Parameter estimation in second-order continuous-time Gaussian autoregressions

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The problem

$(\Omega, \mathcal{F}, \mathbb{P})$, $W = W(t), t \geq 0$.

CAR(N): $X^{(N)}(t) = \sum_{k=0}^{N-1} \theta_{N-k} X^{(k)}(t) + \sigma \dot{W}(t),\ 0 \leq t \leq T$.

σ is known from quadratic variation of $X^{(N-1)}(t)$.

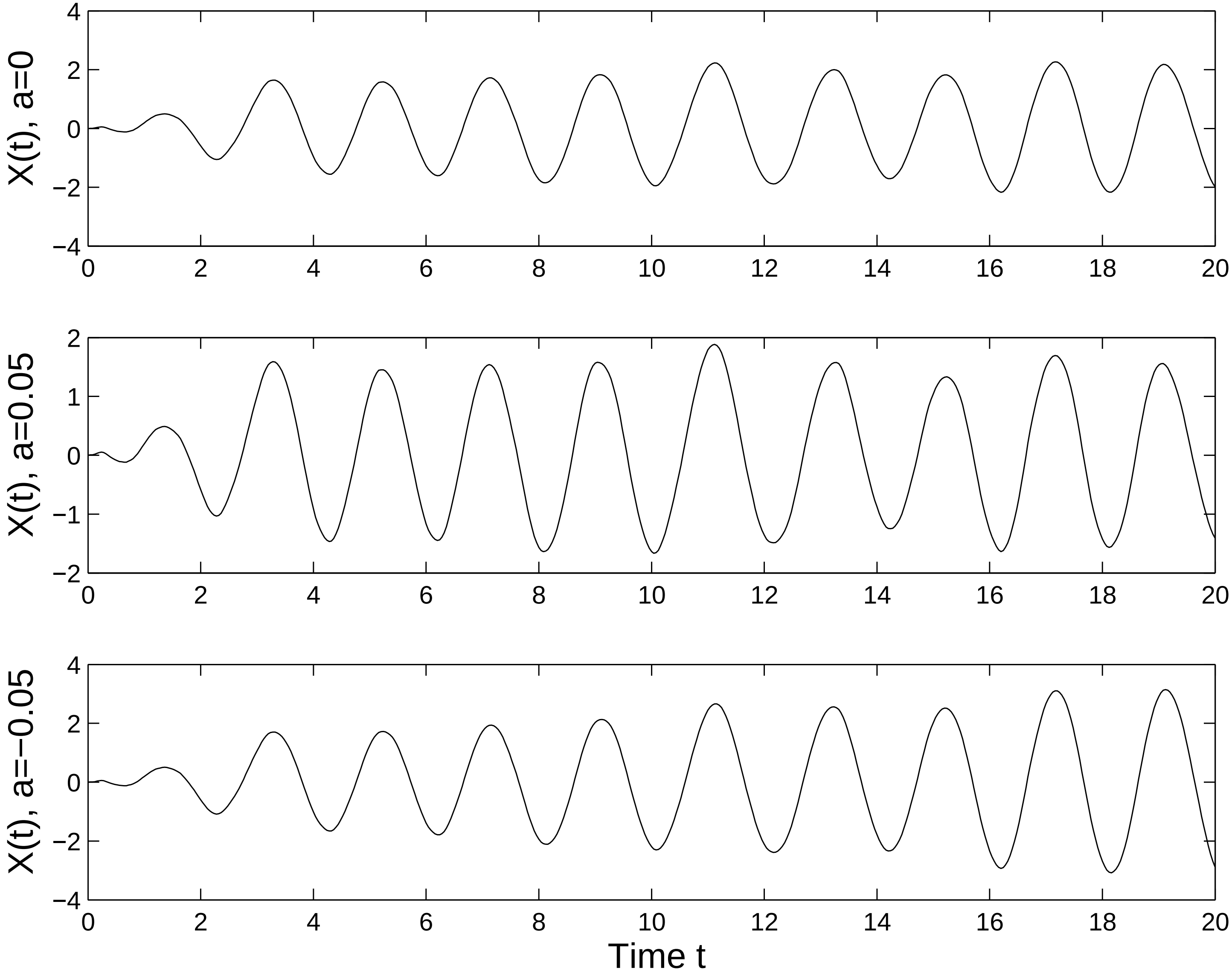
CAR(1): $dX(t) = \theta X(t)dt + \sigma dW(t)$; 3 cases;

CAR(2): $\ddot{X}(t) = \theta_1 \dot{X}(t) + \theta_2 X(t) + \sigma \dot{W}(t)$, or

$$dX = \dot{X}dt,\ d\dot{X} = (\theta_2 X + \theta_1 \dot{X})dt + \sigma dW(t);$$

9 cases, but $9 \neq 3 \times 3$.

