On Multi-step MLE-processes for Some Stochastic Models.

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Motivation

Problem: *Construction of BSDE.* We are given a stochastic differential equation (called *forward*)

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + a(t, X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \le t \le T,$$

and two functions $f\,(t, x, y, z)$ and $\Phi\,(x)$. We have to construct a couple of processes (Y_t, Z_t) such that the solution of the equation

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \;\; Y_0,\ 0 \le t \le T,$$

(called *backward*) has the *final value* $Y_T = \Phi\,(X_T)$.

For the existence and uniqueness of the solution see Pardoux and Peng (1990). The *Markovian case* considered here was discussed by Pardoux and Peng (1992).

Solution: Suppose that $u(t,x)$ is solution of the equation

$$\frac{\partial u}{\partial t} + b(t,x)\frac{\partial u}{\partial x} + \frac{1}{2}a(t,x)^2\frac{\partial^2 u}{\partial x^2} = -f\left(t,x,u,a(t,x)\frac{\partial u}{\partial x}\right),$$

with the final condition $u(T,x) = \Phi(x)$. Then if we put $Y_t = u(t,X_t)$, $Z_t = a(t,X_t)\,u'_x(t,X_t)$. Then by Itô's formula

$$\begin{aligned}\mathrm{d}Y_t &= \left[\frac{\partial u}{\partial t}(t,X_t) + b(t,X_t)\frac{\partial u}{\partial x}(t,X_t) + \frac{1}{2}a(t,x)^2\frac{\partial^2 u}{\partial x^2}(t,X_t)\right]\mathrm{d}t \\ &\qquad + a(t,X_t)\frac{\partial u}{\partial x}(t,X_t)\,\mathrm{d}W_t \\ &= -f(t,X_t,Y_t,Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0 = u(0,X_0).\end{aligned}$$

The final value $Y_T = u(T,X_T) = \Phi(X_T)$.

Statistical problems. We consider this problem in the situations, where the forward equation contains some unknown parameter ϑ:

$$\mathrm{d}X_t = b(\vartheta, t, X_t)\ \mathrm{d}t + a(\vartheta, t, X_t)\ \mathrm{d}W_t, \quad X_0 = x_0,\ 0 \leq t \leq T.$$

Then $u = u\,(t, x, \vartheta)$ and the proposed approximations $\hat Y_t, \hat Z_t$ of the couple Y_t, Z_t are given by the relations

$$\hat Y_t = u(t, X_t, \vartheta_t^*), \qquad\qquad \hat Z_t = u_x'(t, X_t, \vartheta_t^*)\, a(\vartheta_t^*, t, X_t).$$

Here ϑ_t^* is some *good* estimator-process of ϑ with the *small error* of estimation $\mathbf{E}_\vartheta \left(\hat\vartheta_t - \vartheta\right)^2$. This provides us the small errors $\mathbf{E}_\vartheta \left(\hat Y_t - Y_t\right)^2$ and $\mathbf{E}_\vartheta \left(\hat Z_t - Z_t\right)^2$.

Main problem: *how to find a good estimator-process* $\vartheta_t^*, 0 < t \leq T$?

Good means :

- *Depends on observations* $X^t = (X_s, 0 \leq s \leq t)$ *and therefore is stochastic process* $\vartheta^\star = \vartheta_t^\star, 0 < t \leq T$.
- *Easy to calculate for all* $t \in (0, T]$.
- *Asymptpotically efficient for all* $t \in (0, T]$.

The MLE $\hat{\vartheta}_t$ defined by

$$V\left(\hat{\vartheta}_t, X^t\right) = \sup_{\vartheta \in \Theta} V\left(\vartheta, X^t\right), \qquad t \in (0, T]$$

can not be used as *Good* because in non linear case to solve this equation for all $t \in (0, T]$ is computationally very difficult problem.

Our goal is to construct the good estimator-processes in different statements. Kamatani and Uchida [6] recently considered the problem of parameter estimation by the discrete time observations of diffusion process and showed that Mullti-step Newton-Raphson procedure can provide asymptotically efficiennt estimation even if the preliminary estimators have bad rate of convergence.

The general construction is the following. We fix a learning interval $[0, \tau]$ and obtain a preliminary estimator $\bar\vartheta_\tau$. Then we use this estimator to construct one-step and two-step MLE-processes.

Example. In ergodic case we take a learning interval $\left[0,T^\delta\right], \delta\in(\frac{1}{2},1)$ and any consistent estimator $\bar\vartheta_{T^\delta}$ such that $T^{\frac{\delta}{2}}\left(\bar\vartheta_{T^\delta}-\vartheta\right)$ is bounded in probability. The *one-step MLE-process* $\vartheta_t^\star, t\in\left[T^\delta,T\right]$ is

$$\vartheta_t^\star=\bar\vartheta_{T^\delta}+T^{-1}\mathbb{I}\left(\bar\vartheta_{T^\delta}\right)^{-1}\int_{T^\delta}^t\frac{\dot S\left(\bar\vartheta_{T^\delta},X_s\right)}{\sigma\left(X_s\right)^2}\left[{\rm d}X_s-S\left(\bar\vartheta_{T^\delta},X_s\right){\rm d}s\right].$$

This estimator-process is easy to calculate, uniformely on $T^\delta\leq t\leq T$ consistent, asymptotically normal

$$\sqrt{t}\left(\vartheta_t^\star-\vartheta\right)\Longrightarrow\mathcal{N}\left(0,\mathbb{I}\left(\vartheta\right)^{-1}\right)$$

and asymptotically efficient. If the learning interval has $\delta\in(\frac{1}{3},\frac{1}{2}]$, then we construct two-step MLE-process.

Models

- Ergodic diffusion process ($T \to \infty$)

$$\mathrm{d}X_t = S\left(\vartheta, X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T.$$

- Hidden telegraph signal ($T \to \infty$)

$$\mathrm{d}X_t = Y\left(t\right)\mathrm{d}t + \,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T$$

$Y\left(\cdot\right)$ is two-state Markov process depending on $\vartheta = (\lambda, \mu)$.

