

A 'second Le Cam lemma' for the Ibragimov-Khasminskii limit experiment

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introduction: some statistical problems in periodic diffusions

parametric model: unknown $\vartheta \in \Theta$, $\Theta \subset \mathbb{R}^d$, observe solution ξ of

$$d\xi_t = [S(\vartheta, t) + b(\xi_t)] dt + \sigma(\xi_t) dW_t, \quad t \geq 0$$

continuously in time, over a long time interval (asymptotics $n \rightarrow \infty$)

T -periodicity: in the drift, deterministic periodic input governed by $\vartheta \in \Theta$

$$t \longrightarrow S(\vartheta, t) = S(\vartheta, i_T(t)), \quad i_T(t) := t \text{ modulo } T$$

aim: asymptotic inference about ϑ

- local models at ϑ , local scale, convergence of local models
- convergence of estimators

under assumptions which provide periodic ergodicity

- positive Harris recurrence of the grid chain $X := (\xi_{kT})_{k \in \mathbb{N}_0}$
- pos. Harris recurrence of the segment chain $\mathbb{X} := ((\xi_{kT+s})_{0 \leq s \leq T})_{k \in \mathbb{N}_0}$

examples A+B: with some 1-periodic function $S_0(\cdot)$, fixed and known,

$$S(\vartheta, t) = S_0\left(\frac{1}{\vartheta} t\right) : \quad \text{period } T = \vartheta, \Theta = (0, \infty)$$

A: S_0 is continuous (H-K 2009): LAN with **local scale** $\delta_n(\vartheta) = n^{-3/2}$ in the sense of Le Cam (1969), Hájek (1970), Le Cam and Yang (1990), ...

B: S_0 has discontinuities (H-K 2011): obtain with **local scale** $\delta_n(\vartheta) = n^{-2}$ convergence of local models at ϑ to Ibragimov-Khasminskii limit experiment

$$\tilde{\mathcal{E}} := \left\{ \tilde{P}_u : u \in \mathbb{R} \right\}$$

(Golubev (1979), Ibragimov and Khasminskii (1981), Rubin and Song (1995), Dachian (2010), ...) with likelihoods of type

$$\tilde{L}^{u/0} := \frac{d\tilde{P}_u}{d\tilde{P}_0} = \exp\left\{ \tilde{W}_u - \frac{1}{2}|u| \right\}, \quad u \in \mathbb{R}$$

where $\tilde{W} = (\tilde{W}_u)_{u \in \mathbb{R}}$ is double sided standard Brownian motion

experiment $\tilde{\mathcal{E}}$: **unit drift switched off after an unknown amount of time**, with separate branches for 'past' and 'future':

on a probability space carrying independent standard BM's \tilde{W}^+ and \tilde{W}^- , create double sided BM from $\tilde{W}_v := \tilde{W}_v^+$ if $v \geq 0$, $\tilde{W}_v := \tilde{W}_{|v|}^-$ if $v \leq 0$, put

$$\tilde{P}_0 := \mathcal{L} \left(\left(\tilde{W}_v^+, \tilde{W}_v^- \right)_{v \geq 0} \right) \quad \text{on } C([0, \infty), \mathbb{R}^2)$$

and

$$\tilde{P}_u := \begin{cases} \mathcal{L} \left(\left(\tilde{W}_v^+ + v \wedge u, \tilde{W}_v^- \right)_{v \geq 0} \mid \tilde{P}_0 \right) & \text{for } u > 0 \\ \mathcal{L} \left(\left(\tilde{W}_v^+, \tilde{W}_v^- + v \wedge |u| \right)_{v \geq 0} \mid \tilde{P}_0 \right) & \text{for } u < 0 \end{cases}$$

then observation of the bivariate canonical process on $C([0, \infty), \mathbb{R}^2)$ with infinite time horizon $v = +\infty$ leads to likelihood ratios

$$\tilde{L}^{u/0} = \frac{d\tilde{P}_u}{d\tilde{P}_0} = \exp \left\{ \tilde{W}_u - \frac{1}{2}|u| \right\}, \quad u \in \mathbb{R}$$

of experiment $\tilde{\mathcal{E}}$ (use Liptser and Shiryaev 1981, Jacod and Shiryaev 1987)

examples C+D: periodicity T fixed and known, not depending on $\vartheta \in \Theta$

