On Minimum L^p-Distance Estimation for Inhomogeneous Poisson Processes

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Plan

Preliminaries

These previous works are devoted to the study of **BE** and **MLE** in the non regular case.
For this we consider the model of n independent observation of an inhomogeneous Poisson process $X^{(n)} = (X_1, \ldots, X_n)$ where $X_j = \{X_j(t),\, 0 \leq t \in \tau\}$,j= $1, \ldots, n$ are Poisson processes with

$$S(\theta, t) = \theta^{(1)} g(\theta^{(2)} t) + \lambda, \quad 0 \leq t \leq T.$$

We suppose that λ ($\ll$ dark current $\gg$) and the function $g(\cdot)$ are positive and known. The function $g(\cdot)$ is continuously differentiable on over the interval $[0, T]$, except at the point $\tau^* \in (0, T)$. At this the function admit a jump of size r, that-is $g(\tau^*_+) - g(\tau^*_-) = r \neq 0$, where $g(\tau^*_-)$ and $g(\tau^*_+)$ are the right and left limit $g(\cdot)$ at the point τ^* respectively.

Preliminaries

We introduce below the BE, the MLE and the MDE and we are interesting by their asymptotic properties when n tend to infinity. Denote by $L\left(\theta, \theta_o, X^{(n)}\right)$ the likelihood ratio of this problem and by $P_\theta^{(n)}$ the measure induced in the space of observations by n realizations of the Poisson process of intensity measure $\Lambda\left(\theta, [0, T]\right)$. Thus we have

Preliminaries

$$L\left(\theta, \theta_0, X^{(n)}\right) \equiv \frac{\mathrm{d}P^{(n)}_{(\theta^{(1)},\theta^{(2)})}}{\mathrm{d}P^{(n)}_{(\theta_0^{(1)},\theta_0^{(2)})}} =$$

$$= \exp\left\{\sum_{j=1}^{n}\int_0^T \ln\left(\frac{\theta^{(1)}\, g(\theta^{(2)}\, t) + \lambda}{\theta_0^{(1)}\, g(\theta_0^{(2)}\, t) + \lambda}\right)\, \mathrm{d}X_j(t) - \right.$$

$$\left. -n\int_0^T \left(\theta^{(1)}\, g(\theta^{(2)}\, t) - \theta_0^{(1)}\, g(\theta_0^{(2)}\, t)\right)\, \mathrm{d}t\right\}.$$

This function is continuous w.r.t. $\theta^{(1)}$ and discontinuous w.r.t. $\theta^{(2)}$ with jumps at the points

$$\theta_{i_j} = \frac{\tau^*}{t_{i_j}}, \quad i = 1, \ldots, m_j,\ j = 1, \ldots, n.$$

Preliminaries

To describe the properties of all estimators that we will use in the sequel, we need the following notations. Let

$$Z_\theta(v) = \begin{cases} \exp\left\{ \ln \frac{\theta^{(1)}\, g(\tau_+^*)+\lambda}{\theta^{(1)}\, g(\tau_-^*)+\lambda}\, p^-(v) - \frac{\tau^*}{\left(\theta^{(2)}\right)^2}\, \theta^{(1)}\, r\, v \right\}, & v \geq 0 \\ \exp\left\{ \ln \frac{\theta^{(1)}\, g(\tau_-^*)+\lambda}{\theta^{(1)}\, g(\tau_+^*)+\lambda}\, p^+(-v) - \frac{\tau^*}{\left(\theta^{(2)}\right)^2}\, \theta^{(1)}\, r\, v \right\}, & v < 0 \end{cases} \tag{1}$$

where $r = g(\tau_+^*) - g(\tau_-^*)$ and p^+ and p^- are pairwise independent Poisson processes on R_+ of constant intensities

$$Ep^-(v) = \frac{\tau^*}{\left(\theta^{(2)}\right)^2}\, \left(\theta^{(1)}\, g(\tau_-^*) + \lambda\right) v$$

and

$$Ep^+(v) = \frac{\tau^*}{\left(\theta^{(2)}\right)^2}\, \left(\theta^{(1)}\, g(\tau_+^*) + \lambda\right) v.$$

respectively.