- Diffusion process with *small noise* ($\varepsilon \to 0$)

$$\mathrm{d}X_t = S\left(\vartheta, t, X_t\right)\mathrm{d}t + \varepsilon\sigma\left(t, X_t\right)\,\mathrm{d}W_t, \quad x_0,\ 0 \le t \le T.$$

- Discrete time obs. $X^n = (X_{t_0}, X_{t_1}, \ldots X_{t_n})$, $t_i = i\frac{T}{n}, n \to \infty$

$$\mathrm{d}X_t = S\left(t, X_t\right)\mathrm{d}t + \sigma\left(\vartheta, t, X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T.$$

Note that such one-step estimators are used in the problem of the construction of asymptotically distribution free test by the observations of inhomogeneous Poisson process (M. Ben Abdeddaiem) and in the problem of parameter estimation by observations of Markov sequence

$$X_{j+1} = S\left(\vartheta, X_j\right) + \varepsilon_j, j = 1, \ldots, n \to \infty$$

(A. Motrunich).

The both results are presented on the Poster session of this workshop.

Ergodic diffusion

The observed diffusion process is

$$\mathrm{d}X_t = S\left(\vartheta, X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T$$

where $\vartheta \in \Theta \subset \mathcal{R}^d$. The process $X_t, t \geq 0$ has ergodic properties.

Condition $\mathcal{A}_0\left(\Theta\right)$. At particularly : we suppose that the functions $S\left(\vartheta, x\right)$ and $\sigma\left(x\right)^{\pm 1}$ have polynomial majorants and

$$\overline{\lim_{|x|\to\infty}} \sup_{\vartheta\in\Theta} \frac{\operatorname{sgn}\left(x\right)\ S\left(\vartheta, x\right)}{\sigma\left(x\right)^2} < 0.$$

The regularity conditions are: the function $S\left(\vartheta, x\right)$ has three continuous derivatives w.r.t. ϑ and these derivatives have polynomial majorants.

The identifiability condition is: for any $\nu > 0$

$$\inf_{\vartheta_0 \in \Theta} \inf_{|\vartheta - \vartheta_0| > \nu} \mathbf{E}_{\vartheta_0} \left(\frac{S(\vartheta, \xi_0) - S(\vartheta_0, \xi_0)}{\sigma(\xi_0)} \right)^2 > 0,$$

where the r.v. ξ_0 has the density function $f(\vartheta_0, x)$. The Fisher information matrix

$$\mathbb{I}(\vartheta) = \mathbf{E}_\vartheta \left(\frac{\dot{S}(\vartheta, \xi) \dot{S}(\vartheta, \xi)^*}{\sigma(\xi)^2} \right)$$

(here $\dot{S}(\vartheta, x)$ is derivative w.r.t. ϑ) is uniformly non degenerate (below $\lambda \in R^d$)

$$\inf_{\vartheta \in \Theta} \inf_{|\lambda| = 1} \lambda^* \mathbb{I}(\vartheta) \lambda > 0$$

We have to estimate ϑ by $X^t = (X_s, 0 \leq s \leq t)$ for $t \in (0, T]$ and to describe the properties of $\bar{\vartheta}(t) = \bar{\vartheta}(t, X^t), 0 < t \leq T$.

We consider two situations: with learning intervals $[\tau_* T, T]$ and $[\tau_\delta T, T]$, where τ_* is fixed and $\tau_\delta \to 0$. Suppose that we have an estimator $\bar\vartheta_{\tau_* T}$ constructed by the observations $X^{\tau_* T} = (X_t, 0 \leq t \leq \tau_* T)$, which is consistent and asymptotically normal

$$\sqrt{\tau_* T}\left(\bar\vartheta_{\tau_* T} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{D}\left(\vartheta\right)\right).$$

Then for $\tau \in [\tau_*, 1]$ we calculate the one-step MLE-process

$$\vartheta^\star_{\tau T} = \bar\vartheta_{\tau_* T} + \mathbb{I}\left(\bar\vartheta_{\tau_* T}\right)^{-1} \frac{\Delta_{\tau T}\left(\bar\vartheta_{\tau_* T}, X^{\tau T}_{\tau_* T}\right) + \Delta_{\tau_*}\left(\bar\vartheta_{\tau_* T}, X^{\tau_* T}\right)}{\sqrt{\tau\, T}},$$

where for $\tau \in [\tau_*, 1]$

$$\Delta_{\tau T}\left(\vartheta, X_{\tau_* T}^{\tau T}\right) = \frac{1}{\sqrt{\tau T}} \int_{\tau_* T}^{\tau T} \frac{\dot{S}\left(\vartheta, X_t\right)}{\sigma\left(X_t\right)^2} \left[\mathrm{d}X_t - S\left(\vartheta, X_t\right)\, \mathrm{d}t\right],$$

$$\Delta_{\tau_*}\left(\vartheta, X^{\tau_* T}\right) = \frac{A\left(\vartheta, X_{\tau_* T}\right)}{\sqrt{\tau T}} - \frac{1}{2\sqrt{\tau T}} \int_0^{\tau_* T} B_x'\left(\vartheta, X_t\right) \sigma\left(X_t\right)^2 \mathrm{d}t$$

$$- \int_0^{\tau} \frac{\dot{S}\left(\vartheta, X_t\right) S\left(\vartheta, X_t\right)}{\sqrt{\tau T} \sigma\left(X_t\right)^2} \mathrm{d}t,$$

$$B\left(\vartheta, x\right) = \frac{\dot{S}\left(\vartheta, x\right)}{\sigma\left(x\right)^2}, \qquad A\left(\vartheta, x\right) = \int_{x_0}^{x} B\left(\vartheta, z\right)\, \mathrm{d}z.$$

Note that under regularity conditions (K. 2004)

$$\sqrt{\tau T}\left(\vartheta_{\tau T}^{\star} - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}\left(\vartheta\right)^{-1}\right)$$

