Block bootstrap for Poisson sampled almost periodic processes

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I– Almost periodically correlated processes

II– Sampling

III– Estimation of the cyclic mean from Poisson sampling

IV– Bootstrap method

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I − Almost periodically correlated processes (APC)

Examples .

(i) $X_1(t) = \text{sin}(2\pi t/10) + Z(t), \qquad t \in \mathbb{R},$

(ii) $X_2(t) = \text{sin}(2\pi t/10) + Z(t)\,\text{cos}(2\pi t/10), \qquad t \in \mathbb{R},$

where $\{Z(t), t \in \mathbb{R}\}$, zero mean, stationary with continuous covariance.

(iii) $X_3(t) = U(t)\,\text{cos}(t), \qquad t \in \mathbb{R}\,,$

(iv) $X_4(t) = U(t)\,\text{cos}(t) + U(t-1)\,\text{cos}(\pi t), \qquad \forall t \in \mathbb{R},$

where U is an Ornstein Uhlenbeck process (stationary Gaussian process) with $r(\tau) = \text{e}^{-|\tau|}$.

Almost periodically correlated processes (APC)

1 − Definitions

A function $f : \mathbb{R} \to \mathbb{C}$ is **almost periodic** when
$\forall \epsilon > 0$, $\exists L_\epsilon > 0$ s.t. $\forall I$ interval with length(I)$\geq L_\epsilon$, $\exists p_\epsilon \in I$ with

$$\sup_{s\in\mathbb{R}} \Big| f(s+p_\epsilon) - f(s) \Big| < \epsilon.$$

This is equivalent to the existence of trigonometric polynonials $P_n(\,\cdot\,)$, $n \in \mathbb{N}$ such that

$$\sup_{s\in\mathbb{R}} |P_n(s) - f(s)| \to 0 \qquad \text{as} \quad n \to \infty\,.$$

Examples:

$f(s) = \cos(s) + \cos(\pi s)$

$f(s) = \cos(s) + \frac{1}{2}\cos\left(\frac{s}{2}\right) + \frac{1}{4}\cos\left(\frac{s}{4}\right) + \cdots + \frac{1}{2^n}\cos\left(\frac{s}{2^n}\right) + \cdots$

A process $X = \{X(s), s \in \mathbb{R}\}$ is **periodically correlated** (**PC**) (Gudzenko, 1959) (Hurd, 1969) (Dragan, Rozhkov, Javoskyj, 1987) or *wide sense almost cyclostationary* (Gardner, 1994) when

– $\mathsf{E}\Big[X(s)^2\Big] < \infty$, for any $s \in \mathbb{R}$,

– $s \mapsto \mathsf{E}\Big[X(s)\Big]$ and $s \mapsto B(s,\tau) = \mathrm{cov}\Big[X(s), X(s+\tau)\Big]$ are periodic in s for each τ with the same period P.

A process $X = \{X(s), s \in \mathbb{R}\}$ is **almost periodically correlated** (**APC**) (Gladyshev, 1963) or *wide sense almost cyclostationary* (Gardner, 1994) when

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Applications

Mechanical systems (Diagnostic and Vibro-Acoustic): Antoni (2009),

Communication and signal processing : Gardner et al. (1994), Napolitano (2013)

Economics : Lenart and Pipien (2013)

Hydrology, meteorology, ...

Bibliography in Gardner et al. (2006), Serpendin et al. (2005), Hurd and Miamee (2007)

2 − Spectrum

$X = \{X(s) : s \in \mathbb{R}\}$ APC process.

Cyclic mean : for $\lambda \in \mathbb{R}$, and $t \in \mathbb{R}$,

$$m_\lambda := \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \mathsf{E}\Big[X(s)\Big]\, \mathrm{e}^{-i\lambda s}\, \mathrm{d}s.$$

Cyclic covariance : for $\lambda \in \mathbb{R}$, $t, \tau \in \mathbb{R}$

$$a(\lambda, \tau) := \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \mathrm{cov}\Big[X(s), X(s+\tau)\Big]\, \mathrm{e}^{-i\lambda s}\, \mathrm{d}s.$$

Cycle frequency sets of X:

$$\begin{array}{rl} \text{first order} & \Lambda_1 := \{\, \lambda : m_\lambda \neq 0 \,\} \\ \text{second order} & \Lambda_2 := \{\, \lambda : a(\lambda,\tau) \neq 0 : \text{for some } \tau \,\}. \end{array}$$

They are at most countable.

For a **PC process** with period $P > 0$ we have

$$\Lambda_1, \Lambda_2 \subset \frac{2k\pi}{P}\mathbb{Z} = \left\{\frac{2k\pi}{P} : k \in \mathbb{Z}\right\}$$

$$m_\lambda = \frac{1}{P}\int_0^P \mathsf{E}\Big[X(s)X(s+\tau)\Big]\,\mathrm{e}^{-i\lambda s}\,\mathrm{d}s, \qquad \forall\lambda \in \Lambda_1$$

$$a_\lambda(\tau) = \frac{1}{P}\int_0^P \mathrm{cov}\Big[X(s), X(s+\tau)\Big]\,\mathrm{e}^{-i\lambda s}\,\mathrm{d}s, \qquad \forall\lambda \in \Lambda_2$$

and

$$m_\lambda = 0, \qquad a_\lambda(\tau) = 0 \qquad \text{otherwise.}$$

For $\{X(t) : t \in \mathbb{R}\}$ a **stationary process** with mean m and covariance function $r(\tau) = \mathsf{E}[X(s+\tau)X(s)]$, we have

$$\Lambda_1 \subset \{0\}, \qquad \Lambda_2 = \{0\}$$

$$m_0 = m \in \mathbb{R},$$

$$a_0(\tau) = r(\tau)$$

$$m_\lambda = 0, \qquad a_\lambda(\tau) = 0 \qquad \forall \lambda \neq 0$$

3 – Examples

Example (i) $X_1(t) = \sin\left(\frac{2\pi t}{10}\right) + Z(t),\ t \in \mathbb{R}$,

$\{Z(t), t \in \mathbb{R}\}$, zero mean, stationary with continuous covariance $r(\cdot)$.

Then X_1 is PC and $\Lambda_1 = \left\{\pm\frac{\pi}{5}\right\}$, $\Lambda_2 = \{0\}$

$$m_{\pi/5} = \overline{m_{-\pi/5}} = \frac{-i}{2}, \qquad a_0(\tau) = r(\tau).$$

Example (ii) $X_2(t) = \sin\left(\frac{2\pi t}{10}\right) + Z(t)\cos\left(\frac{2\pi t}{10}\right),\ t \in \mathbb{R}$,

Then X_2 is PC and $\Lambda_1 = \left\{\pm\frac{\pi}{5}\right\}$, $\Lambda_2 = \left\{0, \pm\frac{2\pi}{5}\right\}$

$$m_{\pi/5} = \overline{m_{-\pi/5}} = \frac{-i}{2}$$

$$a_0(\tau) = \frac{1}{2}\cos\left(\frac{\pi\tau}{5}\right) r(\tau), \qquad a_{2\pi/5}(\tau) = \overline{a_{2\pi/5}(\tau)} = \frac{1}{4} e^{i\frac{\pi\tau}{5}} r(\tau).$$

Example (iii) $\quad X_3(t) = U(t)\cos(t),\ t \in \mathbb{R}\,,$

where U is an Ornstein Uhlenbeck process (stationary Gaussian process) with $r(\tau) = \mathrm{e}^{-|\tau|}$.

