

# On Minimum $L^p$ -Distance Estimation for Inhomogeneous Poisson Processes

**A.S.Dabye,**

Université Gaston Berger, Saint-Louis, Sénégal

**joint work** : **D.B. Ba**, Université de Thiès, Sénégal

3 avril 2015



## 1 Estimation in no regular case

- Preliminaries
- Main results

## 2 MDE on $L^p$ -distance

- General idea
- Preliminaries
- Main result
  - Consistency
  - Limit distribution

# Plan

## 1 Estimation in no regular case

Preliminaries

Main results

## 2 MDE on $L^p$ -distance

General idea

Preliminaries

Main result

Consistency

Limit distribution

These previous works are devoted to the study of **BE** and **MLE** in the non regular case.

For this we consider the model of  $n$  independent observation of an inhomogeneous Poisson process  $X^{(n)} = (X_1, \dots, X_n)$  where  $X_j = \{X_j(t), 0 \leq t \in \tau\}$ ,  $j= 1, \dots, n$  are Poisson processes with

$$S(\theta, t) = \theta^{(1)}g(\theta^{(2)}t) + \lambda, \quad 0 \leq t \leq T.$$

We suppose that  $\lambda$  ( $\ll$  dark current  $\gg$ ) and the function  $g(\cdot)$  are positive and known. The function  $g(\cdot)$  is continuously differentiable on over the interval  $[0, T]$ , except at the point  $\tau^* \in (0, T)$ . At this the function admit a jump of size  $r$ , that-is  $g(\tau_+^*) - g(\tau_-^*) = r \neq 0$ , where  $g(\tau_-^*)$  and  $g(\tau_+^*)$  are the right and left limit  $g(\cdot)$  at the point  $\tau^*$  respectively.

We introduce below the BE, the MLE and the MDE and we are interesting by their asymptotic properties when  $n$  tend to infinity. Denote by  $L(\theta, \theta_o, X^{(n)})$  the likelihood ratio of this problem and by  $\mathcal{P}_\theta^{(n)}$  the measure induced in the space of observations by  $n$  realizations of the Poisson process of intensity measure  $\Lambda(\theta, [0, T])$ . Thus we have

$$\begin{aligned} L(\theta, \theta_0, \mathbf{X}^{(n)}) &\equiv \frac{d\mathcal{P}_{(\theta^{(1)}, \theta^{(2)})}^{(n)}}{d\mathcal{P}_{(\theta_0^{(1)}, \theta_0^{(2)})}^{(n)}} = \\ &= \exp \left\{ \sum_{j=1}^n \int_0^T \ln \left( \frac{\theta^{(1)} g(\theta^{(2)} t) + \lambda}{\theta_0^{(1)} g(\theta_0^{(2)} t) + \lambda} \right) dX_j(t) - \right. \\ &\quad \left. - n \int_0^T \left( \theta^{(1)} g(\theta^{(2)} t) - \theta_0^{(1)} g(\theta_0^{(2)} t) \right) dt \right\}. \end{aligned}$$

This function is continuous w.r.t.  $\theta^{(1)}$  and discontinuous w.r.t.  $\theta^{(2)}$  with jumps at the points

$$\theta_{ij} = \frac{\tau^*}{t_{ij}}, \quad i = 1, \dots, m_j, \quad j = 1, \dots, n.$$

# Preliminaries

To describe the properties of all estimators that we will use in the sequel, we need the following notations. Let

$$Z_{\theta}(v) = \begin{cases} \exp \left\{ \ln \frac{\theta^{(1)} g(\tau_+^*) + \lambda}{\theta^{(1)} g(\tau_-^*) + \lambda} p^-(v) - \frac{\tau^*}{(\theta^{(2)})^2} \theta^{(1)} r v \right\}, & v \geq 0 \\ \exp \left\{ \ln \frac{\theta^{(1)} g(\tau_-^*) + \lambda}{\theta^{(1)} g(\tau_+^*) + \lambda} p^+(-v) - \frac{\tau^*}{(\theta^{(2)})^2} \theta^{(1)} r v \right\}, & v < 0 \end{cases} \quad (1)$$