One-step MLE ($\delta \in (\frac{1}{2}, 1)$)

Introduce the learning interval $0 \leq t \leq T^{\delta}$, where $\delta \in (1/2, 1)$ and denote by $\bar{\vartheta}_{\tau_\delta}$ an estimator of parameter ϑ which is unifrormly in ϑ on compacts $\mathbb{K} \subset \Theta$ asymptotically normal

$$T^{\delta/2} \left(\bar{\vartheta}_{\tau_\delta} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{D}\left(\vartheta_0\right)\right),$$

where $\tau_\delta = T^{-1+\delta} \to 0$ and the matrix $\mathbb{D}\left(\vartheta_0\right)$ of limit covariance is bounded. Moreover we suppose that we have the convergence of moments too and we have

$$\sup_{\vartheta_0 \in \mathbb{K}} T^{p\delta/2} \mathbf{E}_{\vartheta_0} \left|\bar{\vartheta}_{\tau_\delta} - \vartheta_0\right|^p < C$$

where the constant $C > 0$ does not depend on T. As the preliminary estimator we can take the MLE, minimum distance estimator or any other.

The one-step MLE-process we construct as follows:

$$\vartheta_\tau^\star=\bar\vartheta_{\tau_\delta}+\frac{\mathbb{I}\left(\bar\vartheta_{\tau_\delta}\right)^{-1}}{\sqrt{\tau T}}\Delta_\tau\left(\bar\vartheta_{\tau_\delta},X_{T^\delta}^{\tau T}\right),\qquad \tau\in\left[\tau_\delta,1\right]$$

and

$$\Delta_\tau\left(\vartheta,X_{T^\delta}^{\tau T}\right)=\frac{1}{\sqrt{\tau T}}\int_{T^\delta}^{\tau T}\frac{\dot S\left(\vartheta,X_t\right)}{\sigma\left(X_t\right)^2}\left[{\rm d}X_t-S\left(\vartheta,X_t\right){\rm d}t\right].$$

Introduce the random process

$$\eta_{\tau,T}\left(\vartheta_0\right)=\tau\sqrt{T}\mathbb{I}\left(\vartheta_0\right)^{1/2}\left(\vartheta_\tau^\star-\vartheta_0\right),\qquad\quad \tau_*\leq\tau\leq 1$$

where $\tau_*\in(0,1)$ and measurable space $\left(\mathcal{C}\left[\tau_*,1\right],\mathfrak{B}\right)$ of continuous on $\left[\tau_*,1\right]$ functions. Here $\mathfrak{B}$ is the corresponding borelian σ-algebra. Denote by $W\left(\tau\right),0\leq\tau\leq1$ the d-dimensional standard Wiener process.

Theorem 1 *Suppose that the regularity conditions hold. Then the estimator-process* $\vartheta^\star_\tau, \tau_\delta < \tau \leq 1$ *has the following properties:*

1. *It is uniformly consistent: for any* $\nu > 0$

$$\lim_{T\to\infty} \sup_{\vartheta_0\in\mathbb{K}} \mathbf{P}_{\vartheta_0}\left(\sup_{\tau_\delta\leq\tau\leq 1} \left|\vartheta^\star_\tau - \vartheta_0\right| > \nu\right) = 0.$$

2. *For any* $\tau_* \in (0,1)$ *the random process* $\eta_{\tau,T}\left(\vartheta_0\right), \tau_* \leq \tau \leq 1$ *converges weakly in* $\left(\mathcal{C}\left[\tau_*, 1\right], \mathfrak{B}\right)$ *to the process* $W\left(\tau\right), \tau_* \leq \tau \leq 1$.

3. *It is asymptotically efficient.*

Two-step MLE. $\left(\delta \in (\frac{1}{3},\frac{1}{2}]\right)$ Let us take the *first* estimator $\tilde\vartheta_{\tau_\delta}$ constructed by the observations $X^{T^\delta} = \left(X_t, , 0 \leq t \leq T^\delta\right)$ with $\delta \in (\frac{1}{3},\frac{1}{2}]$. We suppose that this estimator is consistent, asymptotically normal and the moments converge too:

$$\tilde v_{\tau_\delta} = T^{\frac{\delta}{2}}\left(\tilde\vartheta_{\tau_\delta} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{D}\left(\vartheta_0\right)\right), \qquad \sup_{\vartheta_0\in\mathbb{K}} \mathbf{E}_{\vartheta_0}\left|\tilde v_{\tau_\delta}\right|^p \leq C,$$

for any $p > 0$. Here $\mathbb{D}\left(\vartheta_0\right)$ is some matrix and $C > 0$ does not depend on T. It can be the MLE, MDE, BE or the EMM.

Introduce the *second* preliminary estimator, which is estimator-process

$$\bar{\vartheta}_\tau = \tilde{\vartheta}_{\tau_\delta} + (\tau T)^{-1/2}\, \mathbb{I}\left(\tilde{\vartheta}_{\tau_\delta}\right)^{-1} \Delta_{\tau T}\left(\tilde{\vartheta}_{\tau_\delta}, X_{T^\delta}^{\tau T}\right), \qquad \tau \in [\tau_\delta, 1]$$

where $\tau_\delta = T^{-1+\delta}$. Note that $T^\gamma\left(\bar{\vartheta}_\tau - \vartheta_0\right) \rightarrow 0$ for $\gamma \in (1-\delta, 2\delta)$

$$\Delta_{\tau T}\left(\vartheta, X_{T^\delta}^{\tau T}\right) = \frac{1}{\sqrt{\tau T}} \int_{T^\delta}^{\tau T} \frac{\dot{S}\left(\vartheta, X_t\right)}{\sigma\left(X_t\right)^2} \left[\mathrm{d}X_t - S\left(\vartheta, X_t\right)\mathrm{d}t\right].$$