Then X_3 is PC and

$$\begin{aligned} \mathsf{E}\{X(s)X(s+\tau)\} &= \mathrm{e}^{-|\tau|}\cos(s)\cos(s+\tau) \\ &= a(-2,\tau)\,\mathrm{e}^{-i2s} + a(0,\tau) + a(2,\tau)\,\mathrm{e}^{i2s} \end{aligned}$$

$$\Lambda_1 = \emptyset, \qquad \Lambda_2 = \left\{-2, 0, 2\right\},$$

$$a(0,\tau) = \frac{1}{2}\,\mathrm{e}^{-|\tau|}\cos\tau, \qquad a(2,\tau) = \overline{a(-2,\tau)} = \frac{1}{4}\,\mathrm{e}^{-|\tau|}\,\mathrm{e}^{i\tau}.$$

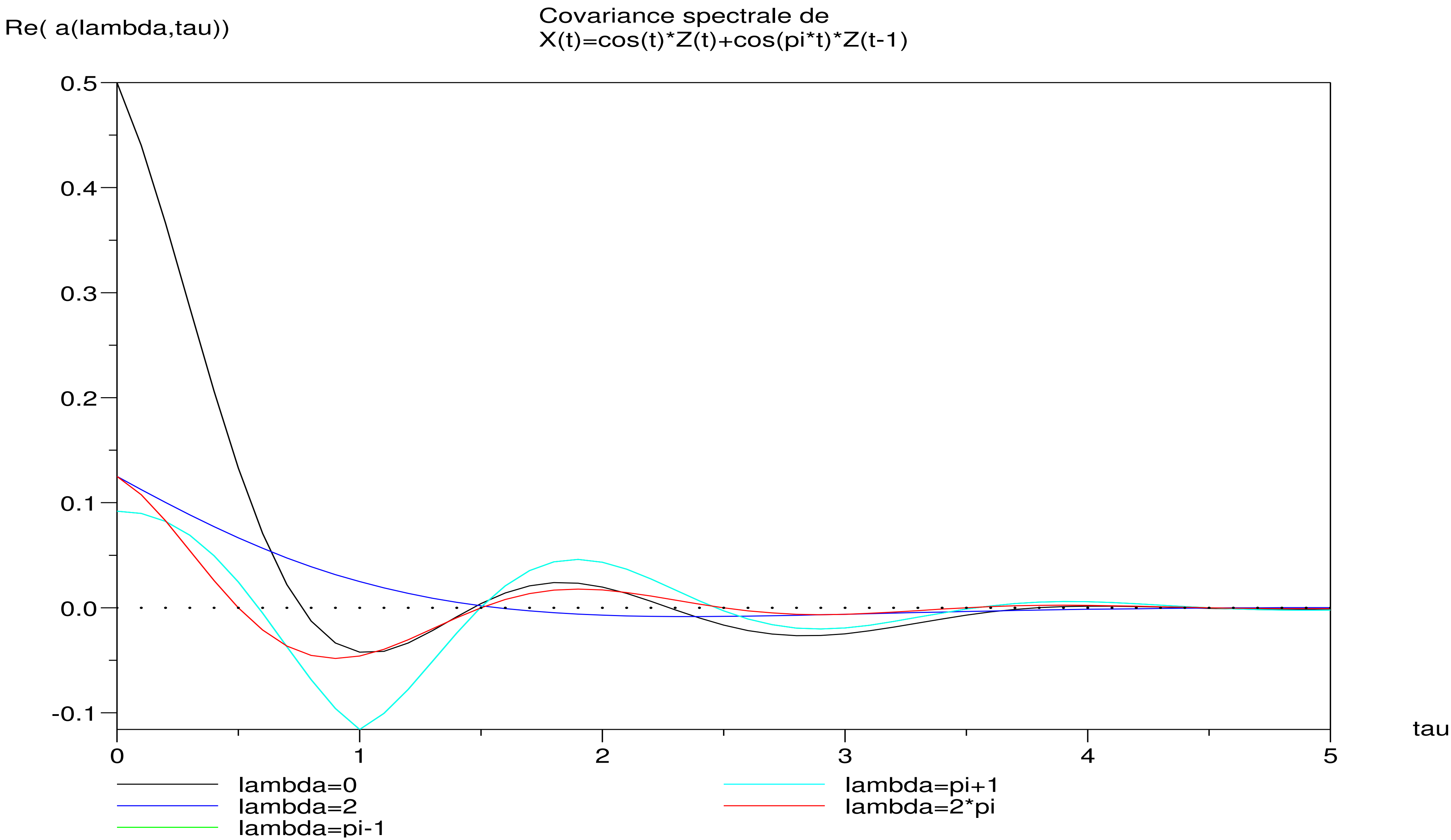
Example (iv)

$$X_4(t) = U(t)\cos(t) + U(t-1)\cos(\pi t), \forall t \in \mathbb{R}\,,$$

U is an Ornstein Uhlenbeck process.

Then the processus X_4 is APC,

$$\Lambda_1 = \emptyset, \quad \Lambda_2 = \Big\{-2\pi, -1-\pi, 1-\pi, -2, 0, 2, \pi-1, \pi+1, 2\pi\Big\}.$$



Real parts of the cyclic covariance

II − Sampling

Aim : estimation of m_λ and determination of $\Lambda_1 = \Big\{\lambda \in \mathbb{R} : m_\lambda \neq 0\Big\}$.

1 − Continuous time observation

Observation : one sample $\{X(t), t \in [0, T]\}$, $\quad T \longrightarrow \infty$.

Estimator of $m(\lambda)$ (Hurd & Miamee, 2007)

$$\widehat{m}_T^{(cont)}(\lambda) := \frac{1}{T}\int_0^T X(t)\, e^{-i\lambda t}\, dt,$$

2 − Discrete time observation of $\{X(t), t \in [0,T]\}$

Observation : $\{X(t_k) : k = 1, \cdots, N(T)\}$

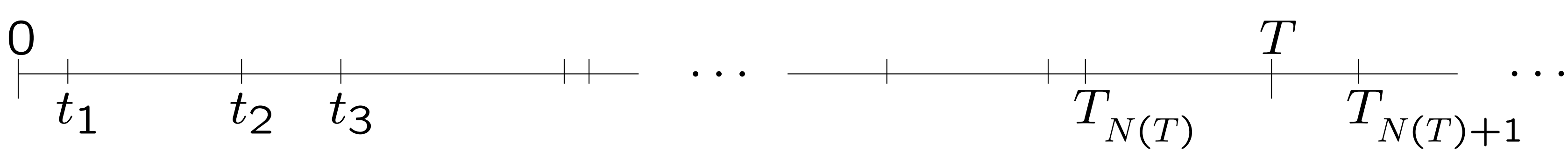
– Either deterministic instants $t_k = k\delta$ where $k = 0, \cdots, N(T) := \left\lfloor \frac{T}{\delta} \right\rfloor$

$\delta = constante \Longrightarrow$ folding and aliasing phenomena

$\delta = \delta_T \to 0$ as $T \to \infty$ with some rate $\Longrightarrow$ consistency

– Or random sampling instants $t_k = T_k$: objective of this work.