where  $r = g(\tau_+^*) - g(\tau_-^*)$  and  $p^+$  and  $p^-$  are pairwise independent Poisson processes on  $R_+$  of constant intensities

$$E p^-(v) = \frac{\tau^*}{(\theta^{(2)})^2} \left( \theta^{(1)} g(\tau_-^*) + \lambda \right) v$$

and

$$E p^+(v) = \frac{\tau^*}{(\theta^{(2)})^2} \left( \theta^{(1)} g(\tau_+^*) + \lambda \right) v.$$

respectively.

We note also the Fisher information w.r.t. the second parameter

$$I(\theta) = \int_0^T \frac{g^2(\theta^{(2)} t)}{\theta^{(1)} g(\theta^{(2)} t) + \lambda} dt. \quad (2)$$

We note  $\theta_u = \theta^{(1)} + \frac{u}{\sqrt{n}}$  and  $\theta_v = \theta^{(2)} + \frac{v}{n}$ .

## Conditions A.

- a) *the function  $g(\cdot)$  and the constant  $\lambda$  are positives and known ;*
- b) *the function  $g(\cdot)$  est continuously differentiable on  $[0, \tau^*) \cup (\tau^*, T]$  and has a jump at the point  $\tau^* \in (0, T)$  with  $g(\tau_+^*) - g(\tau_-^*) = r > 0$  and  $g(\tau_+^*) g(\tau_-^*) \neq 0$ .*



# Definition

The **BE**  $\tilde{\theta}_n = (\tilde{\theta}_n^{(1)}, \tilde{\theta}_n^{(2)})$  of  $\theta_0 = (\theta_0^{(1)}, \theta_0^{(2)})$  for a given density  $p$  and for square loss function  $\ell(\cdot, \cdot) = \ell(x, y) = x^2 + y^2$ , is defined by

$$\tilde{\theta}_n = E(\theta \mid \mathbf{X}^{(n)}) = \int_{\Theta} \theta p(\theta \mid \mathbf{X}^{(n)}) d\theta$$

Furthermore we define **MLE**  $\hat{\theta}_n$  of the parameter  $\theta$  as the solution of the equation

$$\begin{aligned} \max \left\{ L\left(\left(\hat{\theta}_n^{(1)}, \hat{\theta}_n^{(2)}\right) +, \theta_1, \mathbf{X}^{(n)}\right), L\left(\left(\hat{\theta}_n^{(1)}, \hat{\theta}_n^{(2)}\right) -, \theta_1, \mathbf{X}^{(n)}\right) \right\} \\ = \sup_{\theta \in \Theta} L\left(\theta, \theta_1, \mathbf{X}^{(n)}\right). \end{aligned}$$

Also we define the **MDE**  $\theta_n^*$  as a solution of the equation

$$\left\| \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i(\cdot) - \Lambda(\theta_n^*, \cdot) \right\| = \inf_{\vartheta \in \Theta} \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i(\cdot) - \Lambda(\vartheta, \cdot) \right\|.$$

# Theorem

Let  $\xi_1$  a gaussian random variable,  $\xi_1 \sim \mathcal{N}(0, I(\theta)^{-1})$  and  $\xi_2$  defined by

$$\xi_2 = \int_{-\infty}^{\infty} v Z_{\theta}(v) dv \left( \int_{-\infty}^{\infty} Z_{\theta}(v) dv \right)^{-1}$$

and independent of  $\xi_1$ .  $\mathbf{K} \subset \Theta_1 \times \Theta_2$

## Theorem

*Let the condition A be satisfied. Then*

$$\lim_{\delta \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} \inf_{\bar{\theta}_n} \sup_{|\theta - \theta_0| < \delta} E_{\theta} \ell \left( \sqrt{n}(\bar{\theta}_n^{(1)} - \theta^{(1)}), n(\bar{\theta}_n^{(2)} - \theta^{(2)}) \right) \geq \\ \geq E \ell(\xi_1, \xi_2) = I(\theta_0)^{-1} + E(\xi_2)^2. \quad (3)$$