Does it make sense to estimate? $\sigma = 1$, $\theta_1 = -a$, $\theta_2 = -\pi^2$



The questions

- Construction of the MLE $\widehat{\boldsymbol{\theta}}_T = (\widehat{\theta}_{1,T}, \ldots, \widehat{\theta}_{N,T})^\top$.

- (Strong) consistency: $\lim_{T\to\infty} \widehat{\boldsymbol{\theta}}_T = \boldsymbol{\theta}$ with probability one.

- Rate of convergence: $v_k(T) \nearrow +\infty$ such that non-degenerate $\lim\limits_{T\to+\infty} v_k(T)(\widehat{\theta}_{k,T} - \theta_k)$ exists in distribution.

- NLRR: matrix $R = R_T$ such that $\lim\limits_{T\to+\infty} R_T(\widehat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}) \overset{d}{=} \mathcal{N}$.

- The local likelihood ratio: LAN, LAMN, or LABF?

The local likelihood ratio

$$L_T(\vartheta) = \ln \frac{dP_T^{\boldsymbol{\vartheta}}}{dP_T^{\boldsymbol{\theta}}}(X^{\boldsymbol{\theta}}); \quad \ell_T(\boldsymbol{u}) = L_T(\boldsymbol{\theta} + A_T\boldsymbol{u}) \Rightarrow \ell_\infty(\boldsymbol{u}).$$

- LAN:

$$\ell_\infty(\boldsymbol{u}) = \frac{1}{\sigma}\boldsymbol{u}^\top\boldsymbol{\xi} - \frac{1}{2\sigma^2}\boldsymbol{u}^\top\Sigma_{\boldsymbol{\xi}}\boldsymbol{u},\ \xi \sim \mathcal{N}(0, \Sigma_{\boldsymbol{\xi}}).$$

- LAMN:

$$\ell_\infty(\boldsymbol{u}) = \frac{1}{\sigma}\boldsymbol{u}^\top B^{1/2}\boldsymbol{\eta} - \frac{1}{2\sigma^2}\boldsymbol{u}^\top B\boldsymbol{u},\ \eta \sim \mathcal{N}(0, I),\ B \perp\!\!\!\perp \eta.$$

- LABF:

$$\ell_\infty(\boldsymbol{u}) = \frac{1}{\sigma}\int_0^1 \boldsymbol{u}^\top G(t)d\boldsymbol{w}(t) - \frac{1}{2\sigma^2}\int_0^1 \boldsymbol{u}^\top G(t)G^\top(t)\boldsymbol{u}\,dt,\ (G, w)\ \text{Gaussian}.$$

CAR(1)=OU

$dY(t) = \theta Y(t)dt + \sigma dW(t),\ Y(0) = a$ or
$Y(t) = ae^{\theta t} + \sigma \int_0^T e^{\theta(t-s)} dW(s)$.
Observe $Y(t),\ 0 \le t \le T$; σ is known from quadratic variation.
MLE of θ:

$$\widehat{\theta}_T = \frac{\int_0^T Y(t)dY(t)}{\int_0^T Y^2(t)dt}$$

Local likelihood ratio with $A_T = \left(\mathbb{E} \int_0^T Y^2(t)dt\right)^{-1/2}$:

$$\ell_T(u) = \frac{u}{\sigma} \frac{\int_0^T Y(t)dW(t)}{\left(\mathbb{E} \int_0^T Y^2(t)dt\right)^{1/2}} - \frac{u^2}{2\sigma^2} \frac{\int_0^T Y^2(t)dt}{\mathbb{E} \int_0^T Y^2(t)dt}, \quad u \in \mathbb{R}.$$