The *Two-step MLE-process* we define as follows

$$\vartheta_\tau^{\star\star} = \bar\vartheta_\tau + \frac{\mathbb{I}\left(\bar\vartheta_\tau\right)^{-1}}{\sqrt{\tau T}}\hat\Delta_{\tau T}\left(\tilde\vartheta_{\tau_\delta},\bar\vartheta_\tau,X_{T^\delta}^{\tau T}\right),\qquad \tau_\delta\leq\tau\leq 1,$$

where

$$\hat\Delta_{\tau T}\left(\vartheta_1,\vartheta_2,X_{T^\delta}^{\tau T}\right) = \frac{1}{\sqrt{\tau T}}\int_{T^\delta}^{\tau T}\frac{\dot S\left(\vartheta_1,X_t\right)}{\sigma\left(X_t\right)^2}\left[{\rm d}X_t - S\left(\vartheta_2,X_t\right){\rm d}t\right].$$

Note that $\hat\Delta_{\tau T}\left(\vartheta,\vartheta,X_{T^\delta}^{\tau T}\right) = \Delta_{\tau T}\left(\vartheta,X_{T^\delta}^{\tau T}\right)$.

Theorem 2 *Suppose that the conditions of regularity hold. Then the Two-step MLE-process* $\vartheta_\tau^{\star\star}, \tau_\delta \leq \tau \leq 1$ *is consistent, asymptotically normal*

$$\sqrt{T}\left(\vartheta_\tau^{\star\star} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \tau^{-1}\mathbb{I}\left(\vartheta_0\right)^{-1}\right),$$

and asymptotically efficient. The random process

$$\eta_{\tau,T}\left(\vartheta_0\right) = \tau\sqrt{T}\mathbb{I}\left(\vartheta_0\right)^{-1/2}\left(\vartheta_\tau^{\star\star} - \vartheta_0\right), \qquad \tau_* \leq \tau \leq 1$$

for any $\tau_* \in (0,1)$ *converges in distribution to the Wiener process* $W\left(\tau\right), \tau_* \leq \tau \leq 1$.

Example. Suppose that the observed process is

$$\mathrm{d}X_t = -\left(X_t - \vartheta\right)^3 \mathrm{d}t + \mathrm{d}W_t, \quad X_0, \quad 0 \leq t \leq$$

The MLE has no explicite expression and

$$\bar\vartheta_T = \frac{1}{T}\int_0^T X_t \,\mathrm{d}t \longrightarrow \vartheta, \qquad \sqrt{T}\left(\bar\vartheta_T - \vartheta\right) \Longrightarrow \mathcal{N}\,($$

If the learning interval is $\left[0, T^\delta\right]$ with $\delta \in (\frac{1}{2}, 1)$, then th

MLE-process is

$$\vartheta_t^\star = \bar\vartheta_{T^\delta} + \frac{3}{T\,\mathrm{I}}\int_{T^\delta}^t \left(X_s - \bar\vartheta_{T^\delta}\right)^2 \Big[\mathrm{d}X_s + \left(X_s - \bar\vartheta_T\right.$$

It is asymptotically normal: for all $\tau \in (0, 1]$

$$\sqrt{\tau T}\,(\vartheta^\star_{\tau T} - \vartheta) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}^{-1}\right)$$

If the learning interval is $\left[0,T^{\delta}\right]$ with $\delta\in(\frac{1}{3},\frac{1}{2}]$, then the second estimator is

$$\tilde{\vartheta}_t=\bar{\vartheta}_{T^\delta}+\frac{3}{T\,\mathrm{I}}\int_{T^\delta}^{t}\left(X_s-\bar{\vartheta}_{T^\delta}\right)^2\left[\mathrm{d}X_s+\left(X_s-\bar{\vartheta}_{T^\delta}\right)^3\mathrm{d}s\right]$$

and the third estimator is two-step MLE-process

$$\vartheta_t^{\star\star}=\tilde{\vartheta}_t+\frac{3}{T\,\mathrm{I}}\int_{T^\delta}^{t}\left(X_s-\bar{\vartheta}_{T^\delta}\right)^2\left[\mathrm{d}X_s+\left(X_s-\tilde{\vartheta}_t\right)^3\mathrm{d}s\right]$$

It is asymptotically normal: for all $\tau\in(0,1]$

$$\sqrt{\tau T}\left(\vartheta_{\tau T}^{\star\star}-\vartheta\right)\Longrightarrow\mathcal{N}\left(0,\mathrm{I}^{-1}\right)$$

and asymptotically efficient.

Hidden Telegraph Signal

(join work with R.Z. Khasminskii)

We observe a trajectory $X^T = (X_t, 0 \leq t \leq T)$ of stochastic process

$$\mathrm{d}X(t) = Y(t)\,\mathrm{d}t + \mathrm{d}W(t), \quad X_0,$$

where $Y(t), 0 \leq t \leq T$ is a two-state (y_1 and y_2) stationary Markov process with infinitesimal transition matrix

$$\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$

We suppose that the values $\lambda > 0$ and $\mu > 0$ are unknown and we have to estimate the two-dimensional parameter $\vartheta = (\lambda, \mu) \in \Theta$, where $\Theta = (c_0, c_1) \times (c_0, c_1)$ by the observations X^T. Here $c_0 < c_1$ are positive constants.