*and the inf is taken over all possible estimators  $\bar{\theta}_n$  of the parameter  $\theta = (\theta^{(1)}, \theta^{(2)})$ .*

# Results

Using inequality (3), we give the following definition

## Definition 1

Let the conditions **A** be satisfied, we say that an estimator  $\bar{\theta}_n$  is **asymptotically efficient** if

$$\begin{aligned} \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \sup_{\bar{\theta}_n \mid |\theta - \theta_0| < \delta} E_{\theta} \ell \left( \sqrt{n} (\bar{\theta}_n^{(1)} - \theta^{(1)}), n (\bar{\theta}_n^{(2)} - \theta^{(2)}) \right) &= \\ &= I(\theta_0)^{-1} + E(\xi_2)^2. \end{aligned}$$

## Theorem

Let the condition **A** be satisfied, then the **BE**  $\tilde{\theta}_n$  verify uniformly in  $\theta \in \mathbf{K}$  the relations

$$P_{\theta} - \lim_{n \rightarrow \infty} \tilde{\theta}_n = \theta,$$

$$\mathcal{L}_{\theta} \left( \sqrt{n} (\tilde{\theta}_n^{(1)} - \theta^{(1)}), n (\tilde{\theta}_n^{(2)} - \theta^{(2)}) \right) \implies \mathcal{L}(\xi_1, \xi_2),$$

$$\lim_{n \rightarrow \infty} E_{\theta} \left\| \left( \sqrt{n} (\tilde{\theta}_n^{(1)} - \theta^{(1)}), n (\tilde{\theta}_n^{(2)} - \theta^{(2)}) \right) \right\|^p = E \left( |\xi_1|^2 + |\xi_2|^2 \right)^{\frac{p}{2}}$$

for all  $p > 0$  where  $\|\cdot\|$  is the euclidian norm.

# Main result

Let  $\xi_1$  gaussian random variable,  $\xi_1 \sim \mathcal{N}(0, I(\theta)^{-1})$  and let  $\xi_3$  defined by the equation

$$\max \{ Z_\theta(\xi_{3+}), Z_\theta(\xi_{3-}) \} = \sup_{v \in R} Z_\theta(v)$$

and independent of  $\xi_1$ . Let  $\mathbf{K} \subset \Theta_1 \times \Theta_2$  a compact set.

## Theorem

*Let the condition A be satisfied, then the MLE  $\hat{\theta}_n$  verify uniformly in  $\theta \in \mathbf{K}$  the relations*

$$P_\theta - \lim_{n \rightarrow \infty} \hat{\theta}_n = \theta,$$

$$\mathcal{L}_\theta \left( \sqrt{n}(\hat{\theta}_n^{(1)} - \theta^{(1)}), n(\hat{\theta}_n^{(2)} - \theta^{(2)}) \right) \implies \mathcal{L}(\xi_1, \xi_3),$$

$$\lim_{n \rightarrow \infty} E_\theta \left\| \left( \sqrt{n}(\hat{\theta}_n^{(1)} - \theta^{(1)}), n(\hat{\theta}_n^{(2)} - \theta^{(2)}) \right) \right\|^p = E(|\xi_1|^2 + |\xi_3|^2)^{\frac{p}{2}}$$

