = \frac{-i}{2}, \quad a_0(\tau) = r(\tau).$$

Example (ii) $X_2(t) = \sin\left(\frac{2\pi t}{10}\right) + Z(t) \cos\left(\frac{2\pi t}{10}\right), t \in \mathbb{R},$

Then X_2 is PC and $\Lambda_1 = \left\{\pm\frac{\pi}{5}\right\}, \quad \Lambda_2 = \left\{0, \pm\frac{2\pi}{5}\right\}$

$$m_{\pi/5} = \overline{m_{-\pi/5}} = \frac{-i}{2}$$

$$a_0(\tau) = \frac{1}{2} \cos\left(\frac{\pi\tau}{5}\right) r(\tau), \quad a_{2\pi/5}(\tau) = \overline{a_{2\pi/5}(\tau)} = \frac{1}{4} e^{i\frac{\pi\tau}{5}} r(\tau).$$

Example (iii) $X_3(t) = U(t) \cos(t), t \in \mathbb{R},$

where U is an Ornstein Uhlenbeck process (stationary Gaussian process) with $r(\tau) = e^{-|\tau|}$.

Then X_3 is PC and

$$\begin{aligned} \mathbb{E}\{X(s)X(s + \tau)\} &= e^{-|\tau|} \cos(s) \cos(s + \tau) \\ &= a(-2, \tau) e^{-i2s} + a(0, \tau) + a(2, \tau) e^{i2s} \end{aligned}$$

$$\Lambda_1 = \emptyset, \quad \Lambda_2 = \{-2, 0, 2\},$$

$$a(0, \tau) = \frac{1}{2} e^{-|\tau|} \cos \tau, \quad a(2, \tau) = \overline{a(-2, \tau)} = \frac{1}{4} e^{-|\tau|} e^{i\tau}.$$

Example (iv)

$$X_4(t) = U(t) \cos(t) + U(t-1) \cos(\pi t), \forall t \in \mathbb{R},$$

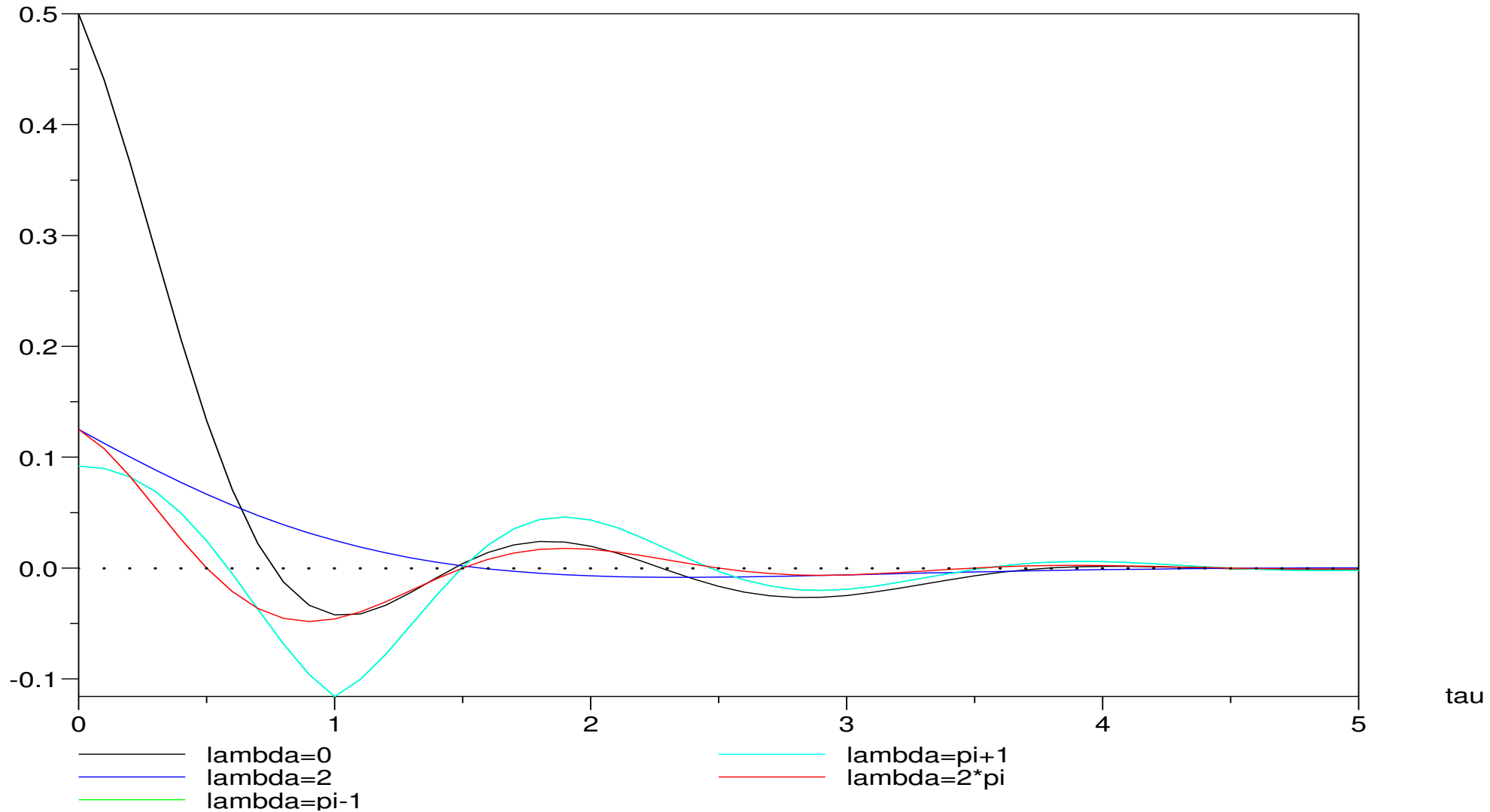
U is an Ornstein Uhlenbeck process.

Then the process X_4 is APC,

$$\Lambda_1 = \emptyset, \quad \Lambda_2 = \{-2\pi, -1 - \pi, 1 - \pi, -2, 0, 2, \pi - 1, \pi + 1, 2\pi\}.$$

Re(a(lambda,tau))

Covariance spectrale de
 $X(t)=\cos(t)*Z(t)+\cos(\pi*t)*Z(t-1)$



Real parts of the cyclic covariance

II – Sampling

Aim : estimation of m_λ and determination of $\Lambda_1 = \{\lambda \in \mathbb{R} : m_\lambda \neq 0\}$.

1 – Continuous time observation

Observation : one sample $\{X(t), t \in [0, T]\}$, $T \rightarrow \infty$.

