Minimax Interpolation of Smooth Random Processes

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This talk deals with recovering a random function

$$f(x),\ \ x \in \mathbb{R}^1 \text{ based on } \quad Y_k, X_k,\ \ k = 1, \dots, n \quad ,$$

where

$$Y_k = f(X_k), \quad k = 1, \dots, n \quad \text{with } X_1 < X_2 < \dots < X_n.$$

The risk of an interpolation $\tilde{f}(x, Y, X)$ is measured by

$$\rho(f, \tilde{f}) = \mathbf{E}\ \big(f(x) - \tilde{f}(x, Y, X) \big)^2$$

and our goal is to find

$$\bar{f} = \arg\min_{\tilde{f}} \rho(f, \tilde{f}).$$

The optimal interpolation can be easily obtained when $f(\cdot)$ is a Gaussian random process with a known covariance function $R(u, v) = \mathbf{E} f(u) f(v)$. In this case

$$\bar{f}(x, X, Y) = \sum_{k=1}^{n} K(x, X_k) Y_k,$$

where $K(\cdot, \cdot)$ is a solution to the Wiener-Hopf-Kolmogorov equation

$$\sum_{k=1}^{n} R(X_k, X_s) K(x, X_k) = R(x, X_s), \ \ s = 1, \dots, n.$$

The interpolation error is computed as follows :

$$
\begin{aligned}
\inf_{\tilde f}(f,\tilde f) &= \mathbf{E}\left(f(x) - \sum_{k=1}^{n} K(x,X_k) f(X_k) \right)^2 \\
&= R(x,x) - \sum_{k=1}^{n} K(x,X_k) R(X_k,x).
\end{aligned}
$$

- *Kolmogorov, A.* Interpolation and extrapolation of stationary sequences. Izv. Akad. Nauk SSSR Ser. Mat. 1941. Vol. 1. P. 3–14.
- *Wiener N.* Extrapolation, Interpolation, and Smoothing of Stationary Time Series. New York: Wiley, 1949.

Let $f(t)$ be a Gaussian stationary random process with spectral density

$$F_{Q,\alpha,m}(\omega) = \int_{-\infty}^{\infty} e^{2\pi i\omega t} R(t,0)\, dt = \frac{Q}{\omega^{2m} + \alpha^{2m}}$$

with an integer $m \geq 1$.

Computing the optimal interpolation is di□cult when α and Q are not known. In this case we have (to make use of kriging)

- to estimate α and Q based on the data $\{X_i, Y_i,\ i = 1, \ldots, n\}$ with the help of the maximum likelihood method;
- to solve the WHK equation.

To avoid the estimation of α, consider the case where $\alpha \to 0$, i.e. let us assume that $f(\cdot)$ has the following spectral density

$$F_{Q,m}(\omega) = \lim_{\alpha \to 0} F_{Q,\alpha,m}(\omega) = \frac{Q}{\omega^{2m}}.$$

So, $f(\cdot)$ is a generalized Gaussian random process, i.e. we can observe solely

$$\zeta(\phi) = \sum_{k=-\infty}^{\infty} f(X_k)\phi_k,$$

where ϕ_k are such that $\sum_{k=-\infty}^{\infty} \phi_k^2 < \infty$ and

$$\sum_{k=-\infty}^{\infty} X_k^p \phi_k = 0, \text{ for all } \quad p = 0, 1, \ldots, m-1.$$

Let $K_{Q,\alpha,m}(\cdot,\cdot)$ be a solution to WHK equation related to the covariance function

$$R(t,s) = \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi i(t-s)} \frac{Q}{\omega^{2m} + \alpha^{2m}} \, d\omega.$$

Denote

$$K_m(\cdot,\cdot) = \lim_{\alpha \to 0} K_{Q,\alpha,m}(\cdot,\cdot).$$

Theorem

Let $n \geq m$. Then $K_m(x, X_k)$ exists and

$$\sum_{k=1}^{n} K_m(x, X_k) X_k^p = x^p, \quad p = 0, \dots, m-1,$$
$$\sum_{k=1}^{n} K_m(x, X_k) d_s^{(m)}(X_k) = d_s^{(m)}(x), \quad s = 1, \dots, n-m,$$

where $d_s^{(m)}(x)$ are computed as follows :

$$d_s^{(0)}(x) = |x - X_s|^{2m-1},$$
$$d_s^{(j+1)}(x) = \frac{d_{s+1}^{(j)}(x) - d_s^{(j)}(x)}{X_{s+j+1} - X_s}.$$

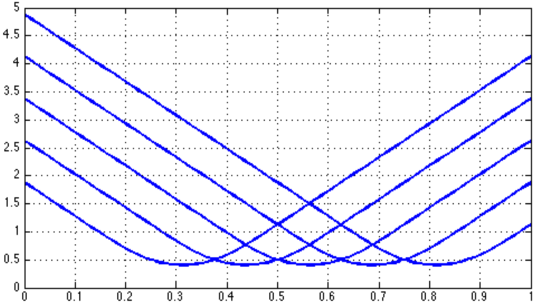

Рис.: Functions admitting the exact interpolation, $m = 2$.