Introduce the estimators

$$\hat{\lambda}_T = \frac{\frac{X_T}{T} - y_1}{y_2 - y_1} \boldsymbol{\beta}_T; \qquad \hat{\mu}_T = \frac{y_2 - \frac{X_T}{T}}{y_2 - y_1} \boldsymbol{\beta}_T$$

where $\boldsymbol{\beta}_T = \alpha_T \, \mathbb{1}_{\{A_T\}} + (c_0 + c_1) \, \mathbb{1}_{\{A_T^c\}}$. Here A_T is the
equation

$$\zeta_T = \left(\frac{X_T}{T} \right)^2 + 2\eta_T \Phi\left(\alpha_T\right)$$

has solution α_T. The function $\Phi\left(x\right) = \frac{1}{x} - \frac{1}{x^2}\left(1 - e^{-x}\right)$

$$\zeta_T = \frac{1}{T} \sum_{i=0}^{T-1} \left[X_{i+1} - X_i\right]^2 - 1,$$

$$\eta_T = \left(\frac{X_T}{T} - y_1 \right) \left(y_2 - \frac{X_T}{T} \right).$$

Theorem 3 *we have for the estimators* $\bar\vartheta_T = \left(\bar\lambda_T, \bar\mu_T\right)$ *the relations*

$$\mathbf{E}\left[\sqrt{T}\left(\bar\lambda_T - \lambda\right)\right]^2 < C, \qquad \mathbf{E}\left[\sqrt{T}\left(\bar\mu_T - \mu\right)\right]^2 < C.$$

Our goal is to obtain asymptotically efficient estimators.

Let us introduce the learning time interval $\left[0, T^\delta\right]$, where $\frac{1}{2} < \delta < 1$, the corresponnding estimators of the method of moments

$$\bar\vartheta_{T^\delta} = \left(\bar\lambda_{T^\delta}, \bar\mu_{T^\delta}\right)$$

Having this estimator we introduce the one-step MLE-process as follows

$$\vartheta_T^\star = \hat{\vartheta}_{T^\delta} + T^{-1}\mathbb{I}_T(\hat{\vartheta}_{T^\delta})^{-1}\int_{T^\delta}^{T} \dot{m}(\hat{\vartheta}_{T^\delta}, s)\ \left[\mathrm{d}X_s - m(\hat{\vartheta}_{T^\delta}, s)\mathrm{d}s\right].$$

Here $m\left(t,\vartheta\right) = \mathbf{E}_\vartheta\left[Y\left(t\right)|\mathcal{F}_t^X\right]$:

$$m\left(t,\vartheta\right) = y_1\mathbf{P}_\vartheta\left(Y\left(t\right) = y_1|\mathcal{F}_t^X\right) + y_2\mathbf{P}_\vartheta\left(Y\left(t\right) = y_2|\mathcal{F}_t^X\right).$$

Let us denote

$$\pi\left(t,\vartheta\right) = \mathbf{P}_\vartheta\left(Y\left(t\right) = y_1|\mathcal{F}_t^X\right), \qquad \mathbf{P}_\vartheta\left(Y\left(t\right) = y_2|\mathcal{F}_t^X\right) = 1 - \pi\left(t,\vartheta\right).$$

Hence

$$m\left(t,\vartheta\right) = y_2 + \left(y_1 - y_2\right)\pi\left(t,\vartheta\right).$$

The random process $\pi\left(t,\vartheta\right), 0\leq t\leq T$ satisfies the following equation

$$\begin{aligned} \mathrm{d}\pi\left(t,\vartheta\right) = & \left[\mu-\left(\lambda+\mu\right)\pi\left(t,\vartheta\right)\right. \\ & \left.+\pi\left(t,\vartheta\right)\left(1-\pi\left(t,\vartheta\right)\right)\left(y_2-y_1\right)\left(y_2+\left(y_1-y_2\right)\pi\left(t,\vartheta\right)\right)\right]\mathrm{d}t \\ & +\pi\left(t,\vartheta\right)\left(1-\pi\left(t,\vartheta\right)\right)\left(y_1-y_2\right)\ \mathrm{d}X\left(t\right) \end{aligned}$$

the vector

$$\dot m(\vartheta,s)=\left(y_1-y_2\right)\frac{\partial\pi_\lambda\left(s,\vartheta\right)}{\partial\vartheta}=\left(y_1-y_2\right)\left(\frac{\partial\pi\left(t,\vartheta\right)}{\partial\lambda},\frac{\partial\pi\left(t,\vartheta\right)}{\partial\mu}\right).$$

The empirical Fisher information matrix is

$$\mathbb{I}_T(\vartheta)=\frac{\left(y_1-y_2\right)^2}{T}\int_{T^\delta}^{T}\frac{\partial\pi_\lambda\left(s,\vartheta\right)}{\partial\vartheta}\frac{\partial\pi_\lambda\left(s,\vartheta\right)^*}{\partial\vartheta}\mathrm{d}s.$$

Theorem 4 *The one-step MLE is asymptotically normal*

$$\sqrt{T}\left(\vartheta_T^\star - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{I}(\vartheta)^{-1}\right)$$

and asymptotically efficient.

Small noise asymptotics

The observed diffusion process is

$$\mathrm{d}X_t = S(\vartheta, t, X_t)\,\mathrm{d}t + \varepsilon\sigma(t, X_t)\,\mathrm{d}W_t, \quad X_0,\ 0 \le t \le T$$

where $\vartheta \in \Theta \subset \mathcal{R}^d$ is unknown parameter.

Recall that $\varepsilon^{-1}\left(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}(\vartheta, x^t)^{-1}\right)$, but to use it can be computantionally difficult problem. Here the matrix function

$$\mathbb{I}\left(\vartheta, x^t(\vartheta)\right) = \int_0^t \frac{\dot{S}(\vartheta, s, x_s(\vartheta))\,\dot{S}(\vartheta, s, x_s(\vartheta))^*}{\sigma(s, x_s(\vartheta))^2}\,\mathrm{d}s, \qquad 0 < t \le T$$

and $\{(x_s(\vartheta), 0 \le s \le T), \vartheta \in \Theta\}$ is solution of ODE

$$\frac{\mathrm{d}x_s}{\mathrm{d}s} = S(\vartheta, s, x_s), \qquad x_0, \quad 0 \le s \le T.$$

It is known that X_s converges to $x_s(\vartheta)$ uniformly in $s \in [0, T]$.

