On the Limit Distributions of the First Moment of the High Boundary Crossing for Random Processes and Sequences

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1. Introduction

Our aim - to give a simple derivation of the limit distribution of the first moment of the high boundary crossing (for wide range of random processes and sequences).

For historical reasons we start exposition with discrete-time sequences and then consider continuous-time processes.

Initial for us is the following well-known "*Anscombe Theorem*".

Let $X_1, X_2, \ldots$ – i.i.d. random variables with

$$\mathbf{E}X_1 = a > 0, \qquad \mathbf{E}|X_1 - a|^2 = \sigma^2 < \infty.$$

Denote $S_n = X_1 + \ldots + X_n, \quad S_0 = 0,$
and introduce the first-passage time moment

$$\tau_B = \min\{n : S_n \geq B\}, \qquad B > 0.$$

For simplicity, we assume $a > 0$ is fixed and $B \to \infty$.
Then with high probability $\tau_B \approx B/a$.

A more accurate result gives

Theorem 1. (Anscombe, 1952)

$$\mathcal{L}\left\{\frac{\tau_B - B/a}{(\sigma^2 B/a^3)^{1/2}}\right\} \Rightarrow \mathcal{N}(0,1) \qquad B \to \infty. \tag{1}$$

(Was proved by (Heyde, 1966) and (Gut, 1974)).

We give simple proof of a more general variant of formula (1).

Let $X_1, X_2, \ldots$ – i.i.d.r.v. with $\mathbf{E}X_1 = a$, $0 < a < \infty$. ,   :

Assumption $\mathbf{A}_1$. There exists a non-random sequence $\{b_n\}$ and parameter $\alpha \in (1, 2]$, such that

$$\mathcal{L}\left\{\frac{S_n - an}{b_n}\right\} \Rightarrow G_\alpha \qquad n \to \infty, \tag{2}$$

where G_α - stable distribution function with parameter α.

It is known:

1) the limit distribution in (2) may be only stable distribution G_α;

2) if $\alpha > 1$, then $b_n = Cn^{1/\alpha}$ with some constant C;

3) distribution G_α has any absolute moment of power $b \in [0, \alpha)$.

Denote by ξ_F random variable with distribution F.

Theorem 2. *For τ_B the stochastic representation holds*

$$\tau_B = \frac{1}{a}(B - b_{B/a}\xi_{G_\alpha})[1 + o_B(1)]. \qquad (3)$$

where $o_B(1) \overset{\mathbf{P}}{\Rightarrow} 0$, $B \to \infty$. In other words, for any fixed x

$$\mathbf{P}\left\{\frac{B - a\tau_B}{b_{B/a}} \le x\right\} \to G_\alpha(x), \qquad B \to \infty. \qquad (4)$$

(Th. 2 was proved by other methods in (Heyde, 1967), (Gut, 1974)).

Corollary 1. *If $\mathbf{E}|X_1 - a|^2 = \sigma^2$, $0 < \sigma < \infty$, then $\alpha = 2$, $b_n = \sigma n^{1/2}$, $G_2(x) = \Phi(x)$ and "Anscombe Theorem" holds.*

Informally speaking, if we have a certain information on the behavior of τ_B, then with a simple trick that information can be improved by one more level. For example, if we know that $\tau_B \overset{\mathbf{P}}{\Rightarrow} B/a$, $B \to \infty$ (see formula (6)), then it is possible to get limiting distribution of τ_B (e.g. Theorem 1). Similarly, using the distribution of τ_B (for example, formula (6)), it is possible to get next term (on B) of limiting distribution of τ_B.

2. Proof of Theorem 2

From assumption $\mathbf{A}_1$ we have

$$S_n = an + b_n \xi_F + b_n o_n(1), \quad n \to \infty,$$

where $o_n(1) \overset{\mathbf{P}}{\Rightarrow} 0$ as $n \to \infty$.

By strong law of large numbers we can get (Heyde, 1966)

$$\frac{\tau_B}{B} \overset{a.s.}{\Longrightarrow} \frac{1}{a}, \qquad B \to \infty, \tag{5}$$

and, in particular,

$$\tau_B = \frac{B[1 + o_B(1)]}{a}, \tag{6}$$

where $o_B(1) \overset{\mathbf{P}}{\Rightarrow} 0$ as $B \to \infty$.

Choose a fixed moment $T = T(B)$ such that:
1) with high probability $\tau_B > T$;
2) T is rather close to B/a (more exactly - below).