Here we consider a Poisson random sampling : $\{T_k\}$ occurence instants of a Poisson point process (Masry 1978)



3 − Poisson random sampling

Sampling scheme : $\{N(t), t \geq 0\}$ stationary Poisson point process, with intensity $\beta > 0$:

independent increments, and $\mathcal{L}\Big(N(s+t) - N(s)\Big) = \mathcal{P}(\beta t)$

$$\mathsf{E}\,[N(t+dt)] = \beta dt, \qquad \mathsf{cov}\,[N(t+dt)\,,\,N(t+u+du)] = \beta\delta_{\{0\}}(du)\,dt$$

T_k, $k = 1, 2, \ldots$ occurence instants of $\{N(t), t \geq 0\}$, and $T_0 \equiv 0$:

$T_k - T_{k-1}$'s independent random variables, and $\mathcal{L}(T_k - T_{k-1}) = \mathcal{E}(\beta)$

Hypothesis : Independence between $\{X(t), t \geq 0\}$ and $\{N(t), t \geq 0\}$.

Observations : $T_1, \ldots, T_{N(T)}$ and $X(T_1), X(T_2), \ldots, X(T_{N(T)})$.

Sampled process : random measure

$$Z(B) := \int_0^T X(t)\, N(t+dt) = \sum_{k=1}^{N(T)} X(T_k)$$

Properties :

$$\mathsf{E}\,[Z(t+dt)] = \beta\, \mathsf{E}\,[X(t)]\, dt$$

$$\begin{aligned}&\mathrm{cov}\,[Z(t+dt), Z(t+u+du)]\\ &\quad = \Big(\beta^2 \mathrm{cov}\,[X(t), X(t+u)]\; du + \beta \left(\mathrm{var}[X(t)] + \mathsf{E}\left[X(t)^2\right]\right) \delta_0(du)\Big) dt.\end{aligned}$$

So the mean measure $\mathsf{E}\,[Z(t+dt)]$ and the covariance measure $\mathrm{cov}\,[Z(t+dt), Z(t+u+du)]$ admits some **almost periodic** properties with respect to t.

Properties of the sampled process :

$$\lim_{T\to\infty} \frac{1}{T} \int_{[0,T]} e^{-i\lambda t} \, \mathsf{E}\{Z(t+dt)\} = \beta \, m_\lambda$$

and

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \int_{[0,T]\times B} & e^{-i\lambda t} \, \mathsf{cov}\{Z(t+dt), Z(t+u+du)\} \\ &= \beta^2 \int_B a_\lambda(u) du + \beta \left(a_\lambda(0) + m_\lambda^{(2)} \right) \delta_0(B)\end{aligned}$$

for any $B \in \mathcal{B}_b$.

Here

$$m_\lambda^{(2)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathsf{E}[X(t)]^2 \, e^{-i\lambda t} dt,$$

$$a_\lambda(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathsf{cov}[X(t), X(t+\tau)] \, e^{-i\lambda t} dt$$

III − Estimation of m_λ from Poisson sampling

$\{X(t), t > 0\}$ and $\{N(t), t \geq 0\} = \{T_k, k = 1, \dots\}$, independent

Observation of $(T_k, X(T_k))$, $k = 1, \dots, N(T)$, $\qquad T \to \infty$

1 − Estimator

$$\widehat{m}_T(\lambda) := \frac{1}{T\beta} \sum_{k=1}^{N(T)} X(T_k) e^{-i\lambda T_k} = \frac{1}{T\beta} \int_0^T X(t)\, N(t + dt)$$

2 − Hypotheses

To establish some results, we assume that the process X satisfies some asymptotic independence. Here we consider that the process X is strongly mixing (α-mixing) (Rosenblatt, 1956), (Doukhan, 1984).

Definition . The processus X is α-mixing when $\alpha_X(t) \stackrel{t\to\infty}{\longrightarrow} 0$, where

$$\alpha_X(t) := \sup\Big\{\Big|\mathsf{P}(A\cap B)-\mathsf{P}(A)\mathsf{P}(B)\Big| : A \in \mathcal{F}^s_{-\infty}(X), B \in \mathcal{F}^{\infty}_{t+s}(X), s \in \mathbb{R}\Big\},$$

$$\mathcal{F}^s_{-\infty}(X) := \sigma\Big\{X(u)\,;\, u \leq s\Big\} \text{ and } \mathcal{F}^{\infty}_{s+t}(X) := \sigma\Big\{X(u)\,;\, s+t \leq s\Big\}.$$

A1 Separability of the frequencies $\displaystyle\sum_{\lambda \in \Lambda_1} \lambda^{-2} < \infty\,.$

A2 *Mixing condition* The process $\{X(t), t \geq 0\}$ is α-mixing with

– either $\{X(t), t \geq 0\}$ is bounded, $\displaystyle\int_0^\infty \tau \alpha_X(\tau)\, d\tau < \infty$

– or there exists $\delta > 0$ such that

$$\sup_t \mathsf{E}\left[X(t)^{4+\delta}\right] < \infty \quad \text{and} \quad \int_0^\infty \tau \alpha_X(\tau)^{\delta/(4+\delta)}\, d\tau < \infty.$$

3 − Properties

– Under **A1** $\qquad \limsup_{T\to\infty} T \left| \mathsf{E}\left[\widehat{m}_T(\lambda)\right] - m_\lambda \right| < \infty, \qquad \lambda \in \mathbb{R}$

– Under **A2** $\qquad$ For $\lambda_1, \lambda_2 \in \mathbb{R}$

$$C_{(\lambda_1,\lambda_2)} := \lim_{T\to\infty} T \,\mathrm{cov}\left[\widehat{m}_T(\lambda_1), \widehat{m}_T(\lambda_2)\right]$$

$$= \frac{m^{(2)}_{(\lambda_1-\lambda_2)}}{\beta} + \frac{a_{(\lambda_1-\lambda_2)}(0)}{\beta} + \int_{\mathbb{R}} a_{(\lambda_1-\lambda_2)}(\tau) e^{i\lambda_2\tau} d\tau.$$

where

$$m^{(2)}_\lambda = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathsf{E}[X(t)]^2 e^{-i\lambda t} dt,\ a_\lambda(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{cov}[X(t), X(t+\tau)] e^{-i\lambda t} dt$$

– Under **A1** and **A2**

$$\lim_{T\to\infty} T\,\mathsf{E}\left[\left|\widehat{m}_T(\lambda) - m_\lambda\right|^2\right] = \sigma^2_\lambda := C_{(\lambda,\lambda)}.$$

Theorem Under **A1** and **A2**

$$\mathcal{L}\Big(\sqrt{T}\Big(\widehat{m}_T(\lambda) - m_\lambda\Big)\Big) \longrightarrow \mathcal{N}_2\Big(0, \Sigma_2(\lambda)\Big) \qquad \text{as} \quad T \to \infty.$$

$$\Sigma_2(\lambda) := \begin{pmatrix} \sigma^2_{1,1}(\lambda) & c_{1,2}(\lambda) \\ \\ c_{2,1}(\lambda) & \sigma^2_{2,2}(\lambda) \end{pmatrix},$$

where

$$\sigma^2_{1,1}(\lambda) = \frac{1}{2}\left(C_{(\lambda,\lambda)} + \Re\left(C_{(\lambda,-\lambda)}\right)\right), \qquad \sigma^2_{2,2}(\lambda) = \frac{1}{2}\left(C_{(\lambda,\lambda)} - \Re\left(C_{(\lambda,-\lambda)}\right)\right),$$

$$c_{1,2}(\lambda) = c_{2,1}(\lambda) = \frac{i}{4}\left(-C_{(\lambda,-\lambda)} + C_{(-\lambda,\lambda)}\right) = \frac{1}{2}\Im\left(C_{(\lambda,-\lambda)}\right).$$