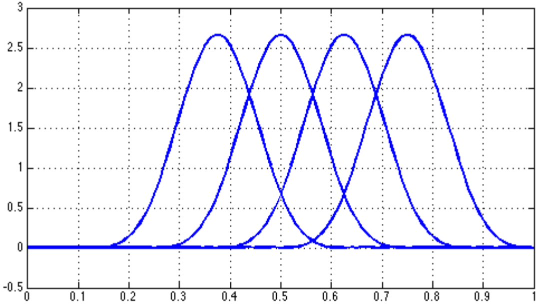

Рис.: Functions admitting the exact interpolation, $m = 2$.

Lemma

Let $n \geq 2m$*. Then* $K_m(x, \cdot)$ *is a solution to*

$$AK_m(x, \cdot) = b(x),$$

where A *is a* $2m - 1$ *diagonal band matrix.*

Interpolating Splines

Recall that interpolating spline is defined as $\lim_{\epsilon\to 0} \bar{S}^{\epsilon}_{Q,m}(x, Y)$, where

$$\bar{S}^{\epsilon}_{Q,m}(x, Y) = \arg\min_{f} \left(\frac{1}{2\epsilon^2} \sum_{k=1}^{n} \left(Y_k - f(X_k) \right)^2 + \frac{1}{2Q} \int_0^1 \left(f^{(m)}(x) \right)^2 dx \right).$$

It is clear that $\bar{S}^{\epsilon}_{Q,m}(x, Y)$ is linear in Y_k, $k = 1, \ldots, n$, and thus

$$\bar{S}^{\epsilon}_{Q,m}(x) = \sum_{k=1}^{n} K^{\epsilon}_{Q,m}(x, X_k) Y_k.$$

Theorem

Let $n \geq m$. Then

$$\lim_{\epsilon \to 0} K^{\epsilon}_{Q,m}(x, y) = K_m(x, y),$$

where $K_m(\cdot, \cdot)$ is defined in Theorem 1.

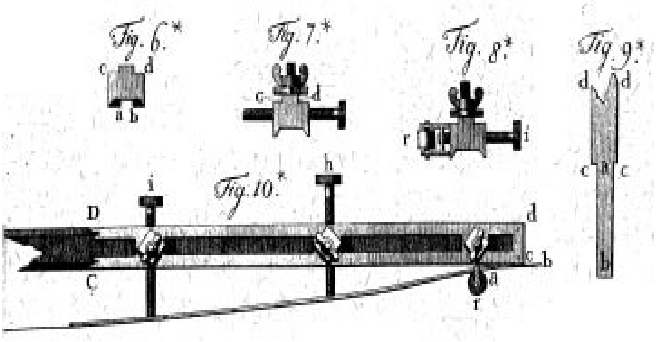

Рис.: The cubic spline.

Duhamel du Monceau H.-L. Eléments de l'Architecture Navale ou Traité de la Construction des Vaissaux. Paris, 1752.

Minimax Interpolation

Assume that $f(\cdot)$ is a stationary Gaussian random process with a spectral density $F(\cdot)$. We begin with the case where $F(\cdot)$ is known. Our goal is recover $f(x),\ x \in (\infty, \infty)$ based on the data

$$Y_k = f(X_k), \quad X_k = kh, \quad k = 0, \pm 1, \pm 2, \ldots.$$

In this case the best interpolation has the following form :

$$\bar{f}(x, X, Y) = h \sum_{k=-\infty}^{\infty} K(x - X_k) Y_k,$$

where $K(\cdot)$ is a symmetric kernel which is defined as a minimizer of the mean-square interpolation error

$$\rho_h(\bar{f}, F) \stackrel{\text{def}}{=} \frac{1}{h} \int_0^h \mathbf{E}[f(x) - \bar{f}(x)]^2 \, dx.$$

Denote by $\hat{G}(\omega)$ the Fourier transform of $g \in L_2(-\infty, \infty)$

$$\hat{G}(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} \omega x} g(x)\, dx.$$