Preliminaries

We note also the Fisher information w.r.t. the second parameter

$$I(\theta) = \int_0^T \frac{g^2\left(\theta^{(2)}\, t\right)}{\theta^{(1)}\, g\left(\theta^{(2)}\, t\right) + \lambda}\, \mathrm{d}t. \tag{2}$$

We note $\theta_u = \theta^{(1)} + \frac{u}{\sqrt{n}}$ and $\theta_v = \theta^{(2)} + \frac{v}{n}$.

Conditions A.

a) *the function $g(\cdot)$ and the constant λ are positives and known ;*

b) *the function $g(\cdot)$ est continuously differentiable on $[0, \tau^*) \cup (\tau^*, T]$ and has a jump at the point $\tau^* \in (0, T)$ with $g(\tau_+^*) - g(\tau_-^*) = r > 0$ and $g(\tau_+^*)\, g(\tau_-^*) \neq 0$.*

Definition

The **BE** $\tilde{\theta}_n = \left(\tilde{\theta}_n^{(1)}, \tilde{\theta}_n^{(2)}\right)$ of $\theta_0 = \left(\theta_0^{(1)}, \theta_0^{(2)}\right)$ for a given density p and for square loss function $\ell(\cdot,\cdot) = \ell(x,y) = x^2 + y^2$,is defined by

$$\tilde{\theta}_n = E\big(\theta \mid X^{(n)}\big) = \int_\Theta \theta\, p\big(\theta \mid X^{(n)}\big)\, \mathrm{d}\theta$$

Furthermore we define **MLE** $\hat{\theta}_n$ of the parameter θ as the solution of the equation

$$\max\left\{L\left((\hat{\theta}_n^{(1)}, \hat{\theta}_n^{(2)}+), \theta_1, X^{(n)}\right), L\left((\hat{\theta}_n^{(1)}, \hat{\theta}_n^{(2)}-), \theta_1, X^{(n)}\right)\right\}$$

$$= \sup_{\theta\in\Theta} L\left(\theta, \theta_1, X^{(n)}\right).$$

Also we define the **MDE** $\theta_n^\star$ as a solution of the equation

$$\left\|\frac{1}{n}\sum_{i=1}^{n} X_i(\cdot) - \Lambda(\theta_n^\star, \cdot)\right\| = \inf_{\vartheta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n} X_i(\cdot) - \Lambda(\vartheta, \cdot)\right\|.$$

Theorem

Let ξ_1 a gaussian random variable, $\xi_1 \sim \mathcal{N}\left(0, I(\theta)^{-1}\right)$ and ξ_2 defined by

$$\xi_2 = \int_{-\infty}^{\infty} v\, Z_\theta(v)\,\mathrm{d}v \left(\int_{-\infty}^{\infty} Z_\theta(v)\,\mathrm{d}v\right)^{-1}$$

and independent of ξ_1. $\mathbf{K} \subset \Theta_1 \times \Theta_2$

Theorem

Let the condition A be satisfied. Then

$$\lim_{\delta\to 0} \varliminf_{n\to\infty} \inf_{\bar\theta_n} \sup_{|\theta-\theta_0|<\delta} E_\theta\, \ell\left(\sqrt{n}\,(\bar\theta_n^{(1)} - \theta^{(1)}), n\,(\bar\theta_n^{(2)} - \theta^{(2)})\right) \geq$$

$$\geq E\ell(\xi_1, \xi_2) = I(\theta_0)^{-1} + E(\xi_2)^2. \quad (3)$$

and the inf *is taken over all possible estimators* $\bar\theta_n$ *of the parameter* $\theta = (\theta^{(1)}, \theta^{(2)})$.

