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Parametric estimation in Non Recurrent Diffusions

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Consider the diffusion process $(X_t, t \geq 0)$:

$$dX_t = a(\theta, X_t)dt + dW_t, \quad X_0 = x_0, \quad t \in [0, T]$$

where
$X_0 = x_0$, $x_0 > 0$
$(W(t), t \geq 0)$ is a Wiener process

The parameter of interest is θ

We consider two cases of drift:

Case (I) : $a(\theta, x) = \theta x^{\alpha}$
where $\theta > 0$ and $\alpha \in]-1, 0[$.

Case (II) : $a(\theta, x) = \theta g(x)$
where $\theta > 0$ and g is a function

Our Aim :
For non recurrent diffusion process
1. **Case (I)** : estimation of α and θ
A Hill estimator and Trajectory Fitting estimator
(Dietz and Kutoyants 2001-2003)
2. **Case (II)** : Trajectory Fitting estimator
(conditions of paper by Keller G. et al. 1984)

Case (I) : on approximation results of the solution of (1) by their corresponding deterministic ones established in Gikhman and Skorokod book 1977 Chap. 5
Case (II) : on approximation results of the solution of (1) by their corresponding deterministic ones in paper by Keller, G. et al. 1984

Remark. In **Case (I)** for $\alpha \in]0,1[$, parametric estimation of θ has been considered by
Dietz and Kutoyants 2003

Plan

Case (I) :

1. α parameter and θ known :
Hill estimator of α : consistency and asymptotic normality

2. θ parameter and α known :
Trajectory Fitting Estimator of θ : consistency and asymptotic normality

3. (θ, α) parameters :
Hill estimator for α and Trajectory Fitting Estimator for θ : consistency property

Case (II) :

θ parameter and g known :
Trajectory Fitting Estimator of θ : consistency and asymptotic normality

Some known results on non-recurrent processes

$$dX(t) = \theta a(X_t)dt + \sigma dW(t), \quad t \geq 0$$

where the parameter $\theta \geq 0$
(If $\theta \leq 0$ and $a(x) = x$ then X_t is positive recurrent).

1. Non-recurrent processes (transient)

we have

$$\mid X_t \mid \to \infty \quad a.s. \quad as \quad t \to \infty$$

at a prescribed rate in 3 cases of drift :

1.

$$a(x) = x$$

2.

$$a(x) = cx + r(x)$$

$$\mid r(x) \mid = K(1+ \mid x \mid^{\gamma}), K > 0, \gamma \in (0,1)$$

3.

$$a(x) = \mid x \mid^{\alpha}, \quad where \quad 0 \leq \alpha < 1$$

Results : consistency and limit law of MLE :

case 1 and case 2 :

$$e^{\theta c T}(\widehat{\theta}_T - \theta) \Rightarrow \frac{\nu}{\chi}$$

with ν, χ rv's

case 3 :

$$T^{\theta c}(\widehat{\theta}_T - \theta) \Rightarrow N(0, 2\theta)$$

Kutoyants's book (1994)
Basawa and Scott (1983)
Dietz and Kutoyants (2003)
Dietz (2001)

2. Null recurrent processes

Hoepfner and Kutoyants (2003) consider

$$dX(t) = \left(\theta \frac{X_t}{1 + X_t^2} + g(X_t)\right) dt + \sigma dW(t), \quad t \geq 0$$

where parameter $\theta \in \Theta = (-\sigma^2/2, \sigma^2/2)$

g a nuisance function

$\sigma > 0$

Results : consistency and limit law of MLE -LAMN condition - efficiency.

$$n^{\frac{\alpha(\theta)}{2}}(\widehat{\theta}_n - \theta) \Rightarrow \quad mixture \quad of \quad normals$$

where

$$\alpha(\theta) = \frac{1}{2} - \frac{\theta}{\sigma^2} < \frac{1}{2}$$

Case (I) : $a(\theta, x) = \theta x^{\alpha}$
We observe $(X_t, t \in [0, T])$
The parameter is α :

$$-1 < \alpha < 0$$

$\theta > 0$ is known
We define a Hill estimator $\widehat{\alpha}_T$ of α

Case (I) :