*for all  $p > 0$  where  $\|\cdot\|$  is the euclidian norm.*

## Theorem

Let the conditions A are satisfied, then the MDE  $\theta_n^*$  verify uniformly in  $\theta \in \mathbf{K}$  the relations

$$P_\theta - \lim_{n \rightarrow \infty} \theta_n^* = \theta,$$

$$\mathcal{L}_\theta \left( \sqrt{n}(\theta_n^{*(1)} - \theta^{(1)}), \sqrt{n}(\theta_n^{*(2)} - \theta^{(2)}) \right) \implies \mathcal{L}(\xi_4, \xi_5),$$

where  $(\xi_4, \xi_5)$  is a gaussian vector, with

$$\xi_4 = \int_0^T W(\Lambda(\theta_0, t)) \tilde{\Gamma}_1(\theta_0, t) dt, \quad \xi_5 = \int_0^T W(\Lambda(\theta_0, t)) \tilde{\Gamma}_2(\theta_0, t) dt$$

# Plan

## 1 Estimation in no regular case

- Preliminaries
- Main results

## 2 MDE on $L^p$ -distance

- General idea
- Preliminaries
- Main result
  - Consistency
  - Limit distribution

# General idea

We consider the problem of parameter estimation by the observations of the inhomogeneous Poisson processes.

We suppose that the intensity function of these processes is a smooth function of the unknown parameter and as a method of estimation we take the minimum distance approach.

We are interested by the behavior of estimators in non-hilbertian situation and we define the MDE with the help of the  $L^p$  metrics. We show that (under regularity conditions) the MDE is consistent and we describe its limit distribution.

# Preliminaries

We observe a sample  $X^{(n)} = (X_1, \dots, X_n)$  of  $n$  independent inhomogeneous Poisson processes, where  $X_j = (X_j(t), t \in [0, T])$  are trajectories of the Poisson process with same mean function

$$\Lambda(\vartheta, t) = E_{\vartheta} X_j(t), \quad \Lambda(\vartheta, t) = \int_0^t \lambda(\vartheta, s) ds, \quad \sup_{\vartheta \in \Theta} \lambda(\vartheta, T) < \infty.$$

Here  $\vartheta \in \Theta = (\alpha, \beta)$  is unknown parameter (the set  $\Theta$  is bounded) and  $\lambda(\vartheta, t)$  is the intensity function.

Let us introduce the empirical mean function

$$\hat{\Lambda}_n(t) = \frac{1}{n} \sum_{i=1}^n X_i(t), \quad 0 \leq t \leq T$$

and denote  $\|\cdot\|_p$ ,  $p \geq 2$ , the following  $L^p$ -norme :

$$\|f(\cdot)\|_p = \left( \int_0^T |f(t)|^p dt \right)^{\frac{1}{p}}.$$

It is easy to see that for any  $\theta \in \Theta$

$$\left\| \hat{\Lambda}_n(\cdot) - \Lambda(\theta, \cdot) \right\|_p < +\infty \text{ a.s.}$$



# Preliminaries

We define the *minimum distance estimator*  $\theta_n^*$  as a solution of the equation

$$\left\| \hat{\Lambda}_n(\cdot) - \Lambda(\theta_n^*, \cdot) \right\|_p = \inf_{\vartheta \in \Theta} \left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|_p.$$

If this equation has many solutions then we can take as  $\theta_n^*$  any one of them. We can write

$$\theta_n^* = \arg \inf_{\vartheta \in \Theta} \left\| \hat{\Lambda}_n(\cdot) - \Lambda(\vartheta, \cdot) \right\|_p.$$

We are interested in the properties of the MDE as  $n$  goes to  $+\infty$ .

## Conditions H.

- **h1.** *The function  $\Lambda(\vartheta, \cdot)$  is two times continuously differentiable with respect to  $\vartheta$  and its derivatives are bounded.*
- **h2.** *For any  $\delta > 0$  and any  $\theta_0 \in \Theta$  we have*

$$g(\delta, \theta_0) \equiv \inf_{|\vartheta - \theta_0| > \delta} \left\| \Lambda(\vartheta, \cdot) - \Lambda(\theta_0, \cdot) \right\|_p > 0.$$

- **h3.** *The condition*

$$\inf_{\vartheta \in \Theta} \left\| \dot{\Lambda}(\vartheta, \cdot) \right\|_p > 0$$

*holds.*

# Consistency

## Theorem

Under the condition **h2**, for any  $\delta > 0$  we have

$$P \{ |\theta_n^* - \theta_0| > \delta \} \xrightarrow{n \rightarrow +\infty} 0, \quad (4)$$

i.e. the MDE  $\vartheta_n^*$  is consistent.

