On the exact MLE in linear models with mixed fractional Brownian motion

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Mixed fractional Brownian motion

$$\xi_t = B_t + B_t^H, \quad t \in [0, T]$$

Theorem (P.Cheridito, '01)

The mixed fBm

1. *is a semi-martingale iff* $H \in \{\frac{1}{2}\} \cup (\frac{3}{4}, 1]$
2. *is equivalent to Bm for* $H \in (\frac{3}{4}, 1]$

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Two classic parameter estimation problems

Estimate $\theta \in \mathbb{R}$ from the sample

$$X^T = \{X_t,\ t \in [0, T]\}$$

for the following models:

1. linear 'regression' (mixed fBm with drift)

$$X_t = \theta t + B_t + B_t^H, \quad t \in [0, T]$$

2. linear 'auto-regression' (mixed OU process):

$$dX_t = -\theta X_t dt + dB_t + dB_t^H, \quad t \in [0, T]$$

Objective: asymptotic distribution of the MLE as $T \to \infty$.

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The purely Brownian case

- $H = 1/2$:
  - regression: elementary
  - auto-regression: M. Arato (‘68), A. Novikov (‘71), R. Liptser & A. Shiryaev (‘74)
- $H > 1/2$
  - regression: A. Le Breton (’98)
  - auto-regression: M. Kleptsyna, A. Le Breton (’02)
- $H < 1/2$
  - regression: M. Kleptsyna et al ‘00
  - auto-regression: reduces to the case $H > 1/2$ by means of a transformation due to C. Jost ‘06

The mixed regression model

Theorem (CCK '2012)

$$\hat{\theta}_T = \frac{\int_0^T g(t,T)dX_t}{\int_0^T g(t,T)dt},$$

where $g(t,T)$ solves

$$g(t,T) + H(2H-1)\int_0^T g(s,T)|s-t|^{2H-2}ds = 1, \quad \forall\, t \in [0,T].$$

The estimation error is normal with (λ_H is an explicit constant)

$$\mathbb{E}\big(\theta - \hat{\theta}_T\big)^2 = \frac{1}{\int_0^T g(t,T)dt} = \frac{\lambda_H}{T^{2-2H}}\Big(1 + o(1)\Big), \quad T \to \infty$$

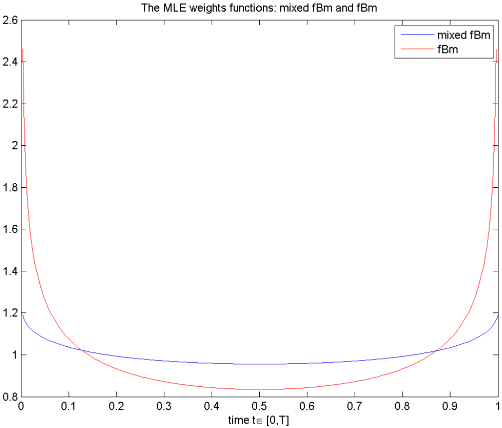
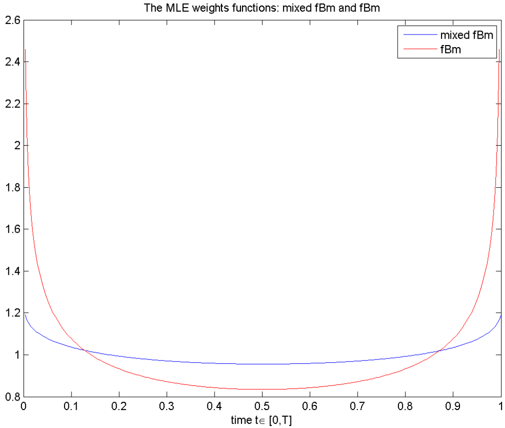


Figure : The MLE weight function: mixed fBm versus fBm $H = 3/4$

- The Brownian part is asymptotically negligible



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A formula for R-N derivative in Cheridito's theorem

Theorem (CCK '2012)

For $H \in (3/4, 1]$ *and* $\xi_t = B_t + B_t^H$

$$\frac{d\mu^\xi}{d\mu^B}(\xi) = \exp\left\{-\int_0^T \varphi_s(\xi)d\xi_s - \frac{1}{2}\int_0^T \varphi_s^2(\xi)ds\right\}$$

where

$$\varphi_s(\xi) = \int_0^s \frac{\dot{g}(r,s)}{g(s,s)}d\xi_r, \quad \text{and} \quad \dot{g}(r,t) = \frac{\partial}{\partial t}g(r,t),$$

and $g(s,t)$ *satisfies the equation(s)*

$$g(s,t) + H(2H-1)\int_0^t g(r,t)|r-s|^{2H-2}dr = 1, \quad 0 \le s \le t \le T.$$

