Vector Hunt-Mackenhoupt-Wheeden Condition and an Estimating Problem

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Stationary process and Hilbert Transform

Let $x(t)$ be a gaussian process with zero mean $\mathbf{E}\,x(t) = 0$ and stationary increments. Denote

$$x[\varphi] = \int_{-\infty}^{\infty} \varphi(t)\, dx(t).$$

The linear operator $x[\varphi]$ is defined in the usual way on the indicator functions:

$$x\left[\mathbf{1}_{[a,b]}\right] = x(b) - x(a),$$

and well defined on linear span S of such functions. We assume that process x has the spectral density f. This means that

$$\int_{-\infty}^{\infty} \frac{f(u)}{1+u^2}\, du < \infty \quad \text{and} \quad \mathbf{E}|x[\varphi]|^2 = \int_{-\infty}^{\infty} |\widehat{\varphi}(u)|^2\, f(u)\, du.$$

Here $\widehat{\varphi}(u)$ is the Fourier transform of φ,

$$\widehat{\varphi}(u) = \int_{-\infty}^{\infty} \varphi(t)e^{itu}\,dt.$$

For a nonnegative function f which if defined on $\mathbb{R}$, denote by L^2_f the Hilbert space with the inner product $(\cdot,\cdot)_f$ and the norm $\|\cdot\|_f$,

$$(h_1,h_2)_f = \int_{-\infty}^{\infty} h_1(u)\overline{h_2(u)}\,f(u)\,du,\ \|h\|^2_f = (h,h)_f\,.$$

The linear operator $x[\varphi]$ defined on S can be extended to $\mathscr{D}_f$,

$$\mathscr{D}_f = \left\{\varphi:\ \varphi\in L^2_{loc},\ \widehat{\varphi}\in L^2_f\right\}, \tag{1}$$

where L^2_{loc} is the set of locally square summable functions.

Denote $H(x)$ the subspace of the space $L^2(dP)$ generated by random variables $x[\varphi]$, $\varphi \in \mathscr{D}_f$. The relation

$$\pi\, x[\varphi] = \widehat{\varphi}$$

determines an isometry $\pi : \; H(x) \; \to \; L^2_f$. This allows to translate many of the problems of geometry in the space $H(x)$ into the appropriate analytical problems in the space L^2_f.

We denote H^2 the corresponding Hardy space of analytic functions. For our purposes, we can think that H^2 is the subspace of L^2 which consists of functions g which is representable in the form

$$g = \int\limits_0^\infty e^{iut}\,\varphi(t)\,dt.$$

Let P_+ be the orthogonal projection in L^2 onto H^2, $P_- = I - P - +$. Let $\mathbb{H}$ denote the Hilbert Transform,

$$\mathbb{H} = -iP_+ + iP_- = iI + 2iP_+.$$

The question we are interested in is under what condition on f the operator $\mathbb{H}$ is bounded in the weighted space L^2. The answer is well known. The famous Hunt-Mackenhoupt-Wheeden theorem (1973) says that the Mackenhoupt condition

$$\sup_I \frac{1}{|I|} \int\limits_I f(u)\, du \, \frac{1}{|I|} \int\limits_I \frac{1}{f(u)}\, du < \infty$$

is necessary and sufficient for the boundedness of $\mathbb{H}$ in weigthed space.

Since $\mathbb{H} = iI - 2iP_+$, then the Mackenhoupt condition is is necessary and sufficient for the boundedness of operator P_+ in weighted space. This result (in terms of the space $H(x)$) has a probabilistic interpretation.