Estimator of $m(\lambda)$ (Hurd & Miamee, 2007)

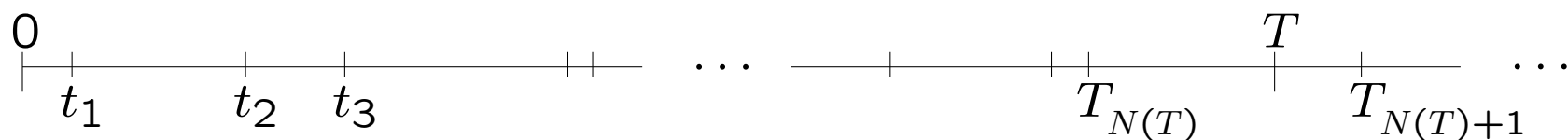
$$\widehat{m}_T^{(cont)}(\lambda) := \frac{1}{T} \int_0^T X(t) e^{-i\lambda t} dt,$$

2 – Discrete time observation of $\{X(t), t \in [0, T]\}$

Observation : $\{X(t_k) : k = 1, \dots, N(T)\}$

- Either deterministic instants $t_k = k\delta$ where $k = 0, \dots, N(T) := \left\lfloor \frac{T}{\delta} \right\rfloor$
 - $\delta = \text{constante} \implies$ folding and aliasing phenomena
 - $\delta = \delta_T \rightarrow 0$ as $T \rightarrow \infty$ with some rate \implies consistency
- Or random sampling instants $t_k = T_k$: objective of this work.

Here we consider a Poisson random sampling : $\{T_k\}$ occurrence instants of a Poisson point process (Masry 1978)



3 – Poisson random sampling

Sampling scheme : $\{N(t), t \geq 0\}$ stationary Poisson point process, with intensity $\beta > 0$:

independent increments, and $\mathcal{L}(N(s+t) - N(s)) = \mathcal{P}(\beta t)$

$$\mathbb{E}[N(t+dt)] = \beta dt, \quad \text{cov}[N(t+dt), N(t+u+du)] = \beta \delta_{\{0\}}(du) dt$$

$T_k, k = 1, 2, \dots$ occurrence instants of $\{N(t), t \geq 0\}$, and $T_0 \equiv 0$:

$T_k - T_{k-1}$'s independent random variables, and $\mathcal{L}(T_k - T_{k-1}) = \mathcal{E}(\beta)$

Hypothesis : Independence between $\{X(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$.

Observations : $T_1, \dots, T_{N(T)}$ and $X(T_1), X(T_2), \dots, X(T_{N(T)})$.

Sampled process : random measure

$$Z(B) := \int_0^T X(t) N(t + dt) = \sum_{k=1}^{N(T)} X(T_k)$$

Properties :

$$\mathbb{E}[Z(t + dt)] = \beta \mathbb{E}[X(t)] dt$$

$$\begin{aligned} & \text{cov}[Z(t + dt), Z(t + u + du)] \\ &= \left(\beta^2 \text{cov}[X(t), X(t + u)] du + \beta \left(\text{var}[X(t)] + \mathbb{E}[X(t)^2] \right) \delta_0(du) \right) dt. \end{aligned}$$

So the mean measure $\mathbb{E}[Z(t + dt)]$ and the covariance measure $\text{cov}[Z(t + dt), Z(t + u + du)]$ admits some **almost periodic** properties with respect to t .

Properties of the sampled process :

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{[0, T]} e^{-i\lambda t} \mathbb{E}\{Z(t + dt)\} = \beta m_\lambda$$

and

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_{[0, T] \times B} e^{-i\lambda t} \text{cov}\{Z(t + dt), Z(t + u + du)\} \\ = \beta^2 \int_B a_\lambda(u) du + \beta \left(a_\lambda(0) + m_\lambda^{(2)} \right) \delta_0(B) \end{aligned}$$

for any $B \in \mathcal{B}_b$.

Here

$$m_\lambda^{(2)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{E}[X(t)]^2 e^{-i\lambda t} dt,$$

$$a_\lambda(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{cov}[X(t), X(t + \tau)] e^{-i\lambda t} dt$$

III – Estimation of m_λ from Poisson sampling

$\{X(t), t > 0\}$ and $\{N(t), t \geq 0\} = \{T_k, k = 1, \dots\}$, independent

Observation of $(T_k, X(T_k)), k = 1, \dots, N(T), \quad T \rightarrow \infty$

1 – Estimator

$$\widehat{m}_T(\lambda) := \frac{1}{T^\beta} \sum_{k=1}^{N(T)} X(T_k) e^{-i\lambda T_k} = \frac{1}{T^\beta} \int_0^T X(t) N(t) dt$$

2 – Hypotheses

To establish some results, we assume that the process X satisfies some asymptotic independence. Here we consider that the process X is strongly mixing (α -mixing) (Rosenblatt, 1956), (Doukhan, 1984).

Definition . The process X is α -mixing when $\alpha_X(t) \xrightarrow{t \rightarrow \infty} 0$, where

$$\alpha_X(t) := \sup \left\{ \left| \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \right| : A \in \mathcal{F}_{-\infty}^s(X), B \in \mathcal{F}_{t+s}^\infty(X), s \in \mathbb{R} \right\},$$

$$\mathcal{F}_{-\infty}^s(X) := \sigma \left\{ X(u) ; u \leq s \right\} \quad \text{and} \quad \mathcal{F}_{s+t}^\infty(X) := \sigma \left\{ X(u) ; s + t \leq u \right\}.$$

A1 Separability of the frequencies $\sum_{\lambda \in \Lambda_1} \lambda^{-2} < \infty$.

A2 *Mixing condition* The process $\{X(t), t \geq 0\}$ is α -mixing with

– either $\{X(t), t \geq 0\}$ is bounded, $\int_0^\infty \tau \alpha_X(\tau) d\tau < \infty$

– or there exists $\delta > 0$ such that

$$\sup_t \mathbb{E} [X(t)^{4+\delta}] < \infty \quad \text{and} \quad \int_0^\infty \tau \alpha_X(\tau)^{\delta/(4+\delta)} d\tau < \infty.$$

3 – Properties

– Under **A1** $\limsup_{T \rightarrow \infty} T \left| \mathbb{E}[\widehat{m}_T(\lambda)] - m_\lambda \right| < \infty, \quad \lambda \in \mathbb{R}$

– Under **A2** For $\lambda_1, \lambda_2 \in \mathbb{R}$

$$\begin{aligned} C_{(\lambda_1, \lambda_2)} &:= \lim_{T \rightarrow \infty} T \operatorname{cov} [\widehat{m}_T(\lambda_1), \widehat{m}_T(\lambda_2)] \\ &= \frac{m_{(\lambda_1 - \lambda_2)}^{(2)}}{\beta} + \frac{a_{(\lambda_1 - \lambda_2)}(0)}{\beta} + \int_{\mathbb{R}} a_{(\lambda_1 - \lambda_2)}(\tau) e^{i\lambda_2 \tau} d\tau. \end{aligned}$$

where

$$m_\lambda^{(2)} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mathbb{E}[X(t)]^2 e^{-i\lambda t} dt, \quad a_\lambda(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \operatorname{cov}[X(t), X(t+\tau)] e^{-i\lambda t} dt$$