CAR(1): Results

Strong consistency: For every $\theta \in \mathbb{R}$, $\lim_{T\to\infty} \widehat{\theta}_T = \theta$ $\mathbb{P}$-a.s.
Limit distributions:
If $\theta < 0$ (asymptotically stable or ergodic case), then

$$\lim_{T\to\infty} \sqrt{|\theta| T}(\widehat{\theta}_T - \theta) \stackrel{d}{=} \sqrt{2}\,|\theta|\,\xi,\ \xi \sim \mathcal{N}(0,1).$$

If $\theta = 0$ (neutrally stable case), then

$$\lim_{T\to\infty} T(\widehat{\theta}_T - \theta) \stackrel{d}{=} \frac{w^2(1)-1}{2\int_0^1 w^2(s)ds},\ w \sim \mathrm{SBM}.$$

If $\theta > 0$ (unstable or explosive case), then

$$\lim_{T\to\infty} e^{\theta T}(\widehat{\theta}_T - \theta) \stackrel{d}{=} 2\theta\,\frac{\eta}{\xi + c}, c = \sqrt{2\theta}\,y_0/\sigma,\ \xi, \eta \text{ iid } \mathcal{N}(0,1).$$

Normal limit with random rate (NLRR): If $\theta \neq 0$, then

$$\lim_{T\to\infty}\left(\int_0^T Y^2(t)dt\right)^{1/2}\left(\widehat{\theta}_T - \theta\right) \stackrel{d}{=} \sigma\,\eta,\ \ \eta \sim \mathcal{N}(0,1).$$

$\ell_T(u)$: LAN if $\theta < 0$, LABF if $\theta = 0$, LAMN if $\theta > 0$.
Summary: one parameter, four questions, three cases.

Table 1: Estimation in CAR(1)

Parameter	Rate	LD	ℓ_T	NLRR
$\theta < 0$	$\sqrt{T}$	$\mathcal{N}$	LAN	Yes
$\theta = 0$	T	$F_1(w)$	LABF	No
$\theta > 0$	$e^{\theta T}$	Ch	LAMN	Yes

CAR(2): The process

$$\ddot{X}(t) = \theta_1 \dot{X}(t) + \theta_2 X(t) + \sigma \dot{W}(t),\ t > 0,\ X(0) = a, \dot{X}(0) = b,$$

or

$$X(t) = a x_1(t) + b x_2(t) + \sigma \int_0^t x_2(t-s) dW(s),$$

where the functions $x_1(t), x_2(t)$ form the fundamental system of solutions for the equation

$$\ddot{x}(t) - \theta_1 \dot{x}(t) - \theta_2 x(t) = 0 \quad (1)$$

In other words, $x_1(0) = 1, \dot{x}_1(0) = 0$, $x_2(0) = 0$, $\dot{x}_2(0) = 1$, and both $x_1 = x_1(t)$ and $x_2 = x_2(t)$ satisfy (1).

Characteristic equation $r^2 - \theta_1 r - \theta_2 = 0.$

Roots $p = \dfrac{\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}}{2}, \quad q = \dfrac{\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}}{2}.$

Then (with possibility of complex p, q)

$$x_1(t) = \begin{cases} \dfrac{qe^{pt} - pe^{qt}}{q - p}, & \text{if } p \neq q, \\ e^{qt}, & \text{if } p = q; \end{cases} \quad x_2(t) = \begin{cases} \dfrac{e^{pt} - e^{qt}}{p - q}, & \text{if } p \neq q, \\ te^{qt}, & \text{if } p = q. \end{cases}$$

11 cases for the solutions, **9 cases** for the estimators:
Ergodic case (3): $\theta_1 < 0, \theta_2 < 0$;
Unstable real roots (6): $-, 0$; $-, +$; $0, +$; $+, +$; $+^2$; 0^2;
Unstable complex roots (2): $\pm\sqrt{-1}\nu$; $\lambda \pm \sqrt{-1}\nu$.