Preliminary estimator. Fix some (small) $\tau > 0$ and introduce the MDE $\bar{\vartheta}_{\tau,\varepsilon}$:

$$\left\|X - x\left(\bar{\vartheta}_{\tau,\varepsilon}\right)\right\|^2 = \inf_{\vartheta\in\Theta} \left\|X - x\left(\vartheta\right)\right\|^2 = \inf_{\vartheta\in\Theta} \int_0^\tau \left[X_t - x_t\left(\vartheta\right)\right]^2 \,\mathrm{d}t.$$

Suppose that the identifiability condition is fulfilled: for any $\nu > 0$

$$\inf_{|\vartheta-\vartheta_0|>\nu} \left\|x\left(\vartheta\right) - x\left(\vartheta_0\right)\right\| > 0.$$

This estimator is consistent and asymptotically normal

$$\varepsilon^{-1}\left(\bar{\vartheta}_{\tau,\varepsilon} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathbb{D}_\tau\left(\vartheta_0\right)\right)$$

where $\mathbb{I}\left(\vartheta, x^\tau\left(\vartheta\right)\right) \geq \mathbb{D}_\tau\left(\vartheta_0\right) > 0$ (K. 1994).

Introduce the *one-step MLE-process*: $\vartheta^*_{t,\varepsilon}, t \in [\tau, T]$

$$\vartheta^*_{t,\varepsilon} = \bar\vartheta_{\tau,\varepsilon} + \varepsilon \mathbb{I}\left(\bar\vartheta_{\tau,\varepsilon}, x^t\left(\bar\vartheta_{\tau,\varepsilon}\right)\right)^{-1} \left[\Delta_t\left(\bar\vartheta_{\tau,\varepsilon}, X^t_\tau\right) + \Delta_\tau\left(\bar\vartheta_{\tau,\varepsilon}, X^\tau\right)\right],$$

where

$$\Delta_t\left(\vartheta, X^t_\tau\right) = \int_\tau^t \frac{\dot S\left(\vartheta, s, X_s\right)}{\varepsilon\sigma\left(s, X_s\right)^2} \left[{\rm d}X_s - S\left(\vartheta, s, X_s\right) {\rm d}s\right], \quad t \in [\tau, T],$$

$$\Delta_\tau\left(\vartheta, X^\tau\right) = A\left(\vartheta, \tau, X_\tau\right) - \int_0^\tau A'_s\left(\vartheta, s, X_s\right) {\rm d}s$$

$$- \frac{\varepsilon^2}{2} \int_0^\tau B'_x\left(\vartheta, s, X_s\right) \sigma\left(s, X_s\right)^2 {\rm d}s - \int_0^\tau \frac{\dot S\left(\vartheta, s, X_s\right) S\left(\vartheta, s, X_s\right)}{\sigma\left(s, X_s\right)^2} {\rm d}s,$$

$$B\left(\vartheta, s, x\right) = \frac{\dot S\left(\vartheta, s, x\right)}{\sigma\left(s, x\right)^2}, \qquad A\left(\vartheta, s, x\right) = \int_{x_0}^x B\left(\vartheta, s, z\right) {\rm d}z$$

Theorem 5 (K. and Zhou) *Suppose that the conditions of regularity hold. Then the one-step MLE-process* $\vartheta^*_{t,\varepsilon}, \tau \leq t \leq T$ *is uniformly consistent: for any* $\nu > 0$

$$\mathbf{P}_{\vartheta_0}\left\{\sup_{\tau\leq t\leq T}\left|\vartheta^*_{t,\varepsilon}-\vartheta_0\right| > \nu\right\} \rightarrow 0,$$

the stochastic process $\eta_{t,\varepsilon}\left(\vartheta_0\right) = \varepsilon^{-1}\left(\vartheta^\star_{t,\varepsilon}-\vartheta_0\right), \tau < \tau_* \leq t \leq T$ *converges weakly to the stochastic process*

$$\xi_t\left(\vartheta_0\right) = \mathbb{I}_t\left(\vartheta_0\right)^{-1}\int_0^t \dot S\left(\vartheta_0, s, x_s\right)\sigma\left(s, x_s\right)^{-1} \mathrm{d}W_s, \qquad \tau_* \leq t \leq T$$

and the estimator-process $\vartheta^*_t, \tau < t \leq T$ *is asymptotically efficient.*

We have good estimator process on the time interval $[\tau, T]$. Let us consider the possibility of the decreasing learning interval $\tau = \tau_\varepsilon = \varepsilon^\delta \to 0$. Suppose that $d = 1$, i.e., ϑ is one-dimensional $\tau = \varepsilon^\delta \in (0, T]$ and the first preliminary estimator $\bar\vartheta_{\tau_\varepsilon}$.

The stochastic process $X_t, 0 \leq t \leq \tau_\varepsilon \to 0$ can be written as follows

$$\mathrm{d}X_t = S\left(\vartheta_0, 0, x_0\right)\mathrm{d}t + \varepsilon\sigma\left(0, x_0\right)\mathrm{d}W_t + O(\tau_\varepsilon)$$

Consider the problem of estimation ϑ by observations $Y_t, 0 \leq t \leq \tau_\varepsilon$ with stochastic differential

$$\mathrm{d}Y_t = S\left(\vartheta_0, 0, x_0\right)\mathrm{d}t + \varepsilon\sigma\left(0, x_0\right)\mathrm{d}W_t, \quad x_0, \quad 0 \leq t \leq \tau_\varepsilon.$$

Suppose that $\left|\frac{\dot S(\vartheta,0,x_0)}{\sigma(0,x_0)}\right| > \kappa > 0$ and define the one-step MLE-process $\vartheta^\star_{t,\varepsilon}, \tau_\varepsilon \leq t \leq T$ as follows

$$\vartheta^\star_{t,\varepsilon} = \bar\vartheta_{\tau_\varepsilon} + \int_{\tau_\varepsilon}^{t} \frac{\dot S\left(\bar\vartheta_{\tau_\varepsilon}, s, X_s\right)}{\mathrm{I}\left(\bar\vartheta_{\tau_\varepsilon}, x_t(\bar\vartheta_{\tau_\varepsilon})\right)\sigma\left(s, X_s\right)^2}\left[\mathrm{d}X_s - S\left(\bar\vartheta_{\tau_\varepsilon}, s, X_s\right)\mathrm{d}s\right]$$