Denote

$$\mathscr{D}_f(T) = \{\varphi : \ \varphi \in \mathscr{D}_f, \ \operatorname{supp} \varphi \subset [-T, T]\} \,.$$

Let $H_T(x)$ be the subspace of $H(x)$, generated by random variables $x[\varphi]$, $\varphi \in \mathscr{D}_f(T)$. Consider the operator P_T, defined on $H(x)$ by the relation

$$P_T \, x[\varphi] = x \left[\mathbf{1}_{[-T,T]}(\cdot) \, \varphi(\cdot) \right] .$$

So, for random variable ξ,

$$\xi = \int\limits_{-\infty}^{\infty} \varphi(t)\, dx(t), \tag{2}$$

we have

$$P_T\, \xi = \int\limits_{-T}^{T} \varphi(t)\, dx(t).$$

Let $\mathscr{P}_T$ be the operator of orthogonal projection in $H(x)$ onto $H_T(x)$. It is clear that

$$\mathbf{E}\left(\xi - \mathscr{P}_T\right)^2 \le \mathbf{E}\left(\xi - P_T\right)^2.$$

Theorem 1. Under the Mackenhoupt condition

$$\mathbf{E}\left(\xi - P_T\right)^2 \le C(f)\mathbf{E}\left(\xi - \mathscr{P}_T\right)^2,$$

where $C(f)$ does not depend on ξ and T.

Vector valued stationary process

Let f be a $d \times d$ matrix weight, that is a function on real line $\mathbb{R}$ whose values are selfadjoint nonnegative matrices. We define a weighted space $L^2(f)$ as the space of all measurable $\mathbb{C}^d$-valued functions on $\mathbb{R}$ satisfying to the condition

$$\|g\|_{L^2(f)} = \int_{-\infty}^{\infty} \big(f(u)g(u), g(u)\big)\, du < \infty. \tag{3}$$

We will use the notation $L^2(\mathbb{C}^d)$, as f is identity matrix. So, $L^2(\mathbb{C}^d)$ is the space of square summable functions on real line with values in $\mathbb{C}^d$. We denote $H^2(\mathbb{C}^d)$ the corresponding Hardy space of analytic functions. For our purposes, we can think that H^2 is the subspace of $L^2(\mathbb{C}^d)$ which consists of functions $g = (g_1, \ldots, g_d)$ with coordinate which is representable in the form

$$g(u) = \int_{0}^{\infty} e^{iut}\, \varphi(t)\, dt.$$

We shall use the same notation, as in the case $d = 1$. Let P_+ be the orthogonal projection in L^2 onto H^2, $P_- = I - P - +$.

Consider the Hilbert Transform,

$$\mathbb{H} = -iP_+ + iP_- = iI + 2iP_+.$$

S. Treil and A. Volberg proved (1995) that vector Muckenhoupt condition

$$\sup_I \left\| \left(\frac{1}{|I|} \int_I f(u)\,du \right)^{1/2} \left(\frac{1}{|I|} \int_I f^{-1}(u)\,du \right)^{1/2} \right\| < \infty. \qquad (4)$$

is necessary and sufficient for the boundedness of Hilbert Transform $\mathbb{H}$ in $L^2(f)$ with matrix weight.

Now consider a gaussian zero mean vector process $x(t) = (x_1(t), \ldots, x_d)$ with stationary and stationary connected increments and spectral density f.

So, for vector function $\varphi = (\varphi_1, \ldots, \varphi_d)^T$,

$$\mathbf{E}\,|x[\varphi]|^2 = \int\limits_{-\infty}^{\infty} \big(f(u)\widehat{\varphi}(u), \widehat{\varphi}(u)\big)\, du. \text{ Here } x[\varphi] = \sum_{j=1}^{d} x_j[\varphi_j].$$

$$\mathscr{D}_f = \left\{\varphi :\ \varphi \in L^2_{loc},\ \widehat{\varphi} \in L^2(f)\right\}, \tag{5}$$

Denote $H(x)$ the subspace of the space $L^2(dP)$ generated by random variables $x[\varphi],\ \varphi \in \mathscr{D}_f$. The relation

$$\pi\, x[\varphi] = \widehat{\varphi}$$

determines an isometry $\pi :\ H(x) \ \to\ L^2(f)$.
Denote

$$\mathscr{D}_f(T) = \{\varphi :\ \varphi \in \mathscr{D}_f,\ \operatorname{supp}\varphi \subset [-T, T]\}\,.$$