CAR(2): MLE

$$d\dot{X}(t) = \theta_1 \dot{X}(t)dt + \theta_2 \left(a + \int_0^t \dot{X}(s)ds \right) dt + \sigma dW(t), \quad \mathbb{P}_T^{\dot{X}} \ll \mathbb{P}_T^W.$$

$$\widehat{\theta}_1(T) = \frac{\int_0^T X^2(t)dt \int_0^T \dot{X}(t)d\dot{X}(t) - \int_0^T X(t)\dot{X}(t)dt \int_0^T X(t)d\dot{X}(t)}{\int_0^T X^2(t)dt \int_0^T \dot{X}^2(t)dt - \left(\int_0^T X(t)\dot{X}(t)dt \right)^2},$$

$$\widehat{\theta}_2(T) = \frac{\int_0^T \dot{X}^2(t)dt \int_0^T X(t)d\dot{X}(t) - \int_0^T X(t)\dot{X}(t)dt \int_0^T \dot{X}(t)d\dot{X}(t)}{\int_0^T X^2(t)dt \int_0^T \dot{X}^2(t)dt - \left(\int_0^T X(t)\dot{X}(t)dt \right)^2}.$$

σ from quadratic variation of $\dot{X}$.

CAR(2): Summary of results

p, q are roots of $r^2 - \theta_1 r - \theta_2 = 0$.

Table 2: Estimation in CAR(2)

Case	v_1	v_2	LD$_1$	LD$_2$	NLRR	ℓ_T
$\theta_1 < 0,\ \theta_2 < 0$	$\sqrt{T}$	$\sqrt{T}$	$\mathcal{N}$	$\mathcal{N}$	Yes	LAN
$q < 0 < p$	$\sqrt{T}$	$\sqrt{T}$	$\mathcal{N}$	$\mathcal{N}$	Yes	DLAMN
$0 < q < p$	e^{qT}	e^{qT}	Ch	Ch	Yes	DLAMN
$0 < q = p$	$T^{-1}e^{qT}$	$T^{-1}e^{qT}$	Ch	Ch	Yes	DLAMN
$q < p = 0$	$\sqrt{T}$	T	$\mathcal{N}$	$F_1(w)$	Yes $(\widehat{\theta}_{1,T})$	LABF/LAN
$q = 0 < p$	T	T	$F_1(w)$	$F_1(w)$	No	DLAMN
$q = p = 0$	T	T^2	$F_1(w)$	$F_1(w)$	No	LABF
$\Re(p) = 0$	T	T	$F_2(w)$	$F_2(w)$	No	LABF
$\Re(p) = \lambda > 0$	$e^{\lambda T}$	$e^{\lambda T}$	Many	Many	Yes	LAMN family

CAR(2): Strong consistency

G. K. Basak and P. Lee (2008):

$$d\boldsymbol{Y}(t) = \Theta \boldsymbol{Y}(t)dt + \boldsymbol{\Sigma} d\boldsymbol{W}(t), \ \boldsymbol{Y}, \boldsymbol{W} \in \mathbb{R}^N,$$

$$\widehat{\Theta}_T = \left(\int_0^T \big(d\boldsymbol{Y}(t)\boldsymbol{Y}^*(t) \big) \right) \left(\int_0^T \boldsymbol{Y}(t)\boldsymbol{Y}^*(t)dt \right)^{-1},$$

strongly consistent in the CAR(N) case.

CAR(2):

$$\Theta = \begin{pmatrix} 0 & 1 \\ \theta_2 & \theta_1 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The second row of $\widehat{\Theta}_T$ is $(\widehat{\theta}_{2,T}, \widehat{\theta}_{1,T})$.
Can you guess the first row?

Asymptotics

p, q are roots of $r^2 - \theta_1 r - \theta_2 = 0$. **Information matrix**

$$\Psi_T = \int_0^T \boldsymbol{X}(t)\boldsymbol{X}^\top(t)dt = \begin{pmatrix} \int_0^T X^2(t)dt & \int_0^T X(t)\dot{X}(t)dt \\ \int_0^T X(t)\dot{X}(t)dt & \int_0^T \dot{X}^2(t)dt \end{pmatrix}.$$