Results

Using inequality (3),we give the following definition

Definition 1

Let the conditions **A** are satisfied,we say that an estimator is $\bar{\theta}_n$ is **asymptotically efficient** if

$$\lim_{\delta\to 0}\varliminf_{n\to\infty}\inf_{\bar{\theta}_n}\sup_{|\theta-\theta_0|<\delta} E_\theta\,\ell\left(\sqrt{n}\,(\bar{\theta}_n^{(1)}-\theta^{(1)}),\,n\,(\bar{\theta}_n^{(2)}-\theta^{(2)})\right) =$$

$$= I(\theta_0)^{-1} + E(\xi_2)^2.$$

Theorem

Let the condition A be satisfied,then the **BE** $\tilde{\theta}_n$ *verify uniformly in* $\theta\in\mathbf{K}$ *the relations*

$$P_\theta - \lim_{n\to\infty}\tilde{\theta}_n = \theta,$$

$$\mathcal{L}_\theta\left(\sqrt{n}\,(\tilde{\theta}_n^{(1)}-\theta^{(1)}),\,n\,(\tilde{\theta}_n^{(2)}-\theta^{(2)})\right) \Longrightarrow \mathcal{L}(\xi_1,\xi_2),$$

$$\lim_{n\to\infty} E_\theta\left\|\left(\sqrt{n}\,(\tilde{\theta}_n^{(1)}-\theta^{(1)}),\,n\,(\tilde{\theta}_n^{(2)}-\theta^{(2)})\right)\right\|^p = E\left(|\xi_1|^2+|\xi_2|^2\right)^{\frac{p}{2}}$$

for all $p>0$ *where* $\|\cdot\|$ *is the euclidian norm.*

Main result

Let ξ_1 gaussian random variable, $\xi_1 \sim \mathcal{N}\left(0, I(\theta)^{-1}\right)$ and let ξ_3 defined by the equation

$$\max\left\{Z_\theta\left(\xi_{3+}\right), Z_\theta\left(\xi_{3-}\right)\right\} = \sup_{v \in R} Z_\theta(v)$$

and independent of ξ_1. Let $\mathbf{K} \subset \Theta_1 \times \Theta_2$ a compact set.

Theorem

Let the condition A be satisfied,then the **MLE** $\hat{\theta}_n$ *verify uniformly in* $\theta \in \mathbf{K}$ *the relations*

$$P_\theta - \lim_{n \to \infty} \hat{\theta}_n = \theta,$$

$$\mathcal{L}_\theta\left(\sqrt{n}\,(\hat{\theta}_n^{(1)} - \theta^{(1)}), n\,(\hat{\theta}_n^{(2)} - \theta^{(2)})\right) \Longrightarrow \mathcal{L}(\xi_1, \xi_3),$$

$$\lim_{n \to \infty} E_\theta \left\|\left(\sqrt{n}\,(\hat{\theta}_n^{(1)} - \theta^{(1)}), n\,(\hat{\theta}_n^{(2)} - \theta^{(2)})\right)\right\|^p = E\left(|\xi_1|^2 + |\xi_3|^2\right)^{\frac{p}{2}}$$

for all $p > 0$ *where* $\|\cdot\|$ *is the euclidian norm.*

Main result

Theorem

Let the conditions A are satisfied,then the **MDE** $\theta_n^\star$ *verify uniformly in* $\theta \in \mathbf{K}$ *the relations*

$$P_\theta - \lim_{n\to\infty} \theta_n^\star = \theta,$$

$$\mathcal{L}_\theta \left(\sqrt{n}\,(\theta_n^{\star(1)} - \theta^{(1)}), \sqrt{n}\,(\theta_n^{\star(2)} - \theta^{(2)}) \right) \Longrightarrow \mathcal{L}(\xi_4, \xi_5),$$

where (ξ_4, ξ_5) *is a gaussian vector, with*

$$\xi_4 = \int_0^T W(\Lambda(\theta_0, t))\tilde{\Gamma}_1(\theta_0, t)\, dt, \quad \xi_5 = \int_0^T W(\Lambda(\theta_0, t))\tilde{\Gamma}_2(\theta_0, t)\, dt$$

Plan

General idea

We consider the problem of parameter estimation by the observations of the inhomogeneous Poisson processes.
We suppose that the intensity function of these processes is a smooth function of the unknown parameter and as a method of estimation we take the minimum distance approach.
We are interested by the behavior of estimators in non-hilbertian situation and we define the MDE with the help of the L^p metrics. We show that (under regularity conditions) the MDE is consistent and we describe its limit distribution.