Tools for the proofs

1) *Some rates of convergence*

$$\left|m_\lambda\right| \le \sup_s \left|\mathsf{E}\left[X(s)\right]\right| < \infty, \qquad \left|a(\lambda,\tau)\right| \le \sup_s 2\mathsf{E}\left[X(s)^2\right] < \infty.$$

$$\begin{aligned}\left|m_\lambda - \frac{1}{T}\int_t^{t+T} \mathsf{E}\left[X(s)\right] \mathrm{e}^{-i\lambda s}\,\mathrm{d}s\right| &\le \frac{1}{T}\sum_{\alpha\in\Lambda^{(1)}\setminus\{\lambda\}} \frac{|m_\alpha|}{|\alpha-\lambda|} \\ &\le \frac{\sup_s \left|\mathsf{E}\left[X(s)\right]\right|}{T}\left(\sum_{\alpha\in\Lambda^{(1)}\setminus\{\lambda\}} \frac{1}{(\alpha-\lambda)^2}\right)^{1/2}\end{aligned}$$

2) *Covariance inequality*
(Wolonski & Rozanov, 1959) (Davydov, 1970)

Let ξ be $\mathcal{F}^s_{-\infty}(X)$-measurable, and ζ be $\mathcal{F}^\infty_{s+t}(X)$-measurable

$$|\mathrm{cov}\{\xi,\zeta)\}| \le 4c_1c_2\,\alpha(t) \qquad \text{if} \quad |\xi| \le c_1,\ |\zeta| \le c_2$$

$$|\mathrm{cov}\{\xi,\zeta)\}| \le 8\,\alpha(t)^{1-\frac{2}{p}}\mathsf{E}\{|\xi|^p\}^{\frac{1}{p}}\mathsf{E}\{|\zeta|^p\}^{\frac{1}{p}} \qquad \text{for} \quad p>2$$

3) *Central limit theorem for (nonstationary) mixing processes* (Guyon 1995)

IV − Boostrap method

Efron (1979) : bootstrap for i.i.d random sequence (see also Giné (1997))

Künsh (1989), Liu and Singh (1992) : moving block bootstrap for dependent (stationary) data : (see Lahiri 2003)

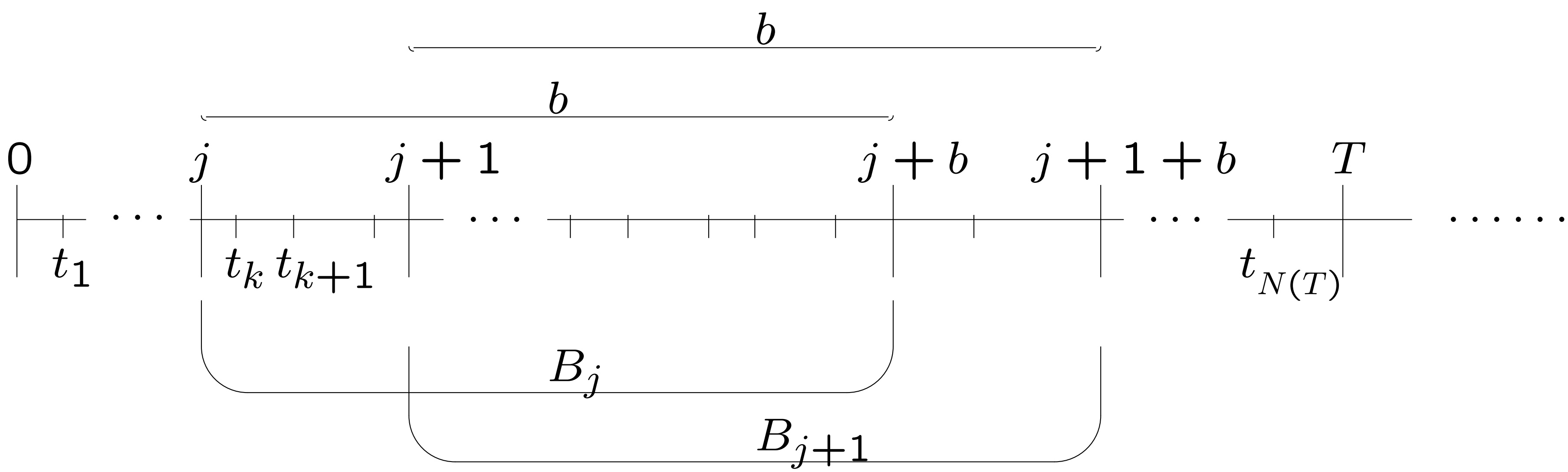
Politis and Romano (1992) : circular block bootstrap (stationary data)

Synowiecki (2007), Leskow et al. (2010), Dudek et al. (2013) : moving block bootstrap for PC and APC time series

1− Circular block bootstrap method : CBB

b block length, $T = lb$,

$$B_j := \{(T_k, X(T_k)) : j < T_k \le j+b,\ k \in \mathbb{N}^*\}, \quad j = 0, \ldots, T-1 \quad \text{(circular)}$$



Random selection with replacement of l blocks from $\{B_0, \ldots, B_{T-1}\}$

$\Longleftrightarrow$

$i_1^*, \ldots, i_l^*$ i.i.d. random variables unif. distrib. on $\{0, 1, \ldots, T-1\}$

$$\mathsf{P}[i_j^* = k] = \frac{1}{T}, \qquad k = 0, \ldots, T-1, \quad j = 1, \ldots, l.$$

Definition CBB sample of the data $\{(T_k, X(T_k)) : k = 1, \ldots, N(T)\}$ is

$$(B_1^*, \ldots, B_l^*), \qquad \text{where} \quad B_j^* := B_{i_j^*}$$

Interest : conservation on some correlation structure in the process

Block length : $b = b_T \to \infty$, $\quad T = lb$,

Problem of choice of b

2 − Bootstrap version of the estimator $\widehat{m}_T(\lambda)$

Let $\widehat{m}_{j,b}$ the estimator of m_λ constructed from $B_j = B_{j,b}$:

– for $j = 0, \ldots, T-b$, we have

$$\widehat{m}_{j,b} := \frac{1}{\beta b} \sum_{k: j < T_k \leq j+b} X(T_k) e^{-i\lambda T_k},$$

– for $j = T-b+1, \ldots, T-1$ as a result of wrapped blocks we get

$$\widehat{m}_{j,b} := \frac{1}{\beta b} \left(\sum_{k: j < T_k \leq T} X(T_k) e^{-i\lambda T_k} + \sum_{k: 0 < T_k \leq T-j+b} X(T_k) e^{-i\lambda T_k} \right).$$

Then $\qquad \widehat{m}_T = \dfrac{1}{l} \displaystyle\sum_{j=0}^{l-1} \widehat{m}_{w+jb,b}$ for any $w = 0, \ldots, b-1$

<u>Bootstrap version of $\widehat{m}_T$</u> : $\quad \widehat{m}_T^* := \dfrac{1}{l} \displaystyle\sum_{j=0}^{l-1} \widehat{m}_{j,b}^*$

Bootstrap version of $\widehat{m}_T$: $\widehat{m}_T^* := \frac{1}{l} \sum_{j=0}^{l-1} \widehat{m}_{j,b}^*$ where

$$\widehat{m}_{j,b}^* := \frac{1}{b\beta} \sum_{k:j^* < T_k \le j^*+1} X(T_k) e^{-i\lambda T_k}.$$

Conditional expectation with respect to the observation :

$$\mathsf{E}^*\Big[\widehat{m}_T^*\Big] := \mathsf{E}\Big[\widehat{m}_T^* \Big| \Big(T_k, X(T_k)\Big) : k = 1, \ldots, N(T)\Big] = \widehat{m}_T.$$

E^* is the expectation conditioned by the observation.