– Under **A1** and **A2**

$$\lim_{T \rightarrow \infty} T \mathbb{E} \left[|\widehat{m}_T(\lambda) - m_\lambda|^2 \right] = \sigma_\lambda^2 := C_{(\lambda, \lambda)}.$$

Theorem Under **A1** and **A2**

$$\mathcal{L}\left(\sqrt{T}\left(\widehat{m}_T(\lambda) - m_\lambda\right)\right) \longrightarrow \mathcal{N}_2\left(0, \Sigma_2(\lambda)\right) \quad \text{as } T \rightarrow \infty.$$

$$\Sigma_2(\lambda) := \begin{pmatrix} \sigma_{1,1}^2(\lambda) & c_{1,2}(\lambda) \\ c_{2,1}(\lambda) & \sigma_{2,2}^2(\lambda) \end{pmatrix},$$

where

$$\sigma_{1,1}^2(\lambda) = \frac{1}{2} \left(C_{(\lambda,\lambda)} + \Re \left(C_{(\lambda,-\lambda)} \right) \right), \quad \sigma_{2,2}^2(\lambda) = \frac{1}{2} \left(C_{(\lambda,\lambda)} - \Re \left(C_{(\lambda,-\lambda)} \right) \right),$$

$$c_{1,2}(\lambda) = c_{2,1}(\lambda) = \frac{i}{4} \left(-C_{(\lambda,-\lambda)} + C_{(-\lambda,\lambda)} \right) = \frac{1}{2} \Im \left(C_{(\lambda,-\lambda)} \right).$$

Tools for the proofs

1) Some rates of convergence

$$|m_\lambda| \leq \sup_s |\mathbb{E}[X(s)]| < \infty, \quad |a(\lambda, \tau)| \leq \sup_s 2\mathbb{E}[X(s)^2] < \infty.$$

$$\left| m_\lambda - \frac{1}{T} \int_t^{t+T} \mathbb{E}[X(s)] e^{-i\lambda s} ds \right| \leq \frac{1}{T} \sum_{\alpha \in \Lambda^{(1)} \setminus \{\lambda\}} \frac{|m_\alpha|}{|\alpha - \lambda|}$$
$$\leq \frac{\sup_s |\mathbb{E}[X(s)]|}{T} \left(\sum_{\alpha \in \Lambda^{(1)} \setminus \{\lambda\}} \frac{1}{(\alpha - \lambda)^2} \right)^{1/2}$$

2) Covariance inequality

(Wolonski & Rozanov, 1959) (Davydov, 1970)

Let ξ be $\mathcal{F}_{-\infty}^s(X)$ -measurable, and ζ be $\mathcal{F}_{s+t}^\infty(X)$ -measurable

$$|\text{cov}\{\xi, \zeta\}| \leq 4c_1 c_2 \alpha(t) \quad \text{if} \quad |\xi| \leq c_1, |\zeta| \leq c_2$$

$$|\text{cov}\{\xi, \zeta\}| \leq 8 \alpha(t)^{1 - \frac{2}{p}} \mathbb{E}\{|\xi|^p\}^{\frac{1}{p}} \mathbb{E}\{|\zeta|^p\}^{\frac{1}{p}} \quad \text{for} \quad p > 2$$

3) Central limit theorem for (nonstationary) mixing processes (Guyon 1995)

IV – Bootstrap method

Efron (1979) : bootstrap for i.i.d random sequence (see also Giné (1997))

Künsh (1989), Liu and Singh (1992) : moving block bootstrap for dependent (stationary) data : (see Lahiri 2003)

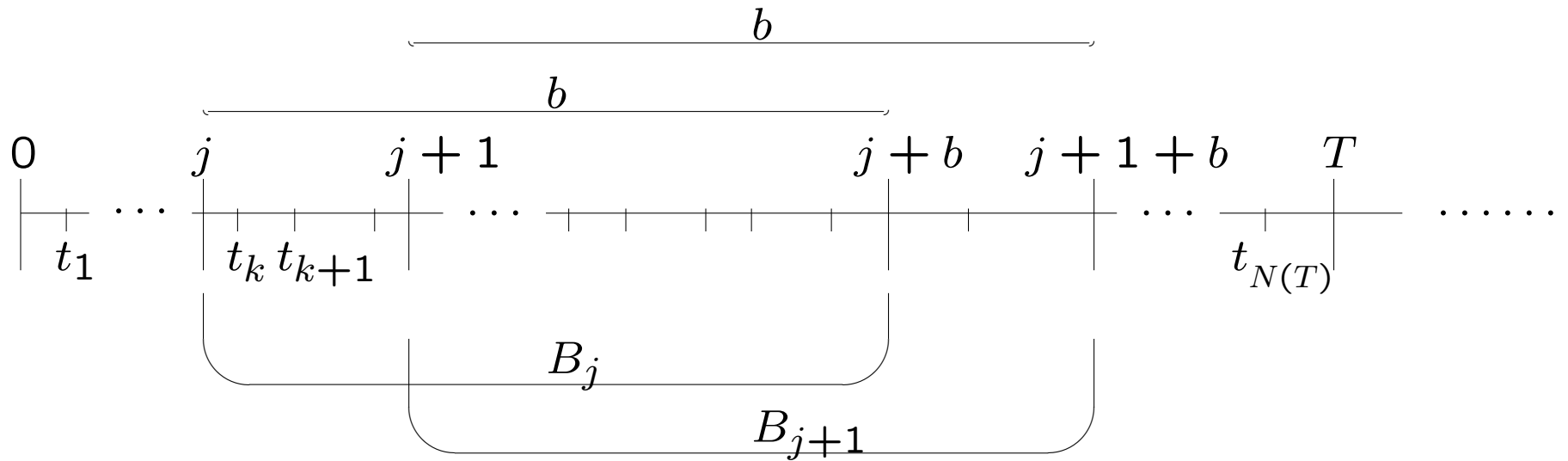
Politis and Romano (1992) : circular block bootstrap (stationary data)

Synowiecki (2007), Leskow et al. (2010), Dudek et al. (2013) : moving block bootstrap for PC and APC time series

1 – Circular block bootstrap method : CBB

b block length, $T = lb$,

$$B_j := \{(T_k, X(T_k)) : j < T_k \leq j + b, k \in \mathbb{N}^*\}, \quad j = 0, \dots, T-1 \quad (\text{circular})$$