Asymptotic behavior of X_t

Th. 5. 17 in Gikhman-Skorohod (1977)

The diffusion process $(X_t, t \geq 0)$ solution of

$$dX(t) = \theta X_t^{\alpha} dt + dW(t), \quad t \geq 0, \quad x_0 > 0$$

with $\theta > 0$, $-1 < \alpha < 0$ is such that

$$X_t \longrightarrow \infty, \quad a.s. \quad t \longrightarrow \infty$$

Precisely there exists a constant $C(\alpha, \theta) > 0$ such that

$$X_t \asymp C(\alpha, \theta) \quad t^{\frac{1}{2-\alpha}}, \quad a.s. \quad t \to \infty$$

A Hill estimator $\widehat{\alpha}_T$
From observation $(X_t, t \in [0,T])$
a decreasing numbers (λ_i) :

$$0 < \lambda_{i+1} < \lambda_i < 1$$

define

$$X_{\lambda_i}^{(T)} := X_{\lambda_i T}$$

where $i = 1,..,k$ and k is given.
Consider the "k largest" observations of the process (X_t) :

$$X_{\lambda_k}^{(T)}, \quad X_{\lambda_{k-1}}^{(T)}, \quad .. \quad , \quad X_{\lambda_1}^{(T)}$$

Set

$$\gamma := \frac{1}{1-\alpha}$$

Define a Hill estimator of γ by

$$\widehat{\gamma}_T = \frac{1}{k} \sum_{i=1}^{k} \frac{\log X_{\lambda_1}^{(T)} - \log X_{\lambda_i}^{(T)}}{\log \lambda_1 - \log \lambda_i}$$

or equivalently

$$\widehat{\gamma}_T = \frac{1}{k} \sum_{i=1}^{k} \frac{\log(\frac{X_{\lambda_1}^{(T)}}{X_{\lambda_i}^{(T)}})}{\log(\frac{\lambda_1}{\lambda_i})}$$

The Hill estimator of α is then

$$\widehat{\alpha}_T := \frac{\widehat{\gamma}_T - 1}{\widehat{\gamma}_T}$$

Theorem

The Hill estimator $\widehat{\gamma}_T$ *satisfies*

$$\widehat{\gamma}_T \to \gamma \quad in \quad probability$$

and

$$T^{\frac{1}{2}(2\gamma-1)}(\widehat{\gamma}_T - \gamma) \Rightarrow N\left(0, (\frac{\gamma}{\beta})^{2\gamma} \sum_{i=1}^{k-1} i^2 \frac{\lambda_i - \lambda_{i+1}}{\lambda_{i+1}^{2\gamma}}\right)$$

as $T \to \infty$

Remark

the rate of convergence is : for $-1 < \alpha < 0$

$$\frac{1}{2}(2\gamma - 1) = \frac{1}{2} + \frac{\alpha}{1-\alpha} < \frac{1}{2}$$

Corollary

The Hill estimator $\widehat{\alpha}_T$ satisfies

$$\widehat{\alpha}_T \to \alpha \quad in \quad probability$$

and

$$T^{\frac{1}{2}+\frac{\alpha}{1-\alpha}}(\widehat{\alpha}_T-\alpha) \Rightarrow N\left((0, \frac{1}{\gamma^4}(\frac{\gamma}{\beta})^{2\gamma}\sum_{i=1}^{k-1} i^2\frac{\lambda_i - \lambda_{i+1}}{\lambda_{i+1}^{2\gamma}}\right)$$

as $T \to \infty$

Questions

1. How to choose λ_i giving a minimal variance in asymptotic law ?

2. How to choose both (λ_i, k) giving a minimal asymptotic variance ?

Case (I) : $a(\theta, x) = \theta x^{\alpha}$

Trajectory Fitting Estimator of θ

α is known

We observe a trajectory $(X_t, t \in [0, T])$ of (1)