Theorem 6 *Suppose that the conditions of regularity hold. Then the one-step MLE-process* $\vartheta^\star_{t,\varepsilon}, \tau_\varepsilon \leq t \leq T$ *is uniformly consistent:*

$$\mathbf{P}_{\vartheta_0}\left\{\sup_{\tau_\varepsilon \leq t \leq T}\left|\vartheta^\star_{t,\varepsilon} - \vartheta_0\right| > \nu\right\} \to 0,$$

the stochastic process $\eta_{t,\varepsilon}\left(\vartheta_0\right) = \varepsilon^{-1}\left(\vartheta^\star_{t,\varepsilon} - \vartheta_0\right), \tau_* \leq t \leq T$ *for any* $\tau_* > 0$ *converges weakly to the stochastic process* $\xi_t\left(\vartheta_0\right), \tau_* \leq t \leq T$ *and the estimator-process* $\vartheta^\star_{t,\varepsilon}$*, is asymptotically efficient for all* $t \in (0, T]$.

If $\vartheta \in \Theta \subset \mathcal{R}^d$ with $d > 1$, then the construction of one-step MLE-process is possible if X_t is k-dimensional and $k \geq d$.

It is possible to take smaller learning interval $[0, \tau_\varepsilon]$ with $\tau_\varepsilon = \varepsilon^\delta$, where $\delta > 1$ but this requires the construction two-step MLE-process.

Unknown volatility (joint work with S. Gasparyan)

The forward equation is

$$\mathrm{d}X_t = S\left(t, X_t\right)\mathrm{d}t + \sigma\left(\vartheta, t, X_t\right)\,\mathrm{d}W_t, \quad X_0,\ 0 \leq t \leq T$$

where $\vartheta \in \Theta = (\alpha, \beta)$. We observe the solution of this equation in discrete times $t_i = i\frac{T}{n}$ and have to study the approximation $\hat{Y}_t = u(t, X_{t_k}, \hat{\vartheta}_{t_k}), k = 1, \ldots, n$, where k satisfies the conditions $t_k \leq t \leq t_{k+1}$ and the estimator $\hat{\vartheta}_{t_k}$ is construct by the observations $X^k = (X_0, X_{t_1}, \ldots, X_{t_k})$. Our goal is to realize the same program as above: we study the one-step pseudo-MLE, which can be relatively easy in calculation and has some properties of optimality.

On parameter estimation in diffusion coefficient. First of all remind that ϑ can be calculated without error if we have continuous time observations. To illustrate it we give two examples.

Example 1. Suppose that $\sigma(\vartheta, t, x) = \sqrt{\vartheta} h(t, x), \vartheta \in (\alpha, \beta)$, $\alpha > 0$, and the observed process is

$$\mathrm{d}X_t = S(t, X_t)\,\mathrm{d}t + \sqrt{\vartheta} h(t, X_t)\,\mathrm{d}W_t, \quad X_0, \quad 0 \le t \le T.$$

We suppose as well that $\int_0^t h(s, X_s)^2\,\mathrm{d}s > 0$.

Let us write the Itô formula for X_t^2:

$$X_t^2 = X_0^2 + 2\int_0^t X_s\,\mathrm{d}X_s + \vartheta \int_0^t h(s, X_s)^2\,\mathrm{d}s, \qquad 0 \le t \le T.$$

Hence, for all $t \in (0, T]$ we have with probability 1 the equality

$$\hat{\vartheta}_t = \frac{X_t^2 - X_0^2 - 2\int_0^t X_s\,\mathrm{d}X_s}{\int_0^t h(s, X_s)^2\,\mathrm{d}s} = \vartheta$$

Example 2. Suppose that $\vartheta = (\vartheta_1, \ldots, \vartheta_d) \in \Theta \subset \mathcal{R}^d$ and

$$\sigma\left(\vartheta, t, x\right)^2 = h_0\left(t, x\right) + \sum_{k=1}^{d} \vartheta_k h_k\left(t, x\right) \geq 0$$

for all $\vartheta \in \Theta$, $t \in [0, T]$ and $x \in \mathcal{R}$. Let us put

$$G\left(X^t\right) = X_t^2 - X_0^2 - 2\int_0^t X_s \mathrm{d}X_s - \int_0^t h_0\left(s, X_s\right)\mathrm{d}s,$$

$$H_k\left(X^t, t\right) = \int_0^t h_k\left(s, X_s\right)\, \mathrm{d}s.$$

Then once more by the same Itô formula for X_t^2 we obtain the family of equations

$$G\left(X^t\right) = \sum_{k=1}^{d} \vartheta_k H_k\left(t, X^t\right), \qquad t \in [0, T].$$

Hence for any $t \in [0, T]$ we can take $(t_1 < t_2 < \ldots < t_d = t)$ and solve the system of equations

$$G\left(X^{t_m}\right) = \sum_{k=1}^{d} \vartheta_k H_k\left(t_m, X^{t_m}\right), \qquad m = 1, \ldots, d.$$

Suppose that the matrix $\mathbb{H} = \left(H_{km}\right)_{d\times d}$, where

$$H_{km} = H_k\left(t_m, X^{t_m}\right),$$

is such that there exists an inverse matrix $\mathbb{H}^{-1}$, then the vector-parameter ϑ can be written as

$$\theta = \mathbb{H}^{-1}\,\mathbf{G},$$

where the vector $\mathbf{G} = \left(G\left(X^{t_1}\right), \ldots, G\left(X^{t_d}\right)\right)$.