Let $H_T(x)$ be the subspace of $H(x)$, generated by random variables $x[\varphi],\ \varphi \in \mathscr{D}_f(T)$. Consider the operator P_T, defined on $H(x)$ by the relation

$$P_T\, x[\varphi] = x\left[\mathbf{1}_{[-T,T]}(\cdot)\, \varphi(\cdot)\right].$$

Let $\mathscr{P}_T$ be the operator of orthogonal projection in the space $H(x)$ onto $H_T(x)$. It is clear that if

$$\xi = \int\limits_{-\infty}^{\infty} \varphi^T(t)\, dx(t),$$

Then

$$P_T \xi = \int\limits_{-T}^{T} \varphi^T(t)\, dx(t).$$

Here $\varphi(t) = (\varphi_1(t), \ldots, \varphi_d(t))^T$ ia a vector function. So,

$$\int\limits_{-\infty}^{\infty} \varphi^T(t)\, dx(t) = \sum_{j=1}^{d} x_j[\varphi_j], \text{ and } \int\limits_{-T}^{T} \varphi^T(t)\, dx(t) = x[\underline{1}_{[-T,T]}(\cdot)\varphi(\cdot)].$$

It is evident that $\mathbf{E}\left(\xi - P_T\right)^2 \geq C(f)\mathbf{E}\left(\xi - \mathscr{P}_T\right)^2.$

Theorem 2. The vector Muckenhoupt condition is necessary and sufficient for the boundedness of operator P_T. Under vector Muckenhoupt condition, for any ξ in $H(x)$

$$\mathbf{E}\left(\xi - P_T\right)^2 \leq C(f)\mathbf{E}\left(\xi - \mathscr{P}_T\right)^2.$$

where $C(f)$ does not depend on ξ and T.

Estimation of pseudoperiodic function

This talk is connected with nonparametric estimation of the function $s(t)$ as the observation process $y(t)$ is given by

$$dy(t) = s(t)dt + dx(t), t \in [-T, T].$$

Here unknown function $s \in \mathscr{L}_* \subset \mathscr{L}$, $\mathscr{L}$ is the Banach space with the norm $\|\cdot\|_{\mathcal{L}}$,

$$\|s\|_{\mathcal{L}}^2 = \sup_x \int_x^{x+1} |s\,(t)\,|^2\,dt < \infty, \tag{6}$$

$\mathscr{L}_*$ is the subset of the Stepanov space $\mathscr{L}(\Lambda)$ of pseudoperiodic functions

$$s(t) = \sum_{u \in \Lambda} a(u) e^{iut}, \text{ defined by } \sum_{u \in \Lambda} (1 + |u|)^{2\beta} |a(u)|^2 \leq C, \tag{7}$$

Λ is a countable subset of real line such that

$$\tau = \tau(\Lambda) = \inf_{u,v \in \Lambda,\, u \neq v} |u - v| > 0. \tag{8}$$

noise process $x(t)$ is the gaussian process with stationary increments with zero mean and with the spectral density f.

The set Λ is called the spectral set of s.

Consider the linear space L^2_T with the norm $\|\cdot\|_T$ and the inner product $(\cdot, \cdot)_T$,

$$\|s\|_T = \left\{ \frac{1}{2T} \int\limits_{-T}^{T} |s(t)|^2 \, dt \right\}^{1/2}, \; (s_1, s_2)_T = \frac{1}{2T} \int\limits_{-T}^{T} s_1(t) \overline{s_2(t)} \, dt.$$

The linear space with the Wiener and Paley (1964) proved that under condition $\tau(\Lambda) > 0$ the following norms are topologically equivalent on $\mathscr{L}(\Lambda)$ for sufficiently large T: $\|s\|_*$, $\|s\|_T$ and $\|s\|_{\mathscr{L}}$. This means that, for $T > T_0$

$$c_1 \|s\|_* \le \|s\|_T \le C_1 \|s\|_*, \quad c_2 \|s\|_* \le \|s\|_{\mathscr{L}} \le C_2 \|s\|_*,$$

where c_1, C_1, c_2, C_2 depend only on τ.