### For The Proof.

Let us denote  $Z_n(\vartheta, t) = \sqrt{n}(\hat{\Lambda}_n(t) - \Lambda(\vartheta, t))$  and  $Z_n(\vartheta) = \|Z_n(\vartheta, \cdot)\|_p$ . We have for any  $\delta > 0$

$$P \{ |\theta_n^* - \theta_0| > \delta \} \leq P \left\{ Z_n^p(\theta_0) > \left( \frac{\sqrt{n}}{2} \right)^p (g(\delta, \theta_0))^p \right\} \leq \frac{2^p E_{\theta_0} Z_n^p(\theta_0)}{n^{\frac{p}{2}} (g(\delta, \theta_0))^p}. \quad (5)$$

Therefore, we have shown that

$$E_{\theta_0} Z_n^p(\theta_0) = \int_0^T E_{\theta_0} |Z_n(\theta_0, t)|^p dt \leq C.$$

# Limit distribution

We introduce the following notations :

$$\vartheta_u = \theta_0 + \frac{u}{\sqrt{n}} \quad \text{for all } u \in U_n = (\sqrt{n}(\alpha - \vartheta_0), \sqrt{n}(\beta - \vartheta_0)).$$

Introduce a Wiener process  $W(\tau)$ ,  $0 \leq \tau \leq \Lambda(\theta_0, T)$  and the random process

$$H(u) = \left\| W(\Lambda(\theta_0, t)) - u \dot{\Lambda}(\theta_0, t) \right\|_p, \quad u \in R.$$

This allow us to define the random variable

$$\zeta = \arg \inf_u H(u)$$

and we suppose that  $\zeta$  is unique.

## Theorem

Let the conditions **H** are fulfilled, then the MDE converges in distribution,

$$\sqrt{n}(\theta_n^* - \vartheta) \implies \zeta.$$

**Proof.** To prove this theorem we apply the method by Ibragimov and Khasminskii (see [8]) and Kutoyants (see [11]) as follows. The estimator  $\theta_n^*$  is a continuous functional of the random function

$$H_n(u) = \left\| \sqrt{n}(\hat{\Lambda}_n(\cdot) - \Lambda(\theta_0 + \frac{u}{\sqrt{n}}, \cdot)) \right\|_p,$$

i.e.,

$$u_n^* = \sqrt{n}(\theta_n^* - \vartheta) = \arg \inf_{u \in U_n} H_n(u).$$

Therefore, if we show the weak convergence of the random functions  $H_n(\cdot)$  to the random function  $H(\cdot)$ , then this yields the convergence  $u_n^* \Rightarrow \zeta$ . Note that  $\sup_u H_n(u) \rightarrow \infty$  as  $n \rightarrow \infty$  and the application of this method is not entirely direct.

# Limit distribution

The proof is based on the following three lemmas

## Lemma

*Under conditions **H** the finite dimensional distributions of the process  $Z_n^p(u, t_1), Z_n^p(u, t_2), \dots, Z_n^p(u, t_k)$  converge to the finite dimensional distributions of the process  $Z^p(u, t_1), Z^p(u, t_2), \dots, Z^p(u, t_k)$  as  $n \rightarrow \infty$ .*

## Lemma

*Let the condition **h1** be fulfilled then the estimate*

$$E_{\theta_0} |Z_n(u, t_2) - Z_n(u, t_1)|^p \leq C |t_2 - t_1|,$$

*holds. Here  $C$  is some positive constant.*

## Lemma

*Under conditions **H** the finite dimensional distributions of the process  $H_n(u_1), H_n(u_2), \dots, H_n(u_k)$  converge to the finite dimensional distributions of the process  $H(u_1), H(u_2), \dots, H(u_k)$  as  $n \rightarrow \infty$ .*