The problem became more interesting if we consider the discrete time observations $X^n = (X_{t_1}, \ldots, X_{t_n})\,, t_j = j\frac{T}{n}$ and the problem of approximation in the *high frequency asymptotics* $(n \to \infty)$. Then in Example 1 we obtain the estimator

$$\hat{\vartheta}_{t,k} = \frac{X_{t_k}^2 - X_0^2 - 2\sum_{j=1}^{k} X_{t_{k-1}}\left(X_{t_k} - X_{t_{k-1}}\right)}{\sum_{j=1}^{k} h\left(t_{j-1}, X_{t_{j-1}}\right)^2 \delta}, \qquad \delta = \frac{T}{n}.$$

It can be easily shown that if $n \to \infty$ then we have $\hat{\vartheta}_{t,n} \to \vartheta$ and we can use it in the approximation of Y_t as follows $\hat{Y}_{t,n} = u(t, X_t, \hat{\vartheta}_{t,n})$. We can describe the distribution of error $\sqrt{n}\left(\hat{Y}_{t,n} - Y_t\right)$, but the estimator is not asymptotically optimal. We consider a different estimator.

Let us introduce the equation

$$X_{t_{j+1}} = X_{t_j} + S\left(t_j, X_{t_j}\right)\delta + \sigma\left(t_j, X_{t_j}, \vartheta\right)\left[W_{t_{j+1}} - W_{t_j}\right].$$

Note that conditional $(X_{t_0}, \ldots, X_{t_j})$ distribution

$$X_{t_{j+1}} - X_{t_j} - S\left(t_j, X_{t_j}\right)\delta \quad \sim \quad \mathcal{N}\left(0, \sigma\left(t_j, X_{t_j}, \vartheta\right)^2 \delta\right),$$

therefore we can itroduce the log pseudo-likelihood ratio

$$L\left(\vartheta, X^k\right) = -\frac{1}{2}\sum_{j=0}^{k-1} \ln\left[2\pi\sigma\left(t_j, X_{t_j}, \vartheta\right)^2 \delta\right] - \frac{1}{2}\sum_{j=0}^{k-1} \frac{\left(X_{t_{j+1}} - X_{t_j} - S\left(t_j, X_{t_j}\right)\delta\right)^2}{\sigma\left(t_j, X_{t_j}, \vartheta\right)^2 \delta}$$

The corresponding contrast function is

$$U_k\left(\vartheta, X^k\right) = \sum_{j=0}^{k-1} \ln a\left(t_j, X_{t_j}, \vartheta\right) + \sum_{j=0}^{k-1} \frac{\left(X_{t_{j+1}} - X_{t_j} - S\left(t_j, X_{t_j}\right)\delta\right)^2}{a\left(t_j, X_{t_j}, \vartheta\right)\delta}$$

where $a\left(t, x, \vartheta\right) = \sigma\left(t, x, \vartheta\right)^2$. The estimator $\hat{\vartheta}_{t,n}$ is define by

$$U_k\left(\hat{\vartheta}_{t,n}, X^k\right) = \inf_{\vartheta\in\Theta} U_k\left(\vartheta, X^k\right)$$

It is known that this estimator is consistent, asymptotically conditionally normal

$$\sqrt{n}\left(\hat{\vartheta}_{t,n} - \vartheta_0\right) \Longrightarrow \mathcal{N}\left(0, \mathrm{I}_t\left(\vartheta_0\right)^{-1}\right),$$

$$\mathrm{I}_t\left(\vartheta_0\right) = 2\int_0^t \frac{\dot{\sigma}\left(s, X_s, \vartheta_0\right)^2}{\sigma\left(s, X_s, \vartheta_0\right)^2}\,\mathrm{d}s$$

and asymptotically efficient (Dohnal(1987), Genon-Catalot, Jacod (1993)).

Note that the approximation $\hat{Y}_t = u(t, X_{t_k}, \hat{\vartheta}_{t,n})$ is computationally difficult to realize. That is why we propose as above the one-step pseudo-MLE. Let us fix some (small) $\tau \in (0, T)$. The PMLE estimator $\hat{\vartheta}_{\tau,n}$ constructed by $X_{t_{0,n}}, X_{t_{1,n}}, \ldots, X_{t_{N,n}}$, where N is chosen from the condition $t_{N,n} \leq \tau < t_{N+1,n}$, is consistent and asymptotically conditionally normal.

Introduce the normalized pseudo score-function and the empirical Fisher information

$$\Delta_{k,n}(\vartheta) = \sum_{j=0}^{k-1} \frac{\left[\left(X_{t_{j+1}} - X_{t_j} - S_j\ \delta\right)^2 - a_j(\theta)\,\delta\right] \dot{a}_j(\vartheta)}{2a_j(\vartheta)^2 \sqrt{\delta}},$$

$$\mathrm{I}_{k,n}(\vartheta) = \frac{1}{2} \sum_{j=0}^{k-1} \frac{\dot{a}_j(\vartheta)^2}{a_j(\vartheta)^2}\,\delta = 2 \sum_{j=0}^{k-1} \frac{\dot{\sigma}\left(t_j, X_{t_j}, \vartheta\right)^2}{\sigma\left(t_j, X_{t_j}, \vartheta\right)^2}\,\delta.$$

We have the stable convergence

$$\Delta_{k,n}\left(\vartheta_0\right) \Longrightarrow \sqrt{2}\int_0^t \frac{\dot\sigma\left(s,X_s,\vartheta_0\right)}{\sigma\left(s,X_s,\vartheta_0\right)}\,{\rm d}w_s$$

and the convergence in probability

$${\rm I}_{k,n}\left(\vartheta_0\right) \rightarrow {\rm I}_t\left(\vartheta_0\right).$$

The approximation of the random function Y_t we will do with the help of the following one-step PMLE

$$\vartheta^\star_{k,n} = \hat\vartheta_{\tau,n} + \sqrt{\delta}\,\frac{\Delta_{k,n}(\hat\vartheta_{\tau,n})}{{\rm I}_{k,n}(\hat\vartheta_{\tau,n})}$$

and show that this estimator is asymptotically efficient and easy calculated for all $t \in [\tau, T]$ (or $N < k \leq n$).

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