Random selection with replacement of l blocks from $\{B_0, \dots, B_{T-1}\}$

\iff

i_1^*, \dots, i_l^* i.i.d. random variables unif. distrib. on $\{0, 1, \dots, T-1\}$

$$P[i_j^* = k] = \frac{1}{T}, \quad k = 0, \dots, T-1, \quad j = 1, \dots, l.$$

Definition CBB sample of the data $\{(T_k, X(T_k)) : k = 1, \dots, N(T)\}$ is

$$(B_1^*, \dots, B_l^*), \quad \text{where } B_j^* := B_{i_j^*}$$

Interest : conservation on some correlation structure in the process

Block length : $b = b_T \rightarrow \infty$, $T = lb$,

Problem of choice of b

2 – Bootstrap version of the estimator $\widehat{m}_T(\lambda)$

Let $\widehat{m}_{j,b}$ the estimator of m_λ constructed from $B_j = B_{j,b}$:

– for $j = 0, \dots, T - b$, we have

$$\widehat{m}_{j,b} := \frac{1}{\beta b} \sum_{k:j < T_k \leq j+b} X(T_k) e^{-i\lambda T_k},$$

– for $j = T - b + 1, \dots, T - 1$ as a result of wrapped blocks we get

$$\widehat{m}_{j,b} := \frac{1}{\beta b} \left(\sum_{k:j < T_k \leq T} X(T_k) e^{-i\lambda T_k} + \sum_{k:0 < T_k \leq T-j+b} X(T_k) e^{-i\lambda T_k} \right).$$

Then $\widehat{m}_T = \frac{1}{l} \sum_{j=0}^{l-1} \widehat{m}_{w+jb,b}$ for any $w = 0, \dots, b - 1$

Bootstrap version of \widehat{m}_T : $\widehat{m}_T^* := \frac{1}{l} \sum_{j=0}^{l-1} \widehat{m}_{j,b}^*$

Bootstrap version of \widehat{m}_T : $\widehat{m}_T^* := \frac{1}{l} \sum_{j=0}^{l-1} \widehat{m}_{j,b}^*$ where

$$\widehat{m}_{j,b}^* := \frac{1}{b\beta} \sum_{k:j^* < T_k \leq j^* + 1} X(T_k) e^{-i\lambda T_k}.$$

Conditional expectation with respect to the observation :

$$\mathbb{E}^* \left[\widehat{m}_T^* \right] := \mathbb{E} \left[\widehat{m}_T^* \mid (T_k, X(T_k)) : k = 1, \dots, N(T) \right] = \widehat{m}_T.$$

\mathbb{E}^* is the expectation conditioned by the observation.

Theorem

assume **A1** and **A2** be satisfied and $b \rightarrow \infty$ as $T \rightarrow \infty$ but $b = o(T)$.
then for any $\epsilon > 0$

$$\lim_{T \rightarrow \infty} \mathbf{P} \left[\text{dist} \left[\mathcal{L}^* \left\{ \sqrt{T}(\widehat{m}_T^* - \widehat{m}_T) \right\}, \mathcal{L} \left\{ \sqrt{T}(\widehat{m}_T - m_\lambda) \right\} \right] > \epsilon \right] = 0$$

for any $\epsilon > 0$ and for any λ .

Here dist is a distance which metrizes the convergence in distribution.

In particular, if the cumulative limit distribution of $\sqrt{T}(\widehat{m}_T^* - m_\lambda)$ is continuous **then** for the functions $\phi(z) = |z|^2$, $\phi(z) = \Re z$, $\phi(z) = \Im z$

$$\sup_{x \in \mathbb{R}} \left| \mathbf{P}^* \left[\phi \left(\sqrt{T}(\widehat{m}_T^* - \widehat{m}_T) \right) \leq x \right] - \mathbf{P} \left[\phi \left(\sqrt{T}(\widehat{m}_T - m_\lambda) \right) \leq x \right] \right| \xrightarrow{\text{P-proba}} 0$$

as $T \rightarrow \infty$.

Idea for the proof

$$Y_k^{(\lambda)} := \frac{1}{\beta} \int_k^{k-1} e^{-i\lambda s} X(x) N(s + ds)$$

$\{Y_k^{(\lambda)}, k \in \mathbb{N}\}$ is APC time series

Radulović (1996), Giné (1997), Synowiecki (2007)

Araujo and Giné (1980)

V – Some simulations

E1 $X(t) = U(t) + C_i \sin(2\pi t/10);$

E2 $X(t) = U(t) \cos(2\pi t/10) + C_i \sin(2\pi t/10),$

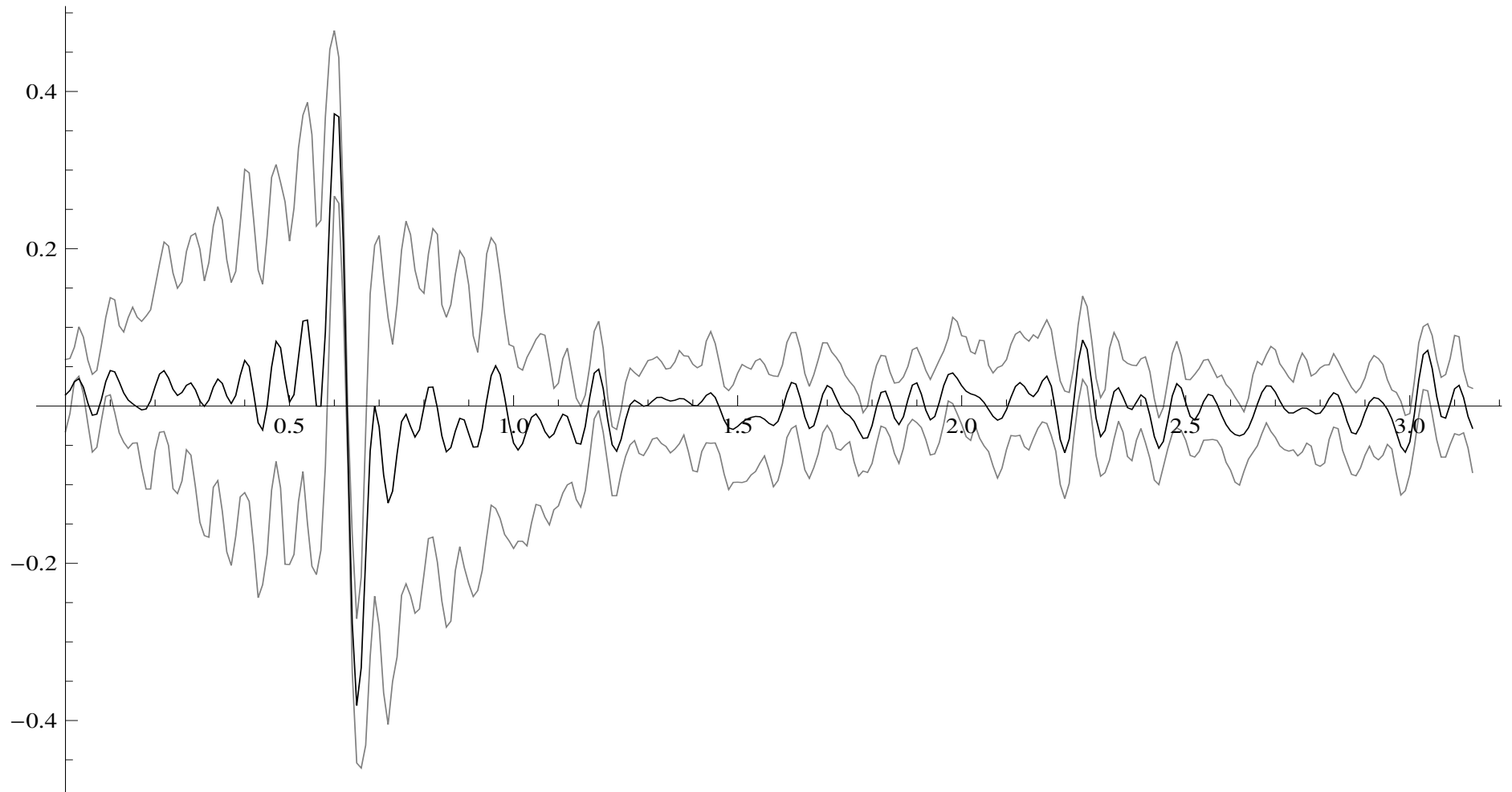
$U = \{U(t) : t \in \mathbb{R}\}$ Ornstein Uhlenbeck process : stationary Markov Gaussian process (cov: $r(\tau) = e^{-|\tau|/2}$)