Consider the statistic : for $t > 0$

$$A_t(\alpha, \theta) = x_0 + \theta \int_0^t X_s^{\alpha} ds$$

Distance process :

$$D_T(\alpha, \theta) = \int_0^T (X_t - A_t(\alpha, \theta))^2 dt$$

Let $\Theta \subset R^{+}$ be the parameter space

Trajectory Fitting Estimator of θ is

$$\widehat{\theta}_{1,T} = \arg\min_{\theta \in \Theta} D_T(\alpha, \theta)$$

We have

$$\widehat{\theta}_{1,T} = \frac{\int_0^T (X_t - x_0)(\int_0^t X_s^\alpha ds)dt}{\int_0^T (\int_0^t X_s^\alpha ds)^2 dt}$$

Consistency and Limit Law
Theorem

Trajectory Fitting Estimator $\widehat{\theta}_{1,T}$ *satisfies :*

$$\widehat{\theta}_{1,T} \longrightarrow \theta \qquad a.s.$$

and

$$T^{\frac{1}{2}+\frac{\alpha}{2-\alpha}}(\widehat{\theta}_{1,T} - \theta) \Rightarrow N(0, \sigma^2(\theta, \alpha))$$

as $T \to \infty$ and where $\sigma^2(\theta, \alpha) > 0$.

Remark

Comparison with the rate of convergence of $\widehat{\alpha}_T$: for $-1 < \alpha < 0$

$$\frac{1}{2} + \frac{\alpha}{1-\alpha} < \frac{1}{2} + \frac{\alpha}{2-\alpha} < \frac{1}{2}$$

We suppose both parameters (α, θ) are unknown
Plugging the Hill estimator of α in distance process $D_T(\alpha, \beta)$
Define a Trajectory Fitting Estimator of θ :

$$\widehat{\theta}_{2,T} = \arg\min_{\theta \in \Theta} D_T(\hat{\alpha}_T, \theta)$$

Theorem

Trajectory Fitting Estimator $\widehat{\theta}_{2,T}$ satisfies :

$$\widehat{\theta}_{2,T} \longrightarrow \theta \quad in \quad probability$$

as $T \to \infty$

Case (II) : $a(\theta, x) = \theta g(x)$
The parameter $\theta > 0$
g is a known function
Define the following statistics :

$$A_t = \int_0^t g(X_s) ds, \quad t \in [0, T]$$

$$X(\theta)_t = x_0 + \theta A_t$$

distance process

$$D(\theta)_T = \int_0^T (X_t - X(\theta)_t)^2 dt$$

Trajectory fitting estimator $\widehat{\theta}_{3,T}$:

$$\widehat{\theta}_{3,T} = \arg\min_{\theta \in \Theta} D(\theta)_T$$

Then

$$\widehat{\theta}_{3,T} = \frac{\int_0^T (X_t - x_0) A_t dt}{\int_0^T A_t^2 dt}, \qquad T > 0$$

μ_t the deterministic solution of the ODE :

$$d\mu_t = a(\theta, \mu_t)dt, \quad \mu_0 = x_0.$$

Introduce :

$$G(x) = \int_{x_0}^{x} \frac{dy}{g(y)}, \quad \psi(x) = \int_{x_0}^{x} \frac{dy}{g^3(y)}, \quad x \geq x_0$$

$$h(t) = \frac{g'(t)}{g^2(t)}$$

For a function f denote

$$\widetilde{f}(t) = f(\mu(t)) = f(\mu_t)$$

Asymptotic behavior of X_t (Keller and al. 1984)

$$dX_t = g(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = 1, \quad t \geq 0$$

Conditions :

(A1) $g : R^+ \to R^+$ is strictly positive, C^2-class function and

$$G(\infty) = \int_1^\infty \frac{dy}{g(y)} = \infty$$

(A2) $h(t) \to 0$, as $t \to \infty$.