Theorem

assume **A1** and **A2** be satisfied and $b \to \infty$ as $T \to \infty$ but $b = o(T)$. **then** for any $\epsilon > 0$

$$\lim_{T\to\infty} \mathsf{P}\Big[\mathsf{dist}\Big[\mathcal{L}^*\Big\{\sqrt{T}(\widehat{m}_T^* - \widehat{m}_T)\Big\}, \mathcal{L}\Big\{\sqrt{T}(\widehat{m}_T - m_\lambda)\Big\}\Big] > \epsilon\Big] = 0$$

for any $\epsilon > 0$ and for any λ.

Here dist is a distance which metrizes the convergence in distribution.

In particular, if the cumulative limit distribution of $\sqrt{T}\left(\widehat{m}_T^* - m_\lambda\right)$ is continuous **then** for the functions $\phi(z) = |z|^2$, $\phi(z) = \Re z$, $\phi(z) = \Im z$

$$\sup_{x\in\mathbb{R}} \Big| \mathsf{P}^*\Big[\phi\Big(\sqrt{T}(\widehat{m}_T^* - \widehat{m}_T)\Big) \le x\Big] - \mathsf{P}\Big[\phi\Big(\sqrt{T}(\widehat{m}_T - m_\lambda)\Big) \le x\Big]\Big| \xrightarrow{\mathsf{P}\text{-proba}} 0$$

as $T \to \infty$.

Idea for the proof

$$Y_k^{(\lambda)} := \frac{1}{\beta}\int_k^{k-1} e^{-i\lambda s} X(x) N(s+ds)$$

$\{Y_k^{(\lambda)}, k \in \mathbb{N}\}$ is APC time series

Radulovìc (1996), Giné (1997), Synowiecki (2007)

Araujo and Giné (1980)

V − Some simulations

E1 $X(t) = U(t) + C_i \sin(2\pi t/10)$;

E2 $X(t) = U(t)\cos(2\pi t/10) + C_i \sin(2\pi t/10)$,

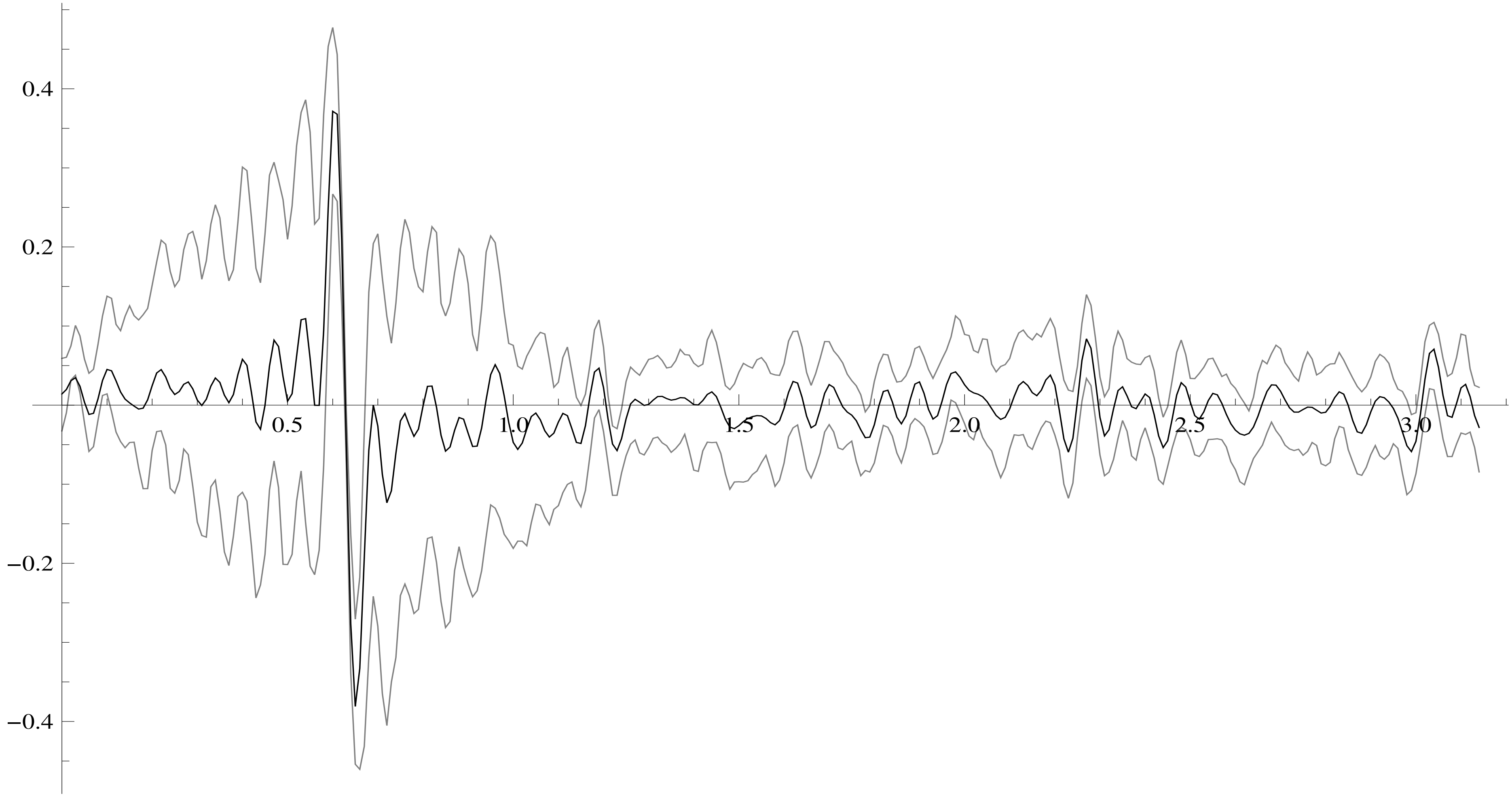
$U = \{U(t) : t \in \mathbb{R}\}$ Orsntein Uhlenbeck process : stationary Markov Gaussian process (cov: $r(\tau) = e^{-|\tau|}/2$)