If $p > 0$, and $p > q$, then

$$\Psi_T \approx \begin{pmatrix} 1 & p \\ p & p^2 \end{pmatrix} e^{2pT},\ 1 \ll \det\Psi_T \ll e^{4pT}.$$

$$\ell_T(\boldsymbol{u}) = \frac{1}{\sigma}\int_0^T \left(\boldsymbol{u}^\top A_T \boldsymbol{X}(t)\right) dW(t) - \frac{1}{2\sigma^2}\boldsymbol{u}^\top A_T^\top \Psi_T A_T \boldsymbol{u}$$

DLAMN: $B = \frac{(1+p^2)^2}{2p}\sigma^2\zeta^2 \begin{pmatrix} 1 & p \\ p & p^2 \end{pmatrix}, \quad A_T \neq \begin{pmatrix} 1/v_1 & 0 \\ 0 & 1/v_2 \end{pmatrix}.$

Two different $\mathcal{N}$ limits

p, q are roots of $r^2 - \theta_1 r - \theta_2 = 0$.

Theorem.

If $q < p < 0$, then

$$\lim_{T\to\infty} \sqrt{T}(\widehat{\theta}_{1,T} - \theta_1) \stackrel{d}{=} \sqrt{2|p+q|}\,\eta_1, \ \lim_{T\to\infty} \sqrt{T}(\widehat{\theta}_{2,T} - \theta_2) \stackrel{d}{=} \sqrt{2pq|p+q|}\,\eta_2,$$

where η_1 and η_2 are iid $\mathcal{N}(0,1)$.

If $q < 0 < p$, then

$$\lim_{T\to\infty} \sqrt{T}(\widehat{\theta}_{1,T} - \theta_1) \stackrel{d}{=} \sqrt{2|q|}\,\eta, \ \lim_{T\to\infty} \sqrt{T}(\widehat{\theta}_{2,T} - \theta_2) \stackrel{d}{=} -p\sqrt{2|q|}\,\eta,$$

where η is $\mathcal{N}(0,1)$.

Zero roots

p, q are roots of $r^2 - \theta_1 r - \theta_2 = 0$.

Theorem

If $q < 0 = p$, then

$$\lim_{T\to\infty} \sqrt{|q|T}\left(\widehat{\theta}_{1,T} - \theta_1\right) \stackrel{d}{=} \sqrt{2}\,|q|\,\eta, \ \lim_{T\to\infty} T\left(\widehat{\theta}_{2,T} - \theta_2\right) \stackrel{d}{=} |q|\,\frac{w^2(1)-1}{2\int_0^1 w^2(s)ds},$$

$\eta \sim \mathcal{N}(0,1)$, $w = w(s)$, $0 \le s \le 1$, is a standard Brownian motion, and η and w are independent.

If $q = 0 < p$, then

$$\lim_{T\to\infty} T\left(\widehat{\theta}_{1,T} - \theta_1\right) \stackrel{d}{=} \frac{w^2(1)-1}{2\int_0^1 w^2(s)ds}, \ \lim_{T\to\infty} T\left(\widehat{\theta}_{2,T} - \theta_2\right) \stackrel{d}{=} -p\,\frac{w^2(1)-1}{2\int_0^1 w^2(s)ds}.$$

If $p = q = 0$, then $X(t) = a + bt + \int_0^t W(s)ds$.

Positive roots

p, q are roots of $r^2 - \theta_1 r - \theta_2 = 0$.

Theorem.

If $p > q > 0$, then $\lim_{T\to\infty} e^{qT}(\widehat{\theta}_{1,T} - \theta_1) \stackrel{d}{=} \frac{2(p+q)q}{p-q} \frac{\eta}{\xi + c}$,

$$\lim_{T\to\infty} e^{qT}\,(\widehat{\theta}_{2,T} - \theta_2) \stackrel{d}{=} -\frac{2(p+q)pq}{p-q} \frac{\eta}{\xi + c},$$

$c = \sqrt{2q}\,(b - ap)/\sigma$; ξ, η iid $\mathcal{N}(0,1)$.

If $p = q > 0$, then

$$\lim_{T\to\infty} \frac{e^{qT}}{qT}(\widehat{\theta}_{1,T} - \theta_1) \stackrel{d}{=} 4\sqrt{2}\,q\,\frac{\eta}{\xi + c}, \quad \lim_{T\to\infty} \frac{e^{qT}}{qT}\,(\widehat{\theta}_{2,T} - \theta_2) \stackrel{d}{=} -4\sqrt{2}\,q^2 \frac{\eta}{\xi + c},$$

$c = \sqrt{2q}\,(b - aq)/\sigma$; ξ, η iid $\mathcal{N}(0,1)$.