# Limit distribution

## Proof.

From the Lemma.1 and Lemma.2 it follows (see [8], Theorem A. 22) that the integrals converge

$$\int_0^T Z_n(u, t)^p dt \implies \int_0^T Z(u, t)^p dt.$$

Hence for any  $u$  we have

$$H_n(u)^p \equiv \int_0^T Z_n(u, t)^p dt \implies \left\| W(\Lambda(\theta_0, t)) - u \dot{\Lambda}(\theta_0, t) \right\|_p^p \equiv H(u)^p. \quad (6)$$

If we consider the vector of random functions

$$\mathbf{H}_n(\mathbf{u}) = (H_n(u_1), H_n(u_2), \dots, H_n(u_k))$$

then the similar consideration provides us the convergence of the vectors too

$$\mathbf{H}_n(\mathbf{u}) \implies \mathbf{H}(\mathbf{u}) = (H(u_1), H(u_2), \dots, H(u_k)).$$

## Lemma

*Under condition **h1**, we have*

$$E_{\theta_0} |H_n^p(u_2) - H_n^p(u_1)|^2 \leq C |u_2 - u_1|^2,$$

*where  $C$  is a positive constant.*

## Lemma

*Under conditions **H** the following estimate holds :*

$$\sqrt{n} \left\| \Lambda\left(\vartheta_0 + \frac{u}{\sqrt{n}}, \cdot\right) - \Lambda(\theta_0, \cdot) \right\|_p \geq \kappa |u|, \quad (7)$$

*where  $\kappa = \min\left(\frac{g(\delta, \theta_0)}{\beta - \alpha}, \frac{\|\dot{\Lambda}(\theta_0, \cdot)\|_p}{2}\right) > 0$ .*

# Limit distribution

This estimate and the proof of the consistency (5) allow us to obtain the following relation (below  $\vartheta_u = \theta_0 + \frac{u}{\sqrt{n}}$ )

$$\begin{aligned} P \{ |\sqrt{n}(\vartheta_n^* - \vartheta_0)| > L \} &= P \{ |u_n^*| > L \} \\ &= P \left\{ \inf_{|u| < L} \|Z_n(u, \cdot)\| > \inf_{|u| \geq L} \|Z_n(u, \cdot)\| \right\} \\ &\leq P \left\{ \sqrt{n} \left\| \hat{\Lambda}_n(\cdot) - \Lambda(\theta_0, \cdot) \right\|_p \geq \frac{\kappa}{2} L \right\} \leq \frac{C}{L^p}. \end{aligned}$$

Hence for any  $\varepsilon > 0$  we can choose such  $L$  that  $P \{ |u_n^*| > L \} \leq \varepsilon$ . Further, from the Lemma.3 and Lemma.4 it follows that the random process  $H_n(u), |u| \leq L$  converges in distribution in the space  $(\mathcal{C}(-L, L), \mathcal{B})$  to the random process  $H(u), |u| \leq L$ . Therefore we have the convergence of the distributions of all continuous in uniform metrics functionals of these processes.



We can write

$$\begin{aligned} P\{\sqrt{n}(\theta_n^* - \theta_0) \leq x\} &= P\left\{\theta_n^* < \theta_0 + \frac{x}{\sqrt{n}}\right\} \\ &= P\left\{\inf_{u < x} \left\|Z_n\left(\theta_0 + \frac{u}{\sqrt{n}}, \cdot\right)\right\|_p < \inf_{u \geq x} \left\|Z_n\left(\theta_0 + \frac{u}{\sqrt{n}}, \cdot\right)\right\|_p\right\} \\ &\leq P\left\{\inf_{u < x} H_n(u) < \inf_{u \geq x} H_n(u), |u_n^*| < L\right\} + P\{|u_n^*| \geq L\} \end{aligned}$$