$C_1 = 1$, $C_2 = 0.01$

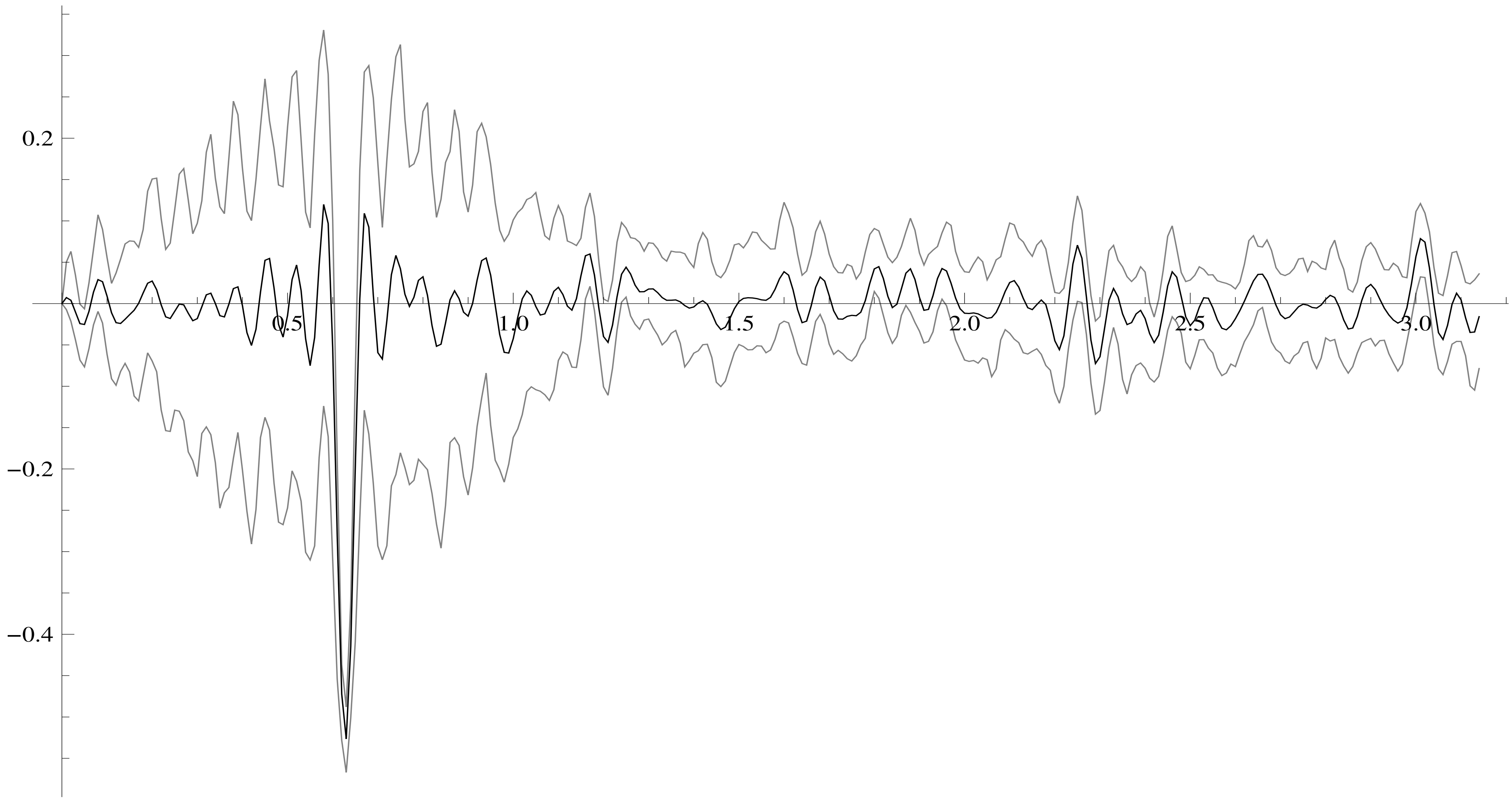
Sampling Scheme : Poisson point process with intensity $\beta = 5$

Observation interval $[0, 100]$, $T = 100$

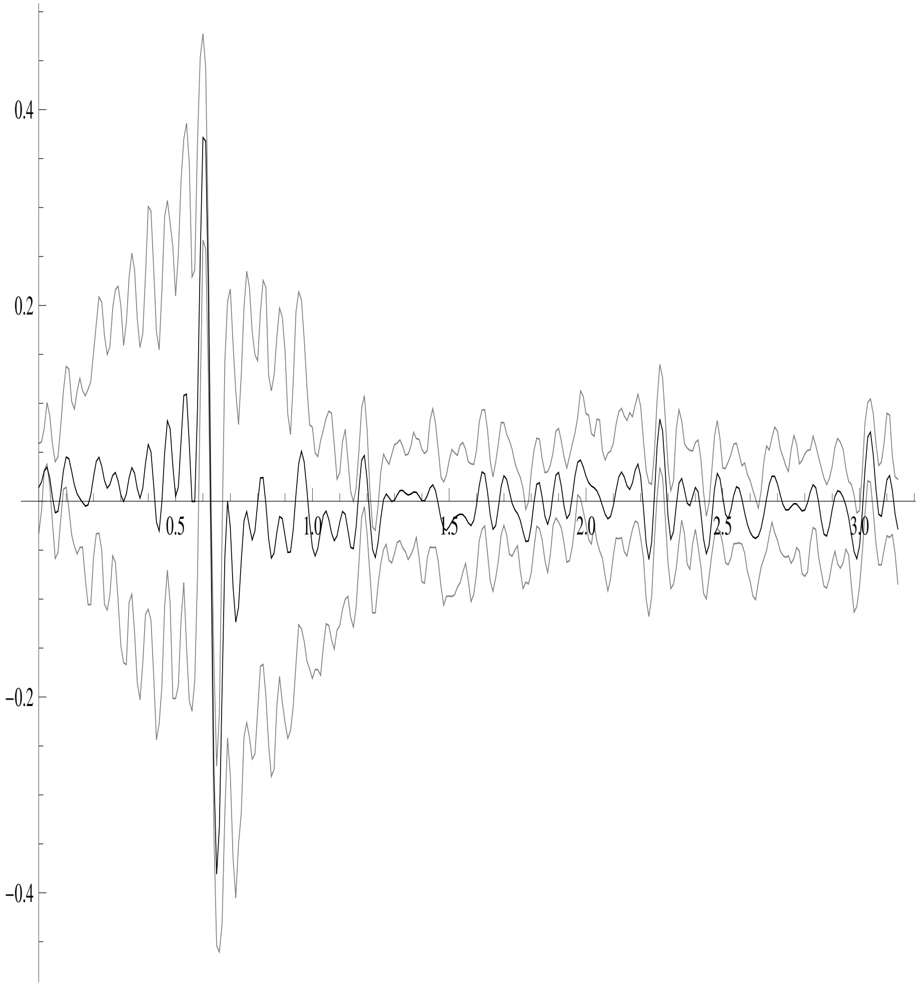
Choice of the block length : $b_1 = \lfloor \sqrt{T} \rfloor$, $b_2 = \lfloor \sqrt[3]{T} \rfloor$