$C_1 = 1, C_2 = 0.01$

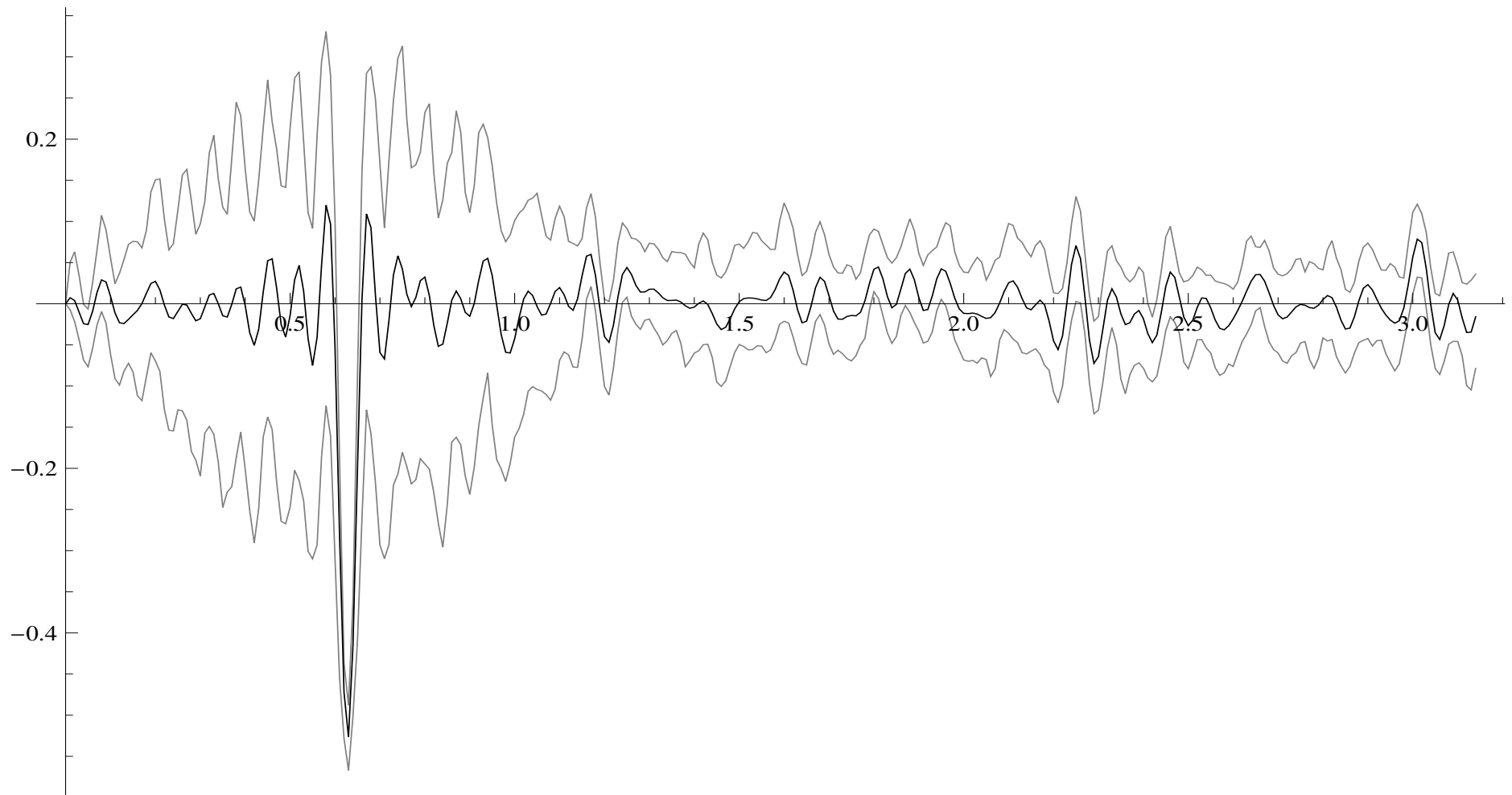
Sampling Scheme : Poisson point process with intensity $\beta = 5$