Proof: the likelihood function

- Let $\mathbb{P}_\theta$ be the probability under which:
  - B_t^H is the fBm with the Hurst parameter H
  - $\widetilde{B}_t = \theta t + B_t$ is the standard Bm with drift θ
  - $\widetilde{B}$ and B^H are independent
- By Girsanov's theorem

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0} = \exp\left(\theta \widetilde{B}_T - \frac{1}{2}\theta^2 T\right).$$

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Proof: the likelihood function

- Let $\mathbb{P}_\theta^X$ be the restriction of $\mathbb{P}_\theta$ to $\mathcal{F}_T^X$, then the likelihood is

$$L_T(X;\theta) := \frac{d\mathbb{P}_\theta^X}{d\mathbb{P}_0^X}(X) = \mathbb{E}_0\left(\frac{d\mathbb{P}_\theta}{d\mathbb{P}_0}(\widetilde{B})\Big|\mathcal{F}_T^X\right) =$$

$$\mathbb{E}_0\left(\exp\left(\theta B_T - \frac{1}{2}\theta^2 T\right)\Big|\mathcal{F}_T^X\right) =$$

$$\exp\left(\theta M_T - \frac{1}{2}\theta^2 \langle M\rangle_T\right),$$

where $M_t := \mathbb{E}_0(B_t|\mathcal{F}_t^X)$ is a Gaussian martingale.

- The MLE and the corresponding MSE are:

$$\hat{\theta}_T = \frac{M_T}{\langle M\rangle_T}, \quad \mathbb{E}_\theta\left(\hat{\theta}_T - \theta\right)^2 = \frac{1}{\langle M\rangle_T}.$$

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Proof: the martingale M_t

- Since M is Gaussian,

$$M_T = \int_0^T g(t, T) dX_t$$

with a deterministic function $g(t, T)$, $t \in [0, T]$.

- Orthogonality of $B_T - M_T$ to linear functionals of X^T gives:

$$g(t, T) + H(2H-1) \int_0^T g(s, T)|s-t|^{2H-2} ds = 1, \; \forall\, t \in [0, T].$$

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Proof: the integral equation

- Let $\mu := T^{2H-1}$ and define

$$g_\mu(u) := T^{2H-1} g(uT, T), \quad u \in [0, 1].$$

- Then

$$\frac{1}{\mu} g_\mu(u) + H(2H-1) \int_0^1 g_\mu(v) |u-v|^{2H-2} dv = 1, \quad u \in [0, 1],$$

and

$$\langle M \rangle_T = \int_0^T g(s, T) ds = T^{2-2H} \int_0^1 g_\mu(u) du.$$

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[Q]: Does the solution of

$$\frac{1}{\mu} g_\mu + K \circ g_\mu = 1$$

converge to the solution of

$$K \circ g_0 = 1 \quad ?$$

(a closed form formula is known for g_0)

[A]: yes, weakly: for $\phi \in C_1[0,1]$

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Proof: singular perturbations problem

The weak convergence is enough for getting the asymptotic:

$$T^{2-2H}\mathbb{E}_\theta\left(\hat{\theta}_T - \theta\right)^2 = \frac{T^{2-2H}}{\langle M\rangle_T} = \frac{T^{2-2H}}{\int g(t,T)dt} = \frac{T^{2-2H}}{T^{2-2H}\int_0^1 g_\mu(u)du} \xrightarrow{\mu\to\infty} \frac{1}{\int_0^1 g_0(u)du} =: \lambda_H.$$

What about the auto-regression (OU) model ?

- Calculations as in M. Kleptsyna, A. Le Breton ('02) show that the MLE in the mixed OU case have the same asymptotic as in the classic case if

$$\lim_{\mu \to \infty} \mu^{\frac{3-4H}{2H-1}} g_\mu^2(0) = \infty$$

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Proof of Theorem 2

- Recall $\xi_t = B_t + B_t^H$
- Let $g(s,t)$ be the solution(s) of the equation(s):

$$g(s,t)+H(2H-1)\int_0^t g(r,t)|r-s|^{2H-2}dr = 1, \quad 0 \le s \le t \le T.$$

- It turns out that

$$\langle M \rangle_t = \int_0^t g(s,t)ds = \int_0^t g^2(s,s)ds,$$

and hence

$$W_t := \int_0^t \frac{1}{g(s,s)} dM_s$$

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