Preliminaries

We observe a sample $X^{(n)} = (X_1, \ldots, X_n)$ of n independent inhomogeneous Poisson processes, where $X_j = \left(X_j\left(t\right), t \in [0,\ T]\right)$ are trajectories of the Poisson process with same mean function

$$\Lambda\left(\vartheta, t\right) = E_\vartheta X_j\left(t\right), \qquad \Lambda\left(\vartheta, t\right) = \int_0^t \lambda\left(\vartheta, s\right) ds, \qquad \sup_{\vartheta\in\Theta} \lambda\left(\vartheta, T\right) < \infty.$$

Here $\vartheta \in \Theta = (\alpha, \beta)$ is unknown parameter (the set Θ is bounded) and $\lambda\left(\vartheta, t\right)$ is the intensity function.

Let us introduce the empirical mean function

$$\hat{\Lambda}_n(t) = \frac{1}{n}\sum_{i=1}^{n} X_i(t), \qquad 0 \le t \le T$$

and denote $\left\|\cdot\right\|_p$, $p \ge 2$, the following L^p-norme :

$$\left\|f(\cdot)\right\|_p = \left(\int_0^T \left|f(t)\right|^p dt\right)^{\frac{1}{p}}.$$

It is easy to see that for any $\theta \in \Theta$

$$\left\|\hat{\Lambda}_n(\cdot) - \Lambda(\theta, \cdot)\right\|_p < +\infty \text{ a.s.}$$

Preliminaries

We define the *minimum distance estimator* $\theta_n^\star$ as a solution of the equation

$$\left\|\hat{\Lambda}_n(\cdot)-\Lambda(\theta_n^\star,\cdot)\right\|_p=\inf_{\vartheta\in\Theta}\left\|\hat{\Lambda}_n(\cdot)-\Lambda(\vartheta,\cdot)\right\|_p.$$

If this equation has many solutions then we can take as $\theta_n^\star$ anyone of them. We can write

$$\theta_n^\star=\arg\inf_{\vartheta\in\Theta}\left\|\hat{\Lambda}_n(\cdot)-\Lambda(\vartheta,\cdot)\right\|_p.$$

We are interested in the properties of the MDE as n goes to $+\infty$.

Conditions H.

- **h1.** *The function $\Lambda(\vartheta,\cdot)$ is two times continuously differentiable with respect to ϑ and its derivatives are bounded.*
- **h2.** *For any $\delta>0$ and any $\theta_0\in\Theta$ we have*

$$g(\delta,\theta_0)\equiv\inf_{|\vartheta-\theta_0|>\delta}\left\|\Lambda(\vartheta,\cdot)-\Lambda(\theta_0,\cdot)\right\|_p>0.$$

- **h3.** *The condition*

$$\inf_{\vartheta\in\Theta}\left\|\dot{\Lambda}(\vartheta,\cdot)\right\|_p>0$$

holds.

Consistency

Theorem

Under the condition **h2**, *for any* $\delta > 0$ *we have*

$$P\left\{|\theta_n^\star - \theta_0| > \delta\right\} \xrightarrow[n\to+\infty]{} 0, \tag{4}$$

i.e. the MDE $\vartheta_n^\star$ *is consistent.*

For The Proof.

Let us denote $Z_n(\vartheta, t) = \sqrt{n}\left(\hat{\Lambda}_n(t) - \Lambda(\vartheta, t)\right)$ and $Z_n(\vartheta) = \|Z_n(\vartheta, \cdot)\|_p$.
We have for any $\delta > 0$

$$\begin{aligned} &P\left\{|\theta_n^\star - \theta_0| > \delta\right\} \le \\ &P\left\{Z_n^p(\theta_0) > \left(\frac{\sqrt{n}}{2}\right)^p \left(g(\delta, \theta_0)\right)^p\right\} \le \frac{2^p E_{\theta_0} Z_n^p(\theta_0)}{n^{\frac{p}{2}} \left(g(\delta, \theta_0)\right)^p}. \end{aligned} \tag{5}$$