With the help of the Poisson summation formula we obtain

$$\mathbf{E}[f(x) - \bar{f}(x)]^2 = \int_{-\infty}^{\infty} F(\omega) \left| 1 - \sum_{k=-\infty}^{\infty} \hat{K}\left(\omega - \frac{k}{h}\right) \mathrm{e}^{2\pi \mathrm{i} x k/h} \right|^2 d\omega.$$

Finally, integrating this equation in $x \in [0, h]$ and using the orthogonality of $\{\exp(2\pi \mathrm{i} kx/h),\ x \in [0, h]\}$, we arrive at

$$\rho_h(\bar{f}, F) = \int_{-\infty}^{\infty} F(\omega) \left[[1 - \hat{K}(\omega)]^2 + \sum_{k \neq 0} \hat{K}^2 \left(\omega + \frac{k}{h} \right) \right] d\omega.$$

Assume that the spectral density of $f(\cdot)$ is unknown but it is such that

$$\mathbf{E}[f^{(m)}(x)]^2 = \int_{-\infty}^{\infty} (2\pi\omega)^{2m} F(\omega)\, d\omega \leq L.$$

Denote the set of all such spectral densities by $\mathcal{F}_m(L)$.

Our goal is to compute the minimax interpolation error

$$\rho_h[\mathcal{F}_m(L)] \stackrel{\text{def}}{=} \inf_{\bar{f}} \sup_{F \in \mathcal{F}_m(L)} \rho_h(\bar{f}, F)$$

and to find the minimax interpolation $\bar{f}_*$ such that

$$\sup_{F \in \mathcal{F}_m(L)} \rho_h(\bar{f}_*, F) = \rho_h[\mathcal{F}_m(L)].$$

Theorem

$$\rho_h[\mathcal{F}_m(L)] = \frac{L}{2}\left(\frac{h}{\pi}\right)^{2m},$$

$$\bar{f}_*(x) = h \sum_{k=-\infty}^{\infty} K_*(x - X_k) f(X_k),$$

where $K_*(\cdot)$ is defined by

$$\hat{K}_*(\omega) = \begin{cases} 1, & \omega \in [0, \omega_*), \\ 2^{m-1}(1-\omega)^m, & \omega \in [\omega_*, 1/2), \\ 1 - 2^{m-1}\omega^m, & \omega \in [1/2, 1-\omega_*), \\ 0, & \omega \geq 1 - \omega_*; \end{cases}$$

here $\omega_* = 1 - 2^{-1+1/m}$.

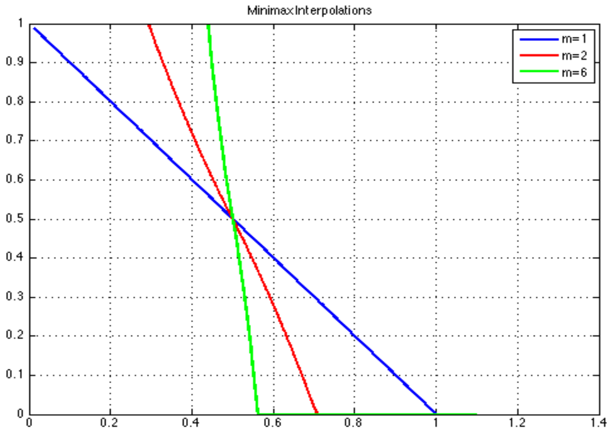

Рис.: The minimax interpolating filters.

The proof is based on the saddle-point theorem for the functional

$$\begin{aligned}\rho_h(\bar{f}, F) =& \Psi(\hat{K}, F) \\ =& \int_{-\infty}^{\infty} F(\omega) \left([1 - \hat{K}(\omega)]^2 + \sum_{k \neq 0} \hat{K}^2 \left(\omega + \frac{k}{h} \right) \right) d\omega,\end{aligned}$$

i.e.

$$\sup_{F \in \mathcal{F}_m(L)} \inf_{\hat{K}} \Psi(\hat{K}, F) = \inf_{\hat{K}} \sup_{F \in \mathcal{F}_m(L)} \Psi(\hat{K}, F).$$

Minimax Efficiency of Interpolating Splines

Our next goal is compare the minimax interpolation with splines over the class $\mathcal{F}_m(L)$. More precisely, we are interested in

$$\rho_h\big(\bar{S}_m, \mathcal{F}_m(L)\big) = \sup_{f\in\mathcal{F}_m(L)} \rho_h(\bar{S}_m, f),$$

where

$$\bar{S}_m = \lim_{\epsilon\to 0} \arg\min_f \left\{ \frac{1}{2\epsilon^2} \sum_{k=1}^{n} \big(Y_k - f(X_k)\big)^2 + \frac{1}{2Q} \int_0^1 \big(f^{(m)}(x)\big)^2 dx \right\}.$$