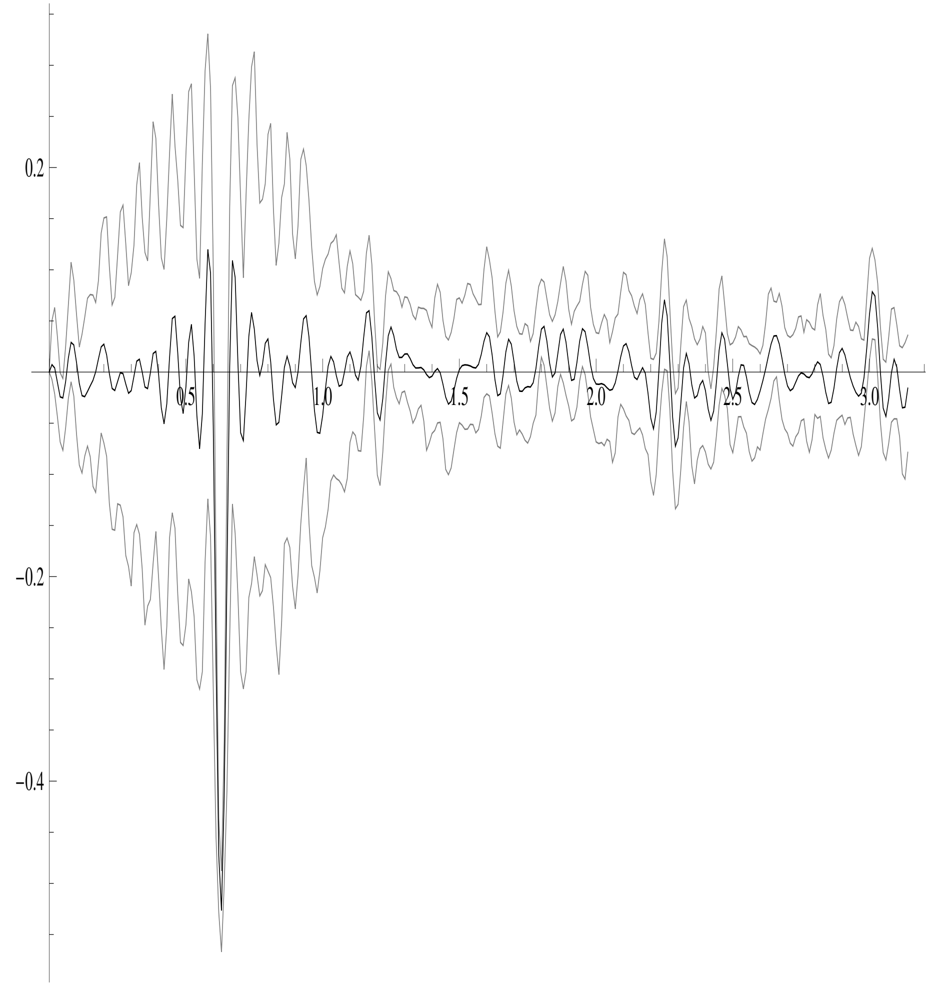


Example **E1** with constant C_1: estimated values of $\Re m_\lambda$ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1.

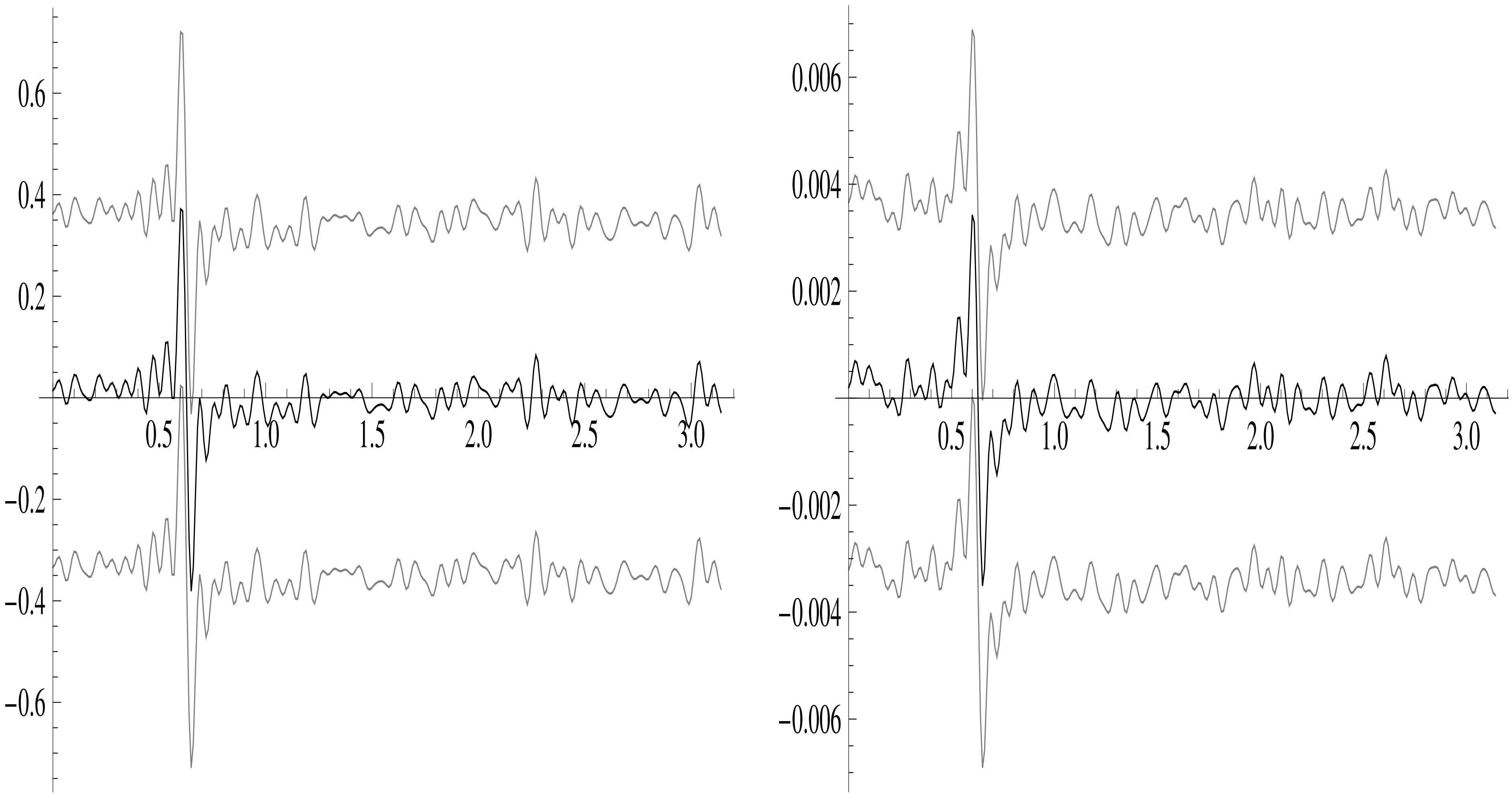


Example **E1** with constant C_1: estimated values of $\Im m_\lambda$ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1.