Therefore, we have shown that

$$E_{\theta_0} Z_n^p(\theta_0) = \int_0^T E_{\theta_0} \left|Z_n(\theta_0, t)\right|^p \, dt \le C.$$

Limit distribution

We introduce the following notations :

$$\vartheta_u = \theta_0 + \frac{u}{\sqrt{n}} \qquad \text{for all} \quad u \in U_n = \left(\sqrt{n}(\alpha - \vartheta_0), \sqrt{n}(\beta - \vartheta_0)\right).$$

Introduce a Wiener process $W(\tau), 0 \leq \tau \leq \Lambda(\theta_0, T)$ and the random process

$$H(u) = \left\| W(\Lambda(\theta_0, t)) - u\,\dot\Lambda(\theta_0, t) \right\|_p, \qquad u \in R.$$

This allow us to define the random variable

$$\zeta = \arg\inf_u H(u)$$

and we suppose that ζ is unique.

Limit distribution

Theorem

Let the conditions **H** *are fulfiled, then the MDE converges in distribution,*

$$\sqrt{n}\left(\theta_n^\star - \vartheta\right) \Longrightarrow \zeta.$$

Proof. To prove this theorem we apply the method by Ibragimov and Khasminskii (see [8])and Kutoyants (see [11]) as follows. The estimator $\theta_n^\star$ is a continuous functional of the random function

$$H_n(u) = \|\sqrt{n}(\hat{\Lambda}_n(\cdot) - \Lambda(\theta_0 + \frac{u}{\sqrt{n}}, \cdot))\|_p,$$

i.e.,

$$u_n^\star = \sqrt{n}\left(\theta_n^\star - \vartheta\right) = \arg \inf_{u \in U_n} H_n\left(u\right).$$

Therefore, if we show the weak convergence of the random functions $H_n\left(\cdot\right)$ to the random function $H\left(\cdot\right)$, then this yields the convergence $u_n^\star \Rightarrow \zeta$. Note that $\sup_u H_n\left(u\right) \to \infty$ as $n \to \infty$ and the application of this method is not entirely direct.

Limit distribution

The proof is based on the following three lemmas

Lemma

Under conditions **H** *the finite dimensional distributions of the process* $Z_n^p(u,t_1), Z_n^p(u,t_2), \ldots, Z_n^p(u,t_k)$ *converge to the finite dimensional distributions of the process* $Z^p(u,t_1), Z^p(u,t_2), \ldots, Z^p(u,t_k)$ *as* $n \to \infty$.

Lemma

Let the condition **h1** *be fulfilled then the estimate*

$$E_{\theta_0}|Z_n(u,t_2) - Z_n(u,t_1)|^p \leq C|t_2 - t_1|,$$

holds. Here C *is some positive constant.*

Lemma

Under conditions **H** *the finite dimensional distributions of the process* $H_n(u_1), H_n(u_2), \ldots, H_n(u_k)$ *converge to the finite dimensional distributions of the process* $H(u_1), H(u_2), \ldots, H(u_k)$ *as* $n \to \infty$.

Proof.
From the Lemma.1 and Lemma.2 it follows (see [8], Theorem A. 22) that the integrals converge

$$\int_0^T Z_n(u,t)^p\,dt \Longrightarrow \int_0^T Z(u,t)^p\,dt.$$

Hence for any u we have

$$H_n(u)^p \equiv \int_0^T Z_n(u,t)^p\,dt \Longrightarrow \left\| W\left(\Lambda(\theta_0,t)\right) - u\,\dot{\Lambda}(\theta_0,t)\right\|_p^p \equiv H(u)^p. \tag{6}$$

If we consider the vector of random functions

$$\mathbf{H}_n(\mathbf{u}) = \left(H_n(u_1), H_n(u_2), \ldots, H_n(u_k)\right)$$

then the similar consideration provides us the convergence of the vectors too

$$\mathbf{H}_n(\mathbf{u}) \Longrightarrow \mathbf{H}(\mathbf{u}) = \left(H(u_1), H(u_2), \ldots, H(u_k)\right).$$

Limit distribution

Lemma

Under condition **h1***, we have*

$$E_{\theta_0}|H_n^p(u_2) - H_n^p(u_1)|^2 \leq C|u_2 - u_1|^2,$$

where C is a positive constant.