C: parametric families $\{S(\vartheta, \cdot) : \vartheta \in \Theta\} \subset C_{\text{per}}([0, T])$ [S&D '10]:

we obtain LAN at ϑ with **local scale** $\delta_n(\vartheta) = n^{-1/2}$ provided parametrization is locally at ϑ smooth in $L^2([0, T], \lambda^{(\vartheta)})$ sense, for a measure $\lambda^{(\vartheta)}$ on $[0, T]$ which involves $\sigma^{-2}(\cdot)$ and the invariant law of the T -segment chain \mathbb{X} under ϑ

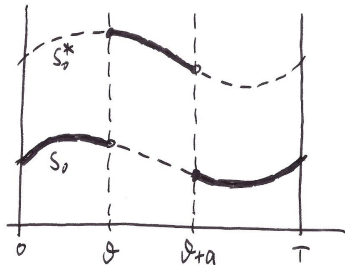
D: $\vartheta \in \Theta$ parametrizing points of discontinuity of $S(\vartheta, \cdot)$ [SISP '10]:

known functions $S_0 < S_0^* \in C_{\text{per}}([0, T])$

known $0 < a < T$, put $\Theta := (0, T - a)$

the function $S(\vartheta, \cdot)$ sketched switches from S_0 at time ϑ to S_0^* , and back at time $\vartheta + a$

when $\sigma^2(\cdot)$ is bounded away from zero, with **local scale** $\delta_n(\vartheta) = n^{-1}$, we obtain convergence of local models at ϑ to an IK limit experiment $\tilde{\mathcal{E}}$



linking Ibragimov-Khasminkii and Gaussian shift limit experiment

Lévy process $(X_u)_{u \geq 0}$ on $(\Omega, \mathcal{A}, P_0)$, $X_0 = 0$, with LT existing on $\Theta := \mathbb{R}$

$$\vartheta \longrightarrow E_{P_0}(e^{\vartheta X_u}) = e^{u \phi_0(\vartheta)}, \quad \vartheta \in \Theta, \quad u > 0$$

induces two types of one-parametric statistical models:

for $u = 1$ fixed, an exponential family $\{Q^{(1, \vartheta)} : \vartheta \in \mathbb{R}\}$ parametrized by ϑ

$$dQ^{(1, \vartheta)} := e^{\vartheta X_1 - \phi_0(\vartheta)} dP_0, \quad \vartheta \in \mathbb{R}$$

for $\vartheta = 1$ fixed, an experiment $\{Q^{(u, 1)} : u > 0\}$ parametrized by u

$$dQ^{(u, 1)} := e^{X_u - u \phi_0(1)} dP_0, \quad u > 0$$

of Ibragimov-Khasminkii type; double-sided version parametrized by $u \in \mathbb{R}$ can be defined using an independent copy X^* of the Lévy process X

X BM: have $\phi_0(\vartheta) = +\frac{1}{2} \vartheta^2$ and get a link between Gaussian shift (in $\vartheta \in \mathbb{R}$) and Ibragimov-Khasminkii (in $u \in \mathbb{R}$) limit experiment

X PP: have $\phi_0(\vartheta) = \lambda(e^\vartheta - 1)$ and get (in $u \in \mathbb{R}$) Poisson approximations to the Ibragimov-Khasminkii limit experiment in the sense of Dachian (2010)

limit theorems in Markov processes with T -periodic semigroup

general setting: $(\xi_t)_{t \geq 0}$ time inhomogeneous strong Markov, càdlàg, Polish state space, $(P_{s,t})_{s < t}$ semigroup of transition probabilities; we assume

$$\underline{T\text{-periodicity}} : P_{s,t}(x, dy) = P_{s+kT, t+kT}(x, dy) \quad \text{for arbitrary } k .$$

need limit theorems as $t \rightarrow \infty$ for functionals $A = (A_t)_{t \geq 0}$ of $(\xi_t)_{t \geq 0}$

$$A_t = \int_0^t f(\xi_s) \Lambda_T(ds) \quad \text{or} \quad A_t = \int_0^t H(s) f(s, \xi_s) \Lambda_T(ds)$$