Observation interval $[0, 100], T = 100$

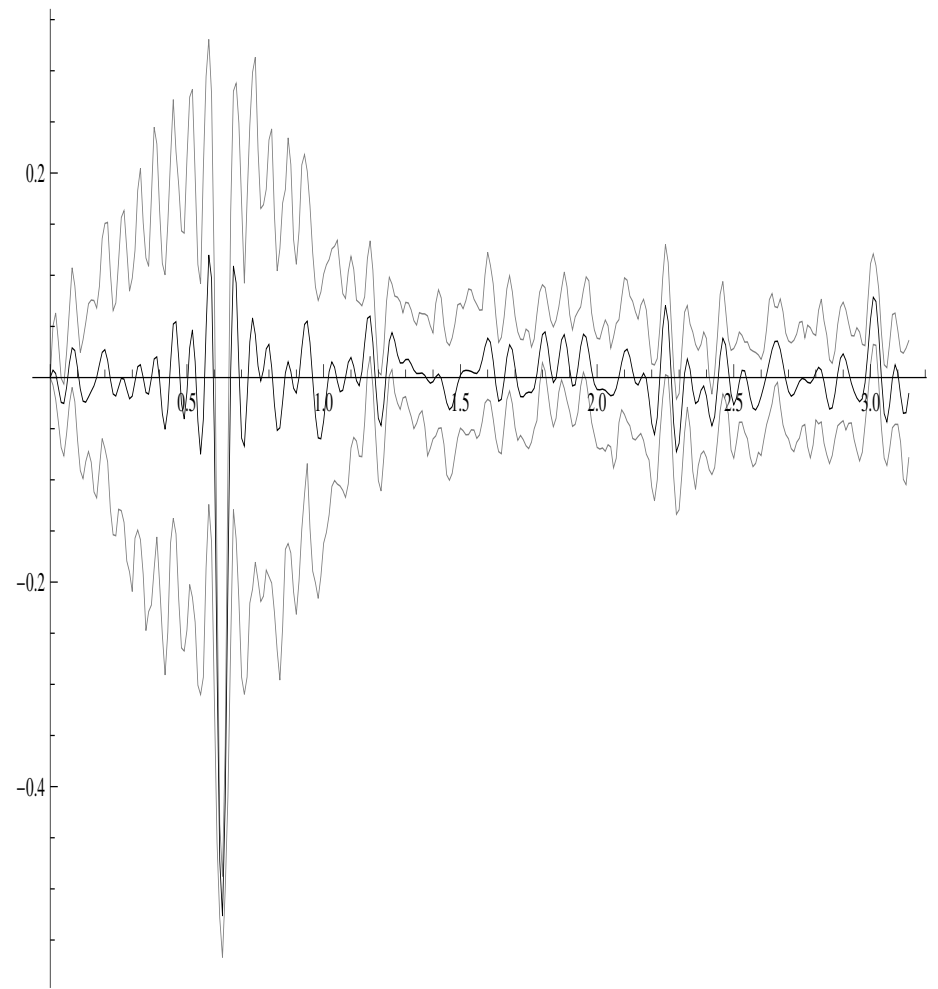
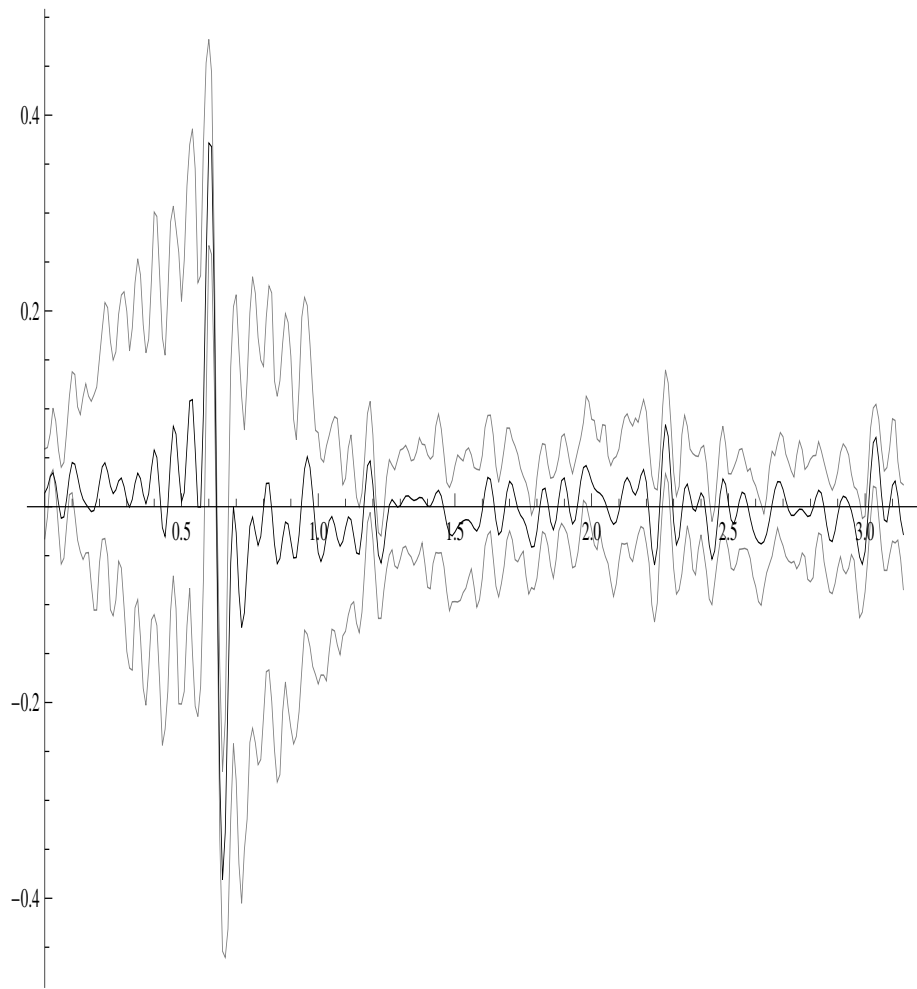
Choice of the block length : $b_1 = \lfloor \sqrt{T} \rfloor, b_2 = \lfloor \sqrt[3]{T} \rfloor$