Complex roots

p, q are roots of $r^2 - \theta_1 r - \theta_2 = 0$.

Theorem.

If $p = \sqrt{-1}\nu$, $\nu > 0$,

then $\theta_1 = 0, \ \lim\limits_{T\to\infty} T\widehat{\theta}_{1,T} \stackrel{d}{=} \dfrac{2 - w_1^2(1) - w_2^2(1)}{\int_0^1 w_1^2(t)dt + \int_0^1 w_2^2(t)dt}$,

$$\theta_2 = -\nu^2, \ \lim_{T\to\infty} T(\widehat{\theta}_{2,T} - \theta_2) \stackrel{d}{=} 2\nu \frac{\int_0^1 w_1(t)dw_2(t) - \int_0^1 w_2(t)dw_1(t)}{\int_0^1 w_1^2(t)dt + \int_0^1 w_2^2(t)dt},$$

where w_1, w_2 are independent standard Brownian motions.

If $p=\lambda+\sqrt{-1}\nu,\ \lambda>0$, then, for $i=1,2$, the families $\{e^{\lambda T}(\widehat{\theta}_{i,T}-\theta_i),\ T\geq 0\}$ are relatively compact, with the limit distributions of the form

$$\frac{\xi_c\eta_c+\xi_s\eta_s}{\xi_c^2+\xi_s^2},$$

where the bivariate normal vectors (ξ_c,ξ_s) and (η_c,η_s) are independent and $\mathbb{E}\eta_c=\mathbb{E}\eta_s=0$.
NLRR: $(\mathfrak{u}_s,\mathfrak{u}_c)\sim\mathcal{N}$,

$$A_T=\begin{pmatrix}\nu & 0\\ \lambda & -1\end{pmatrix}e^{-\lambda T},\quad B=\frac{1}{\mathfrak{u}_s^2+\mathfrak{u}_c^2}\begin{pmatrix}\mathfrak{u}_s & \mathfrak{u}_c\\ -\mathfrak{u}_c & \mathfrak{u}_s\end{pmatrix}.$$

Then the family $BA_T\Psi_T\left(\widehat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}\right),\ T>0$, is relatively compact, with normal limit points independent of $(\mathfrak{u}_s,\mathfrak{u}_c)$.

Some general conclusions

- Dominant eigenvalue determines the structure of LR.
- Non-dominant eigenvalue determines convergence rate.
- Nonzero correlation of the limits in the exponentially unstable cases.
- New effects: DLAMN, no particular limit.
- Next steps: optimality results for DLAMN, inference from discrete observations.

Discrete time: $X_k = X(kh)$

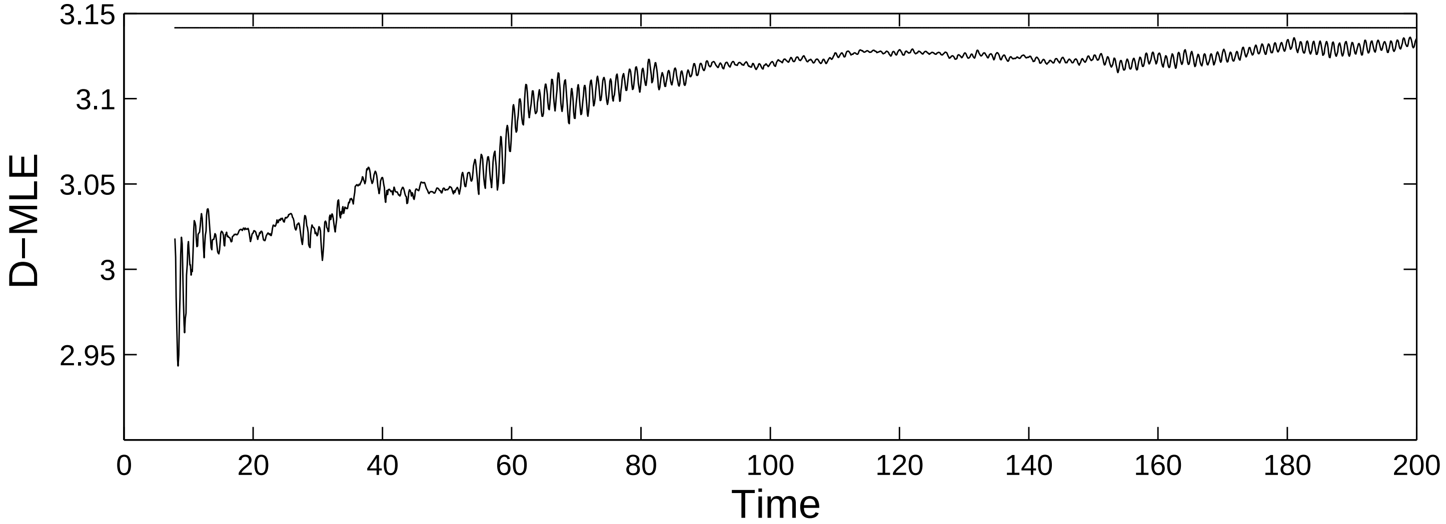
Option 1. D-MLE: approximate integrals with sums.
Option 2. LSE: from