We shall use the estimator

$$\widehat{s}_T(t) = \sum_{|u| \le N} y[\overline{\varphi_u}]\varphi_u^*(t).$$

Here $N = N(T)$ is the appropriately chosen integer, $\varphi_u(t) = \mathbf{1}_{[-T,T]}(t)e^{itu}$, the collection $\{\varphi_u^*(t),\ a \in \Lambda\}$ is conjugate to $\{\varphi_u(t),\ a \in \Lambda\}$ in the space L_T^2,

$$\frac{1}{2T} \int_{-T}^{T} \varphi_v^*(t)\overline{\varphi_u}(t)\,dt = \delta_{u,v}.$$

For an estimator $\widehat{s}_T$ of unknown function s we denote

$$R_T\left(\widehat{s}_T, f\right) = \sup_{s \in \mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|^2_{\mathscr{L}}.$$

Let $R_T\left(f\right)$ be the minimax risk,

$$R_T\left(f\right) = \inf_{\widehat{s}_T} \sup_{s \in \mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|^2_{\mathscr{L}}.$$

Vector valued case

Now consider the case, as we observe vector valued process $y(t) = (y_1(t), y_2(t))$ which is given by

$$dy_1(t) = s_1(t)dt + dx_1(t), t \in [-T, T],$$

$$dy_2(t) = s_2(t)dt + dx_2(t), t \in [-T, T],$$

Here unknown functions $s_j \in \mathscr{L}_*(j) \subset \mathscr{L}(\Lambda_j),\ j = 1, 2,$

$\mathscr{L}_*(j)$ is the subset of the Stepanov space $\mathscr{L}(\Lambda_j)$ of pseudoperiodic functions

$$s(t) = \sum_{u \in \Lambda} a(u) e^{iut}, \text{ defined by } \sum_{u \in \Lambda} (1 + |u|)^{2\beta_j} |a(u)|^2 \leq C, \tag{9}$$

The noise process $x(t) = (x_1(t), x_2(t))$ is the gaussian process with stationary increments with zero mean and with the spectral density f.

We consider the problem of estimating function s_1 with nuisance parameters s_2, and denote by $\mathscr{R}_T(f)$ the minimax risk of this estimation problem.

More precisely, we consider a simple but non-trivial case when

$$f(u) = \begin{pmatrix} 1 & p(u) \\ \overline{p(u)} & 1 \end{pmatrix}. \tag{10}$$

For nonnegative function h, denote by $R_T(h)$ the minimax risk in the estimation problem, as we observe only

$$dy_1(t) = s_1(t)dt + dx_1(t), t \in [-T, T],$$

and h is the spectral density of x_1. Suppose function h satisfies, for some

$\gamma > -1$ and $\varepsilon = N^{-(1+\gamma+\beta)}$, to condition

$$\sum_{|u|\le N} h_\varepsilon(u) \le C()N^{1+\gamma}, \tag{11}$$

where

$$h_\varepsilon(u) = \frac{1}{2\varepsilon} \int\limits_{u-\varepsilon}^{u+\varepsilon} h(t)\,dt$$

Then under Muckenhoupt condition on h

$$R_T\,(h) \asymp T^{-\frac{2\beta}{2\beta+\gamma+1}}.$$

Lemma 1. Suppose we observe $y(t) = (y_1(t), y_2(t))$ which is given by

$$dy_1(t) = s_1(t)dt + dx_1(t), t \in [-T, T],$$

$$dy_2(t) = dx_2(t), t \in [-T, T].$$

Then under vector Muckenhoupt condition minimax risk R_T for this estimation problem satisfies the inequality

$$R_T \geq C(p) R_T (1 - |p|^2).$$

Theorem 1. Suppose $\Lambda_1 \cap \Lambda_2 = 0$, spectral density f satisfies to the vector Muckenhoupt condition. Then

$$\mathscr{R}_T(f) \leq C R_T (1 - |p|^2).$$