We show that finding the distribution of τ_B is approx. equivalent to finding the distribution of $B - S_T$ (for appropriate choice of T).

Using strong Markovian property of $\{S_n\}$, moment τ_B can be represented as

$$\tau_B = (T + \tau_{B-S_T})\, I_{\{\tau_B \geq T\}} + \tau_B I_{\{\tau_B < T\}} =$$
$$= T + \tau_{B-S_T} - \left[(T + \tau_{B-S_T} - \tau_B) I_{\{\tau_B < T\}} \right].$$

Assume that we can choose T such that $[\,] \ll T + \tau_{B-S_T}$. Then

$$\tau_B \approx T + \tau_{B-S_T}.$$

We set $T = B(1-\varepsilon)/a$ with small (but fixed) $\varepsilon > 0$. Then after standard algebra we get Theorem 2. $\square$

The same approach can be applied to non-identically distributed r.v. or weakly dependent, etc.

3. Random Processes

The same approach can be applied to continuous-time case.

We limit ourselves to illustrative example:

$$\xi(t) = t + \sigma W(t), \qquad t \geq 0,$$

$W(t)$, $t \geq 0$, – standard Wiener process; $\sigma > 0$ – given constant.

Introduce the first-passage time moment

$$\tau_A = \min\{t > 0 : \xi(t) = A\}, \qquad A > 0.$$

Characteristic function of τ_A: for any λ random process

$$\eta(t) \stackrel{\text{def}}{=} \exp\left\{-\frac{\lambda^2\sigma^2 t}{2} + \lambda\sigma W(t)\right\} = \exp\left\{-\frac{\lambda^2\sigma^2 t}{2} - \lambda t + \lambda\xi(t)\right\}$$

is martingale. Therefore $\mathbf{E}\eta(\tau_A) = 1$, and then (*Wald fundamental identity*)

$$\mathbf{E}\exp\left\{-\left(\frac{\lambda^2\sigma^2}{2} + \lambda\right)\tau_A\right\} = \exp\{-\lambda A\},$$

or, equivalently,

$$\mathbf{E}\exp\{\mu\tau_A\} = \exp\left\{\frac{2\mu A}{1+\sqrt{1-2\mu\sigma^2}}\right\}.$$

After changing $\mu = s/(\sigma\sqrt{A})$

$$\mathbf{E}\exp\left\{s\frac{\tau_A - A}{\sigma\sqrt{A}}\right\} = \exp\left\{\frac{2s^2}{\left[1+\sqrt{1-2s\sigma/\sqrt{A}}\right]^2}\right\}$$

and then

$$\lim_{A\to\infty}\mathbf{E}\exp\left\{s\frac{\tau_A - A}{\sigma\sqrt{A}}\right\} = \exp\left\{\frac{s^2}{2}\right\}$$

Therefore we get

Proposition 1.

$$\mathcal{L}\left\{\frac{\tau_A - A}{\sigma\sqrt{A}}\right\} \Rightarrow \mathcal{N}(0,1), \qquad A\to\infty. \tag{7}$$

We get the same result by simpler method. Clearly,

$$\frac{\tau_A}{A} \overset{a.s.}{\Longrightarrow} 1, \qquad A \to \infty, \tag{8}$$

and, in particular,

$$\tau_A = A[1 + o_A(1)], \tag{9}$$

where $o_A(1) \overset{\mathbf{P}}{\Rightarrow} 0$ as $A \to \infty$.

Choose a fixed moment $T = T(A)$ such that:

1) with high probability $\tau_A > T$;

2) T is rather close to A (more exactly - below).

We show that finding the distribution of τ_A is approx. equivalent to finding the distribution of $A - \xi(T)$ (for appropriate choice of T).

Again, moment τ_A can be represented as

$$\tau_A = \left[T + \tau_{A-\xi(T)}\right] I_{\{\tau_A \geq T\}} + \tau_A I_{\{\tau_A < T\}} =$$
$$= T + \tau_{A-\xi_T} - \left[(T + \tau_{A-\xi(T)} - \tau_A) I_{\{\tau_A < T\}}\right].$$

Assume that we can choose T such that $[\ ] \ll T + \tau_{A-\xi(T)}$. Then

$$\tau_A \approx T + \tau_{A-\xi(T)}.$$

We set $T = A(1 - \varepsilon)$ with small (but fixed) $\varepsilon > 0$. Then after standard algebra we get Proposition 1. $\square$