In the case of the equidistant design

$$\bar{S}_m(x, Y) = h \sum_{k=-\infty}^{\infty} K_m(x - X_k) f(X_k),$$

where the Fourier transform of $K_m(\cdot)$ is given by

$$\hat{K}_m(\omega) = \left(1 + \sum_{k \neq 0} \left(1 + \frac{k}{\omega}\right)^{-2m}\right)^{-1}.$$

Theorem

$$\rho_h\left(\mathcal{F}_m(L), \bar{S}_m\right) = \frac{L}{2}\left(\frac{h}{\pi}\right)^{2m} \kappa_m,$$

where

$$\kappa_m(\omega) = \max_{\omega} \frac{1}{2^{2m-1}\omega^{2m}} \left(\left(1 - \hat{K}_m\left(\frac{\omega}{h}\right)\right)^2 + \sum_{k \neq 0} \hat{K}_m^2\left(\frac{\omega + k}{h}\right)\right).$$

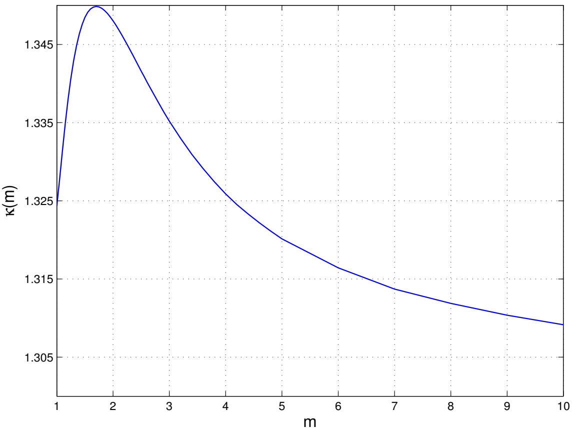

Рис.: Interpolating spline minimax efficiency (m).

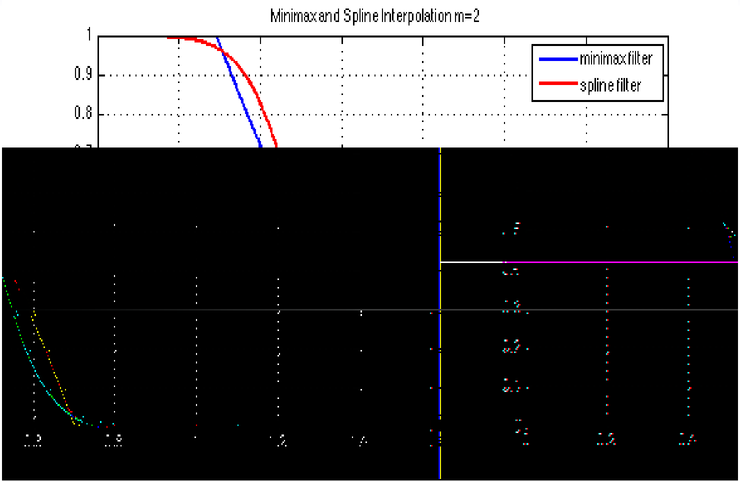

Рис.: The minimax and spline interpolating filters.

Minimax Functional Interpolation

Let

$$\mathcal{W}_m^T(L) = \left\{ f : \int_0^T [f^{(m)}(u)]^2 \, du \le LT \right\}$$

be Sobolev's class on $[0, T]$.

It is assumed that we have at our disposal

$$Y_k^U = f(X_k^U), \; k = 0, 1, \ldots, \lfloor T/h \rfloor,$$

where

$$X_k^U = kh + U, \quad X_k^U \in [0, T]$$

and U is a random variable uniformly distributed on $[0, h]$.

Our goal is to compute the minmax interpolation error

$$\rho[\mathcal{W}_m(L), h] = \lim_{T\to\infty} \inf_{\tilde f} \sup_{f\in\mathcal{W}_m^T(L)} \mathbf{E}^U \frac{1}{T}\int_0^T \left(f(x) - \tilde f(x, Y^U, X^U) \right)^2 dx,$$

where $\mathbf{E}^U$ is the expectation w.r.t. U and *inf* is taken over all interpolations.

Theorem

$$\rho[\mathcal{W}_m(L), h] = \frac{L}{2}\left(\frac{h}{\pi}\right)^{2m}.$$

Pinsker, M.S. (1980). Optimal filtration of square-integrable signals in Gaussian noise. *Problems Inform. Transmission.* **16**, 120–133.

THANK YOU!