with

- σ -finite measures $\Lambda_T(ds)$ which are T -periodic:

$$\Lambda_T(B) = \Lambda_T(B + kT) \quad \text{for all } k ,$$

- measurable nonnegative functions, either $x \rightarrow f(x)$ state dependent or $(t, x) \rightarrow f(t, x)$ T -periodic in the time variable
- càdlàg nondecreasing fcts $H \in \mathcal{RV}_\rho$ (regular var. at ∞ with index $\rho > 0$)

this class is by far larger than the class of all additive functionals of $(\xi_t)_{t \geq 0}$ allows to filter out periodic phenomena

example: fix r or $r_1 < r_2$ in $[0, T)$, take $H(s) = s^2$, consider

$$\Lambda_T(ds) := 1_{(r_1, r_2)}(i_T(s)) ds \quad , \quad A_t := \int_0^t s^2 f(s, \xi_s) 1_{(r_1, r_2)}(i_T(s)) ds$$

$$\Lambda_T(ds) := \sum_{k \in \mathbb{Z}} \epsilon_{(kT+r)}(ds) \quad , \quad A_t := \sum_{k: kT+r \leq t} (kT+r)^2 f(kT+r, \xi_{kT+r})$$

we will obtain limit theorems for $A = (A_t)_{t \geq 0}$ as $t \rightarrow \infty$ under a periodic ergodicity condition below, working with the T -segment chain

$$\mathbb{X} = (\mathbb{X}_k)_k : \quad \mathbb{X}_k := (\xi_{kT+s})_{0 \leq s \leq T} \quad , \quad k \in \mathbb{N}_0 \quad , \quad D([0, T])\text{-valued}$$

and with the T -grid chain

$$X = (X_k)_k : \quad X_k := \xi_{kT} \quad , \quad k \in \mathbb{N}_0$$

which both are time-homogeneous (by T -periodic semigroup of $(\xi_t)_{t \geq 0}$)

assumption : (periodic ergodicity condition)

(H) the T -grid chain X is positive Harris recurrent with invariant prob. μ

(H-Löcherbach 2010): a sufficient condition for (H) is

- $x b(x) + \frac{1}{2} \sigma^2(x) + (x^- \max S^- + x^+ \max S^+) < -\varepsilon$ outside K_1
- $\sigma^2(x) > 0$ on some K_2 which is large in comparison to K_1

which allows for Nummelin splitting in the grid chain

theorem 1: (H-K 2010) condition (H) is equivalent to

the segment chain \mathbb{X} is positive Harris recurrent

then for \mathbb{X} , the invariant probability m on $D([0, T])$ is specified by

$$\left\{ \begin{array}{l} \text{for arbitrary } 0 = t_0 < t_1 < \dots < t_\ell < t_{\ell+1} = T \text{ and arbitrary } A_i, \\ m(\{\pi_{t_i} \in A_i, 0 \leq i \leq \ell+1\}) \text{ is given by} \\ \int \dots \int \mu(dx_0) 1_{A_0}(x_0) \prod_{i=0}^{\ell} P_{t_i, t_{i+1}}(x_i, dx_{i+1}) 1_{A_{i+1}}(x_{i+1}). \end{array} \right.$$

theorem 2 : under (H), for increasing processes $A = (A_t)_{t \geq 0}$ such that

$$\left\{ \begin{array}{l} \text{there is an } m\text{-integrable function } F : D([0, T]) \rightarrow [0, \infty) \\ \text{of form } F(\alpha) = \int_0^T \Lambda_T(ds) f(s, \alpha(s)) \text{ as above} \\ \text{such that } A_{kT} = \sum_{j=1}^k F(\mathbb{X}_j) \text{ for all } k = 1, 2, \dots \end{array} \right.$$

we have

i) almost sure convergence as $t \rightarrow \infty$

$$\frac{1}{t} A_t \longrightarrow \frac{1}{T} m(F) = \frac{1}{T} \int_0^T \Lambda_T(ds) \int [\mu P_{0,s}](dy) f(s, y)$$

ii) for every $H \in \mathcal{RV}_\rho$, $\rho > 0$, almost sure convergence as $t \rightarrow \infty$

$$\frac{1+\rho}{t H(t)} \int_0^t H(s) dA_s \longrightarrow \frac{1}{T} m(F)$$

(H-K 2010, 2011), for regular variation: Bingham, Goldie and Teugels (1987)

convergence of local models: $S(\cdot; t) = S_0\left(\frac{1}{\vartheta} t\right)$, example B

assumptions: • $S_0(\cdot)$ fixed and known, 1-periodic, piecewise continuous, jumps of height ρ_j at times r_j , $0 < r_1 < \dots < r_\ell < 1$