Lemma

Under conditions **H** *the following estimate holds :*

$$\sqrt{n}\left\|\Lambda(\vartheta_0 + \frac{u}{\sqrt{n}}, \cdot) - \Lambda(\theta_0, \cdot)\right\|_p \geq \kappa|u|, \tag{7}$$

where $\kappa = \min\left(\frac{g(\delta, \theta_0)}{\beta - \alpha}, \frac{\left\|\dot{\Lambda}(\theta_0, \cdot)\right\|_p}{2}\right) > 0.$

Limit distribution

This estimate and the proof of the consistency (5) allow us to obtain the following relation (below $\vartheta_u = \theta_0 + \frac{u}{\sqrt{n}}$)

$$
\begin{aligned}
P\left\{\left|\sqrt{n}\left(\vartheta_n^\star - \vartheta_0\right)\right| > L\right\} &= P\left\{\left|u_n^\star\right| > L\right\} \\
&= P\left\{\inf_{|u|<L} \left\|Z_n\left(u,\cdot\right)\right\| > \inf_{|u|\geq L} \left\|Z_n\left(u,\cdot\right)\right\|\right\} \\
&\leq P\left\{\sqrt{n}\left\|\hat{\Lambda}_n(\cdot) - \Lambda(\theta_0,\cdot)\right\|_p \geq \frac{\kappa}{2}\,L\right\} \leq \frac{C}{L^p}.
\end{aligned}
$$

Hence for any $\varepsilon > 0$ we can choose such L that $P\left\{\left|u_n^\star\right| > L\right\} \leq \varepsilon$. Further, from the Lemma.3 and Lemma.4 it follows that the random process $H_n\left(u\right), \left|u\right| \leq L$ converges in distribution in the space $\left(\mathcal{C}\left(-L, L\right), \mathcal{B}\right)$ to the random process $H\left(u\right), \left|u\right| \leq L$. Therefore we have the convergence of the distributions of all continuous in uniform metrics functionals of these processes.

Limit distribution

We can write

$$
\begin{aligned}
P\left\{\sqrt{n}(\theta_n^\star - \theta_0) \le x\right\} &= P\left\{\theta_n^\star < \theta_0 + \frac{x}{\sqrt{n}}\right\} \\
&= P\left\{\inf_{u<x}\left\|Z_n(\theta_0 + \frac{u}{\sqrt{n}}, \cdot)\right\|_p < \inf_{u\ge x}\left\|Z_n(\theta_0 + \frac{u}{\sqrt{n}}, \cdot)\right\|_p\right\} \\
&\le P\left\{\inf_{u<x} H_n(u) < \inf_{u\ge x} H_n(u), |u_n^\star| < L\right\} + P\left\{|u_n^\star| \ge L\right\}
\end{aligned}
$$

Limit distribution

Note that

$$
\begin{aligned}
&P\left\{\inf_{u<x} H_n(u) < \inf_{u\geq x} H_n(u), |u_n^\star| < L\right\} \\
&\qquad = P\left\{\inf_{u<x} H_n(u) < \inf_{u\geq x} H_n(u), \inf_{|u|<L} H_n(u) < \inf_{|u|\geq L} H_n(u)\right\} \\
&\qquad = P\left\{\inf_{|u|<L, u<x} H_n(u) < \inf_{|u|<L, u\geq x} H_n(u)\right\}.
\end{aligned}
$$

From the weak convergence of $H_n(\cdot)$ it follows that

$$
\begin{aligned}
&P\left\{\inf_{|u|<L, u<x} H_n(u) < \inf_{|u|<L, u\geq x} H_n(u)\right\} \\
&\qquad \longrightarrow P\left\{\inf_{|u|<L, u<x} H(u) < \inf_{|u|<L, u\geq x} H(u)\right\} \\
&\qquad = P\left(\zeta < x, |\zeta| < L\right).
\end{aligned}
$$

Now the convergence in distribution follows from the presented above estimates.

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THANKS FOR YOUR ATTENTION ! ! !