# Limit distribution






Note that







$$\begin{aligned} P \left\{ \inf_{u < x} H_n(u) < \inf_{u \geq x} H_n(u), |u_n^*| < L \right\} \\ &= P \left\{ \inf_{u < x} H_n(u) < \inf_{u \geq x} H_n(u), \inf_{|u| < L} H_n(u) < \inf_{|u| \geq L} H_n(u) \right\} \\ &= P \left\{ \inf_{|u| < L, u < x} H_n(u) < \inf_{|u| < L, u \geq x} H_n(u) \right\}. \end{aligned}$$






From the weak convergence of  $H_n(\cdot)$  it follows that

$$\begin{aligned} P \left\{ \inf_{|u| < L, u < x} H_n(u) < \inf_{|u| < L, u \geq x} H_n(u) \right\} \\ \longrightarrow P \left\{ \inf_{|u| < L, u < x} H(u) < \inf_{|u| < L, u \geq x} H(u) \right\} \\ = P(\zeta < x, |\zeta| < L). \end{aligned}$$

Now the convergence in distribution follows from the presented above estimates.

-  Aubry, C. and Dabye, A.S., Asymptotic normality of the minimum distance estimators for a Poisson process with a discontinuous intensity function, *J. Statist. Planning and Inference*, **99**, pp. 3–23, 2001.
-  Ba, D.B., *Estimation Paramétrique par la Méthode de la Distance Minimale pour les Processus de Poisson d'Intensité Discontinue*, Thèse PhD de l'Université Gaston Berger, Saint-Louis, 2008.
-  Ba D. B. and Dabye A. S, On Regular properties of the MDE for non smooth model of Poisson process, *Indian Journal Mathematics*, **55**, I, pp. 145–162, 2009.
-  Cox, D.R., and Lewis, P.A.W., *The Statistical Analysis of Series of Events*, Methuen, London, 1966.
-  Dabye, A.S., *Estimation Paramétrique pour un Processus de Poisson d'Intensité Discontinue*, Thèse PhD de l'Université du Maine, Le Mans, 1999.

-  Dabye, A.S., Propriétés de l'EDM pour un processus de Poisson d'intensité discontinue, *C. R. Acad. Sci. Paris, Ser. I*, **346**, pp. 431-436, 2006.
-  Daley, D.J., and Vere-Jones, D., *An Introduction to the Theory of Point Processes*, Springer, New York, 1988.
-  Ibragimov I.A. and Khasminskii R.Z. *Statistical Estimation. Asymptotic Theory*, Springer, New York, 1981.
-  Karr, A.F., *Point Processes and Their Statistical Inference*, (2-nd ed.), Marcel Decker, New York, 1991.
-  Kutoyants Yu. A., Parameter estimation of intensity of inhomogeneous Poisson processes, *Problems of Control and Information Theory*, 8, 137-149, 1979.
-  Kutoyants, Yu.A., *Statistical Inference for Spatial Poisson Processes*, Lecture Notes in Statistics 134, Springer, N.Y. 1998.

-  Kutoyants, Yu.A., and Liese F., On minimum distance estimation for spatial Poisson processes, *Ann. Academiae Scient. Fennicae, ser.A, I*, **17**, pp. 65–71, 1992.
-  Lewis, P.A.W., (Ed.) *Stochastic Point Processes*, Wiley, N.Y., 1972.
-  Millar, W.P. The minimax principle in asymptotic statistical theory. *Lecture Notes in Mathematics*, vol. 976, 76–262, 1983.
-  Millar, W.P., A general approach to the optimality of MDE, *Trans. Amer. Math. Soc.*, **286**, pp. 377–418, 1984.
-  Snyder, D.R. and Miller, M.I., *Random Point Processes in Time and Space*, Springer, New-York, 1991.



THANKS FOR YOUR ATTENTION!!!