- $\sigma(\cdot) > 0$ Lipschitz, bounded, bounded away from 0; $b(\cdot)$ Lipschitz
- periodic ergodicity condition **(H)** holds for solution of

$$(*) \quad d\xi_t = \left[S_0\left(\frac{t}{\vartheta}\right) + b(\xi_t) \right] dt + \sigma(\xi_t) dW_t, \quad t \geq 0$$

for every $\vartheta \in \Theta$, $\Theta = (0, \infty)$: unknown parameter ϑ is length of the period

notations: • P^ϑ law of solution to **(*)** on canonical path space $(C, \mathcal{C}, \mathbb{F})$

- $\eta = (\eta_t)_{t \geq 0}$ canonical process on $(C, \mathcal{C}, \mathbb{F})$, $m^{(\vartheta)}$ its P^ϑ -martingale part
- $L^{\zeta/\vartheta}$ likelihood ratio process of P^ζ to P^ϑ relative to \mathbb{F} :

$$L^{\zeta/\vartheta} = \mathcal{E}_\vartheta \left(\int_0^\cdot \frac{S_0\left(\frac{t}{\zeta}\right) - S_0\left(\frac{t}{\vartheta}\right)}{\sigma^2(\eta_t)} dm_t^{(\vartheta)} \right) = \mathcal{E}_\vartheta \left(\int_0^\cdot \frac{S_0\left(\frac{t}{\zeta}\right) - S_0\left(\frac{t}{\vartheta}\right)}{\sigma(\eta_t)} dB_t^{(\vartheta)} \right)$$

with P^ϑ -Brownian motion $B^{(\vartheta)}$

localization at ϑ : local scale $\delta_n(\vartheta) = n^{-2}$, observe $(\eta_t)_t$ on $[0, n]$, local model:

$$\left\{ P^{\vartheta+h/n^2} \mid \mathcal{F}_n : h \in \Theta_{\vartheta,n} \right\} \quad \text{parametrized by the local parameter } h$$

write $L_n^{(\vartheta+h/n^2)/\vartheta}$ likelihood of local parameter h to local parameter 0

theorem 3: (H-K 2011) as $n \rightarrow \infty$, we have convergence of local models at ϑ

$$\left(L_n^{(\vartheta+h/n^2)/\vartheta} \right)_{h \in \Theta_{\vartheta,n}}, \quad \Theta_{\vartheta,n} := \{h : \vartheta+h/n^2 > 0\},$$

$$\log L_{\bullet n}^{(\vartheta+h/n^2)/\vartheta} = \mathcal{E}_{\vartheta} \left(\int_0^{\bullet n} \left[S_0\left(\frac{s}{\vartheta+h/n^2}\right) - S_0\left(\frac{s}{\vartheta}\right) \right] \frac{1}{\sigma(\eta_s)} dB_s^{\vartheta} \right)$$

in the sense of finite dimensional distributions to

$$\left(\tilde{L}^{h/0} \right)_{h \in \mathbb{R}}, \quad \tilde{L}^{h/0} = \exp \left\{ \tilde{W}(hJ) - \frac{1}{2}|hJ| \right\}, \quad h \in \mathbb{R}$$

with some scaling factor $J := \frac{1}{2\vartheta^2} \sum_{j=1}^{\ell} \rho_j^2 [\mu^{(\vartheta)} P_{0,r_j\vartheta}^{(\vartheta)}](\frac{1}{\sigma^2})$ which depends on $\sigma(\cdot)$, the sequence of jump times/heights of S_0 , and on the semigroup under ϑ

we shall deduce this theorem from a stronger assertion which has the structure of a '2nd Le Cam lemma' for the Ibragimov-Khasminskii limit experiment

notations: $S_0(\cdot)$ has jumps of height ρ_j at times