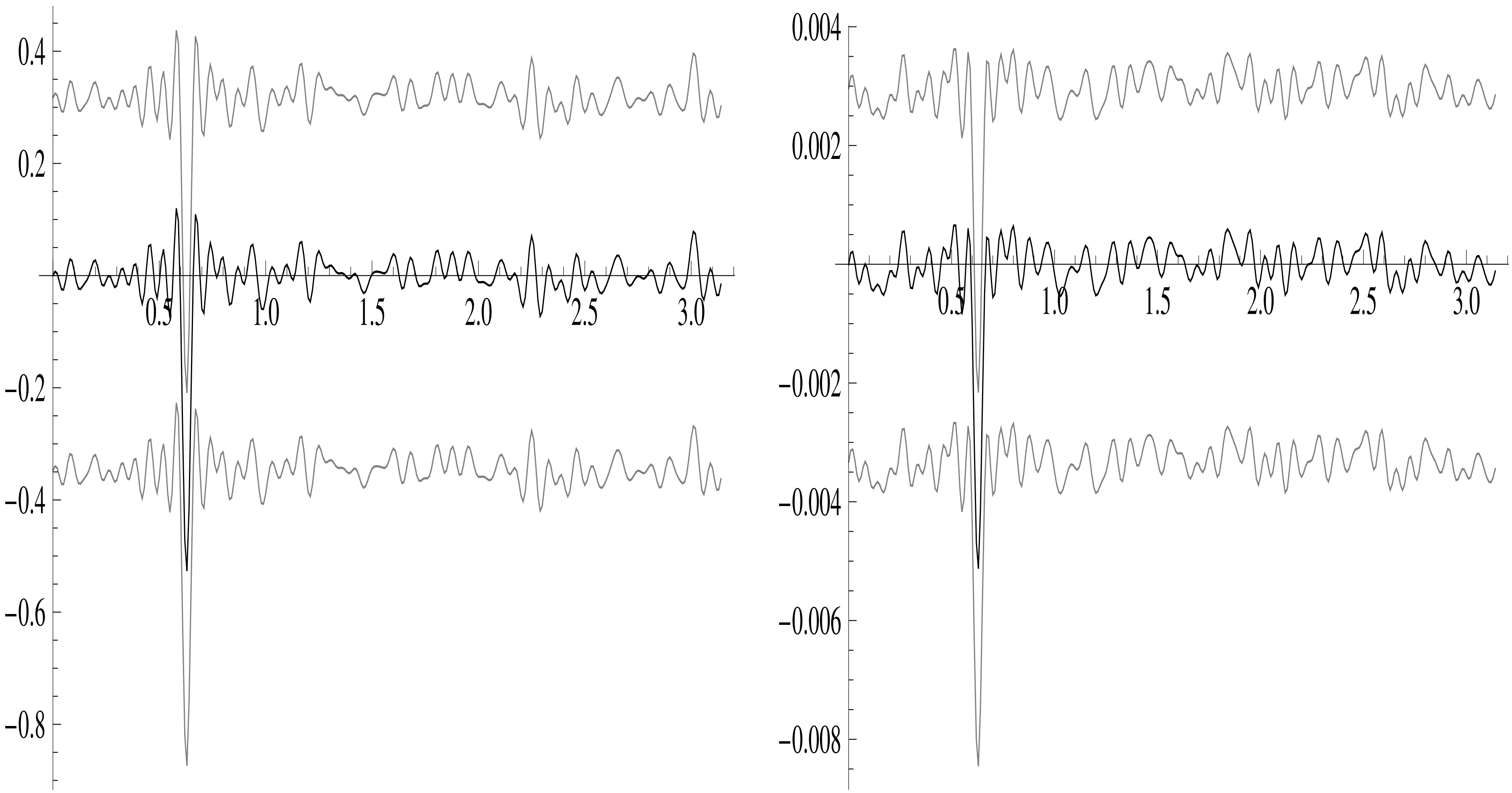




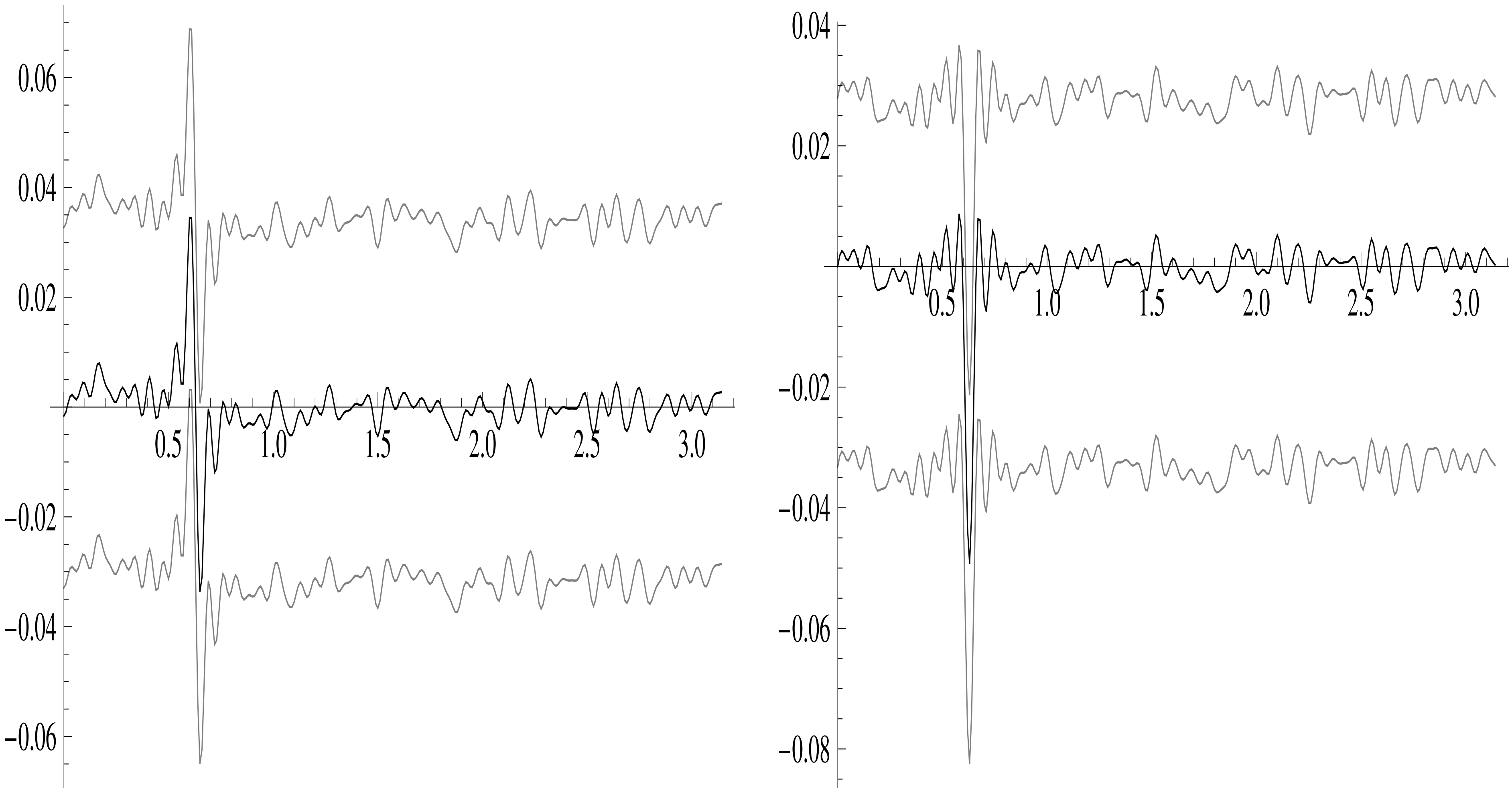
Example **E1** with constant $C_1 = 1$: estimated values of m_λ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1. Real and imaginary parts of m_λ are left- and the right-hand side, respectively.



Example **E1**: estimated values of $\Re m_\lambda$ (gray line) together with the 95% bootstrap simultaneous equal-tiled confidence intervals (black line) constructed with block length equal to b_1. The model with constant $C_1 = 1$ is in the left, and the model with $C_2 = 0.01$ is in the right



Example **E1**: estimated values of $\Im m_\lambda$ (gray line) together with the 95% bootstrap simultaneous equal-tiled confidence intervals (black line) constructed with block length equal to b_1. The model with constant $C_1 = 1$ is in the left, and the model with $C_2 = 0.01$ is in the right



Example **E2** with constant $C_2 = 0.01$: estimated values of m_λ (gray line) together with the 95% bootstrap pointwise equal-tiled confidence intervals (black line) constructed with block length equal to b_1. Real and imaginary parts of m_λ are left- and the right-hand side, respectively.

Open problems

Choice of b

Other schemes of random sampling. For example :
jitter phenomena in the sampling moments : $T_k = k\delta + U_k$

...

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THANK YOU FOR YOUR ATTENTION !