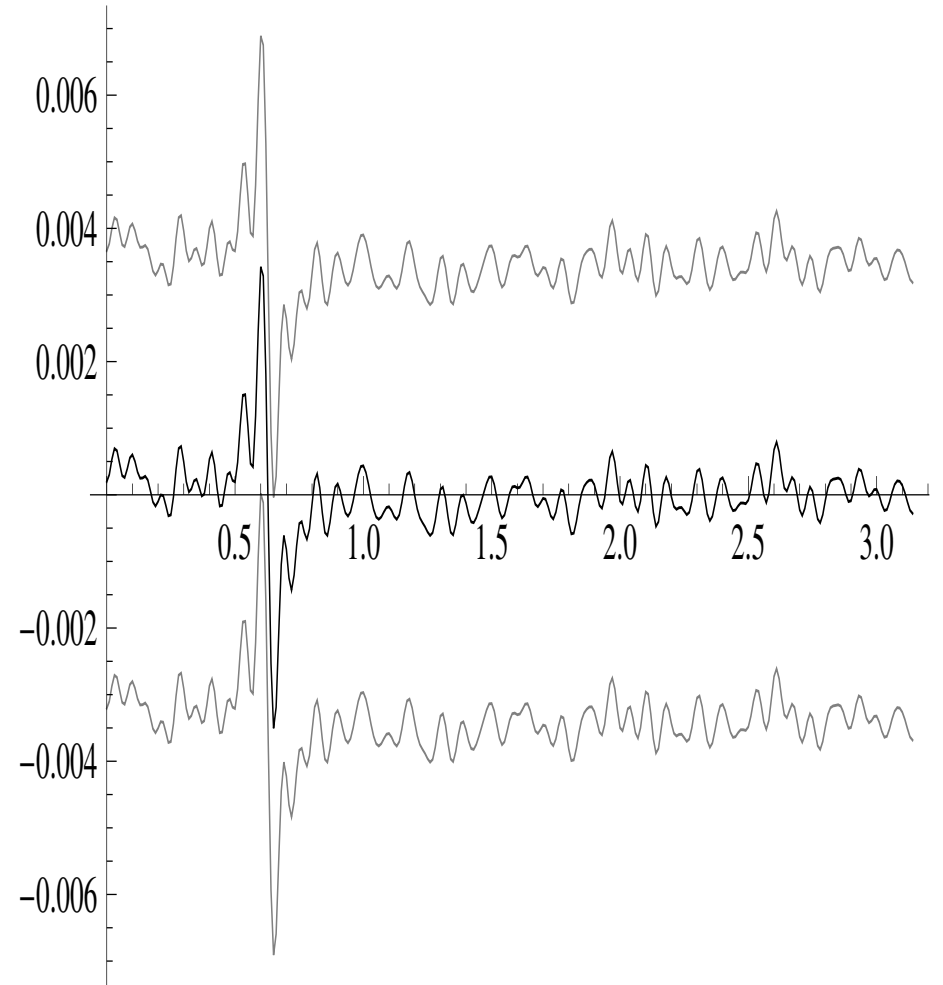
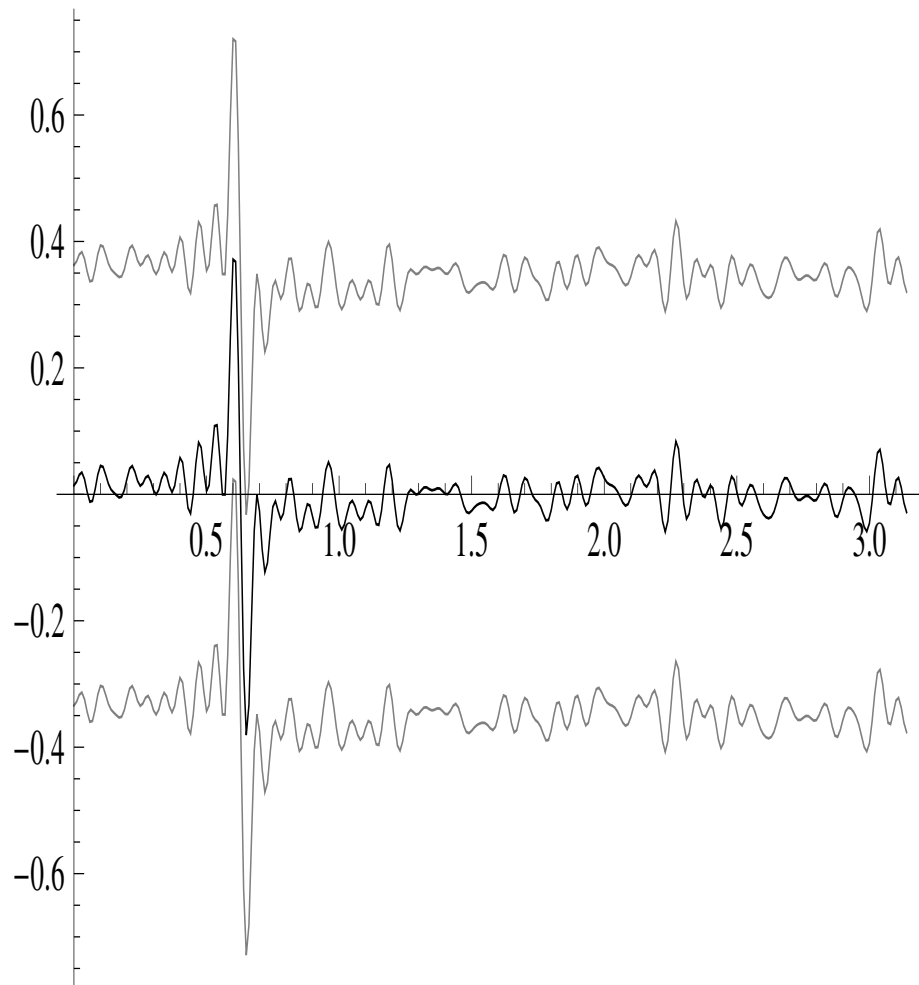
Example **E1** with constant C_1 : estimated values of $\Re m_\lambda$ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1 .



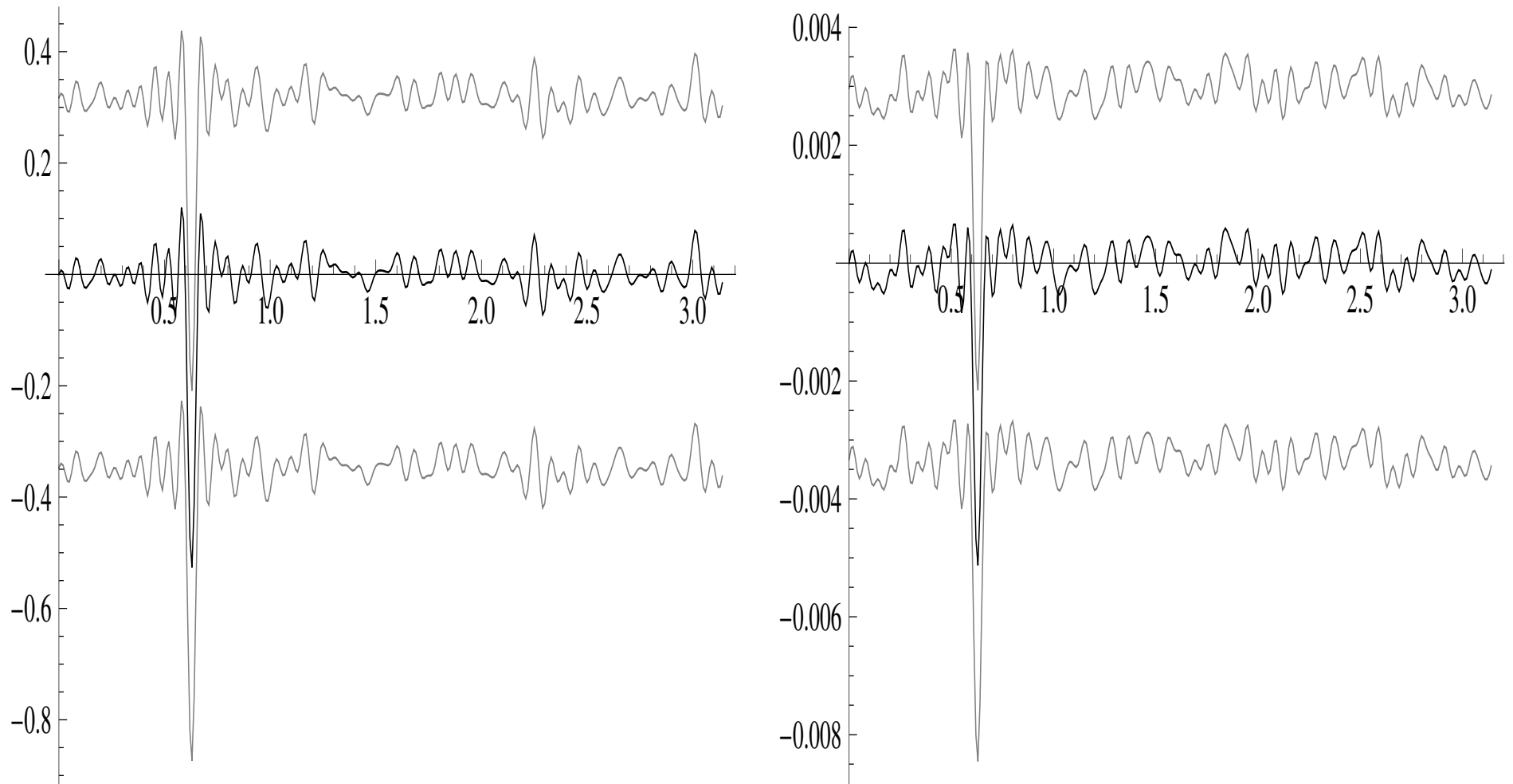
Example **E1** with constant C_1 : estimated values of $\Im m_\lambda$ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1 .