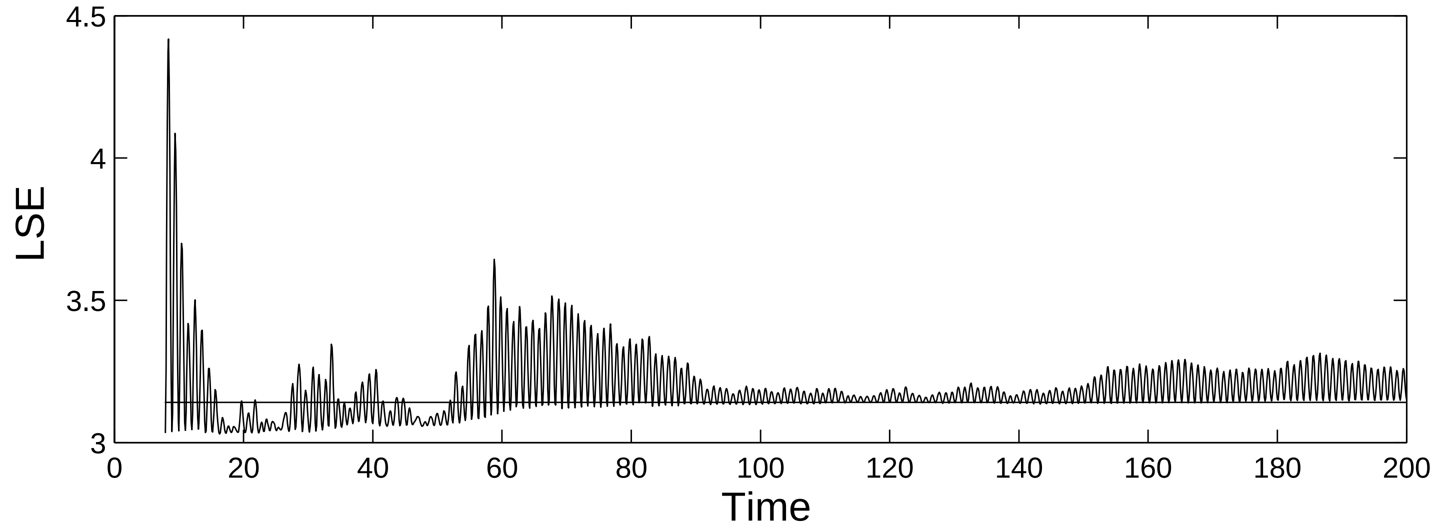
$$X_{k+1} = AX_k + BX_{k-1} + \xi_{k+1},\ A = A(\theta_1, \theta_2),\ B = B(\theta_1, \theta_2)$$

to the least-squares estimator of A and B.

Complications: messy formulas
Difficulties: ξ_k and ξ_{k+1} are dependent.

An example: $\ddot{X} + \pi^2 X = \dot{W}$ $(\theta_1 = 0,$ $\theta_2 = -\pi^2)$





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