(A3) $\sigma : R^+ \to R^+$ is strictly positive, C^2-class function and

$$\int_0^\infty \frac{\widetilde{\sigma}^2(t)}{t^2} dt < \infty$$

(A4) The functions g, g', $\widetilde{\sigma}^2$ and $\widetilde{h}$ are ultimately concave or convex.

If $\psi(\infty) = \infty$, we require the same behavior for the function $\widetilde{h} \circ \widetilde{\psi}^{-1}$.

Theorem. (Th. 2 Keller and al. 1984)
Assume **(A1)**-**(A4)**.
Then the following statements are equivalent
i) $t^{-1}g(t) = o(\psi^{-1/2}(t))$
ii) $X_t/\mu_t \to 1$ in probability on $\{X_t \to \infty\}$ as $t \to \infty$.
iii)There are positive numbers β_t, $t \geq 0$, such that
$X_t/\beta_t \to 1$ in probability on $\{X_t \to \infty\}$

We impose the following conditions

Conditions :

(C1). $g : R^+ \to R^+$ is strictly positive, C^2-class function and regularly varying function with index α, $RV(\alpha)$ where $0 < |\alpha| < 1$.

(C2). The functions g, g', $\frac{1}{\widetilde{g}^2}$ and $\widetilde{h}$ are ultimately concave or convex.
If $\psi(\infty) = \infty$, we require the same behavior for the function $\widetilde{h} \circ \widetilde{\psi}^{-1}$.

Consistency

Theorem.
Suppose Case (II) where g satisfies **(C1)** *and* **(C2)** *and* $\theta_0 > 0$.
Then on $\{X_t \to \infty\}$, *TFE estimator is strongly consistent :*

$$\widehat{\theta}_{3,T} \longrightarrow \theta_0 \quad a.s. \qquad \texttt{as } T \to \infty$$

Limit Law

Define the functions :

$$\rho_t = \int_0^t g(\mu_s)ds, \quad \chi_t = \int_0^t \rho_s^2 ds$$

$$\Phi_t = \int_0^t \rho_s ds, \quad \sigma_T^2 = \int_0^T (\Phi_T - \Phi_t)^2 dt$$

Theorem.

Suppose Case (II) where g satisfies **(C1)** *and* **(C2)** *and* $\theta_0 > 0$*.*

Then on $\{X_t \to \infty\}$

$$\kappa_T(\widehat{\theta}_{3,T} - \theta_0) \Longrightarrow \xi \sim \mathcal{N}(0,1) \qquad \textbf{as } T \to \infty$$

where

$$\kappa_T = \frac{\chi_T}{\sigma_T}$$

Remark

a. The functions
$g(x) = x^{\alpha}$ and $g(x) = x^{\alpha} \log(2+x)$, $0 < |\alpha| < 1$ satisfy the conditions **(C1)** and **(C2)**.

b. *The rate in law convergence is such that :*

$$\kappa_T \sim T^{\frac{1}{2}+\frac{\alpha}{1-\alpha}}.\ l(T)$$

where $l(T)$ is slowly varying function.
So the rate of convergence is

$$\frac{1}{2} + \frac{\alpha}{1-\alpha} < 1/2$$

up a slowly varying function factor.

Case (III) : general case $a(\theta, x)$.

The parameter $\theta > 0$
Define the following statistics :

$$A_t(\theta) = x_0 + \int_0^t a(\theta, X_s)ds, \quad t \in [0, T]$$

Distance process

$$D(\theta)_T = \int_0^T (X_t - A_t(\theta))^2 dt$$

Trajectory fitting estimator $\widehat{\theta}_{4,T}$:

$$\widehat{\theta}_{4,T} = \arg\min_{\theta \in \Theta} D(\theta)_T$$

(work in progress)

Remark

The conditions **(C1)(C2)** *are strengthened by the condition of regularly varying function on g* $RV(\alpha)$ *where* $0 < |\alpha| < 1$. *This condition on g makes possible notable reductions on the assumptions in our study. However if we remove this condition one would require more conditions.*

It remains the study of the MLE in both Case (I) for $\alpha \in]-1, 0[$ *and Case (II) !*

Thank You

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