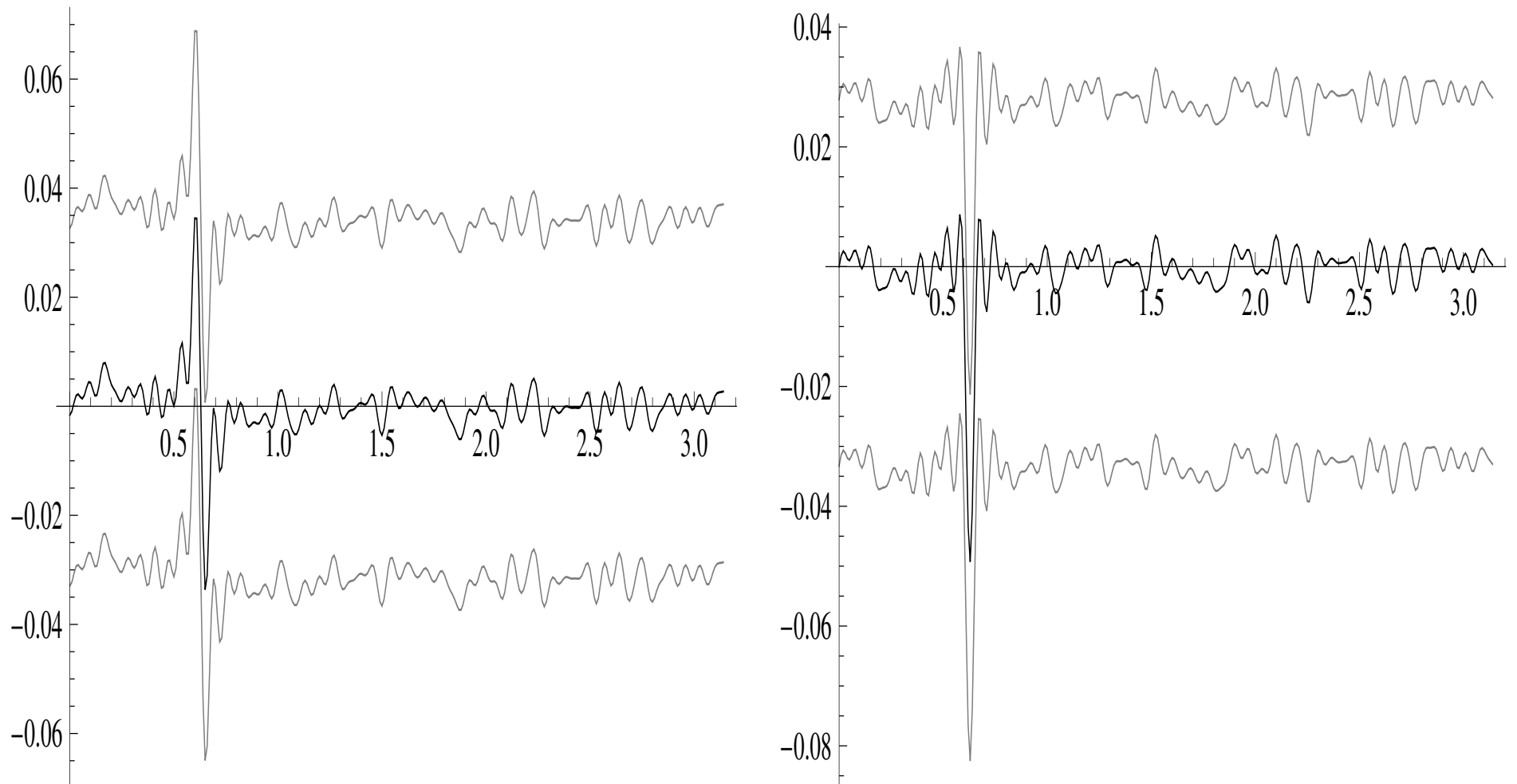
Example **E1** with constant $C_1 = 1$: estimated values of m_λ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1 . Real and imaginary parts of m_λ are left- and the right-hand side, respectively.



Example **E1**: estimated values of $\Re m_\lambda$ (gray line) together with the 95% bootstrap simultaneous equal-tiled confidence intervals (black line) constructed with block length equal to b_1 . The model with constant $C_1 = 1$ is in the left, and the model with $C_2 = 0.01$ is in the right



Example **E1**: estimated values of $\mathfrak{S}m_\lambda$ (gray line) together with the 95% bootstrap simultaneous equal-tiled confidence intervals (black line) constructed with block length equal to b_1 . The model with constant $C_1 = 1$ is in the left, and the model with $C_2 = 0.01$ is in the right



Example **E2** with constant $C_2 = 0.01$: estimated values of m_λ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1 . Real and imaginary parts of m_λ are left- and the right-hand side, respectively.

Open problems

Choice of b

Other schemes of random sampling. For example :
jitter phenomena in the sampling moments : $T_k = k\delta + U_k$

...

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THANK YOU FOR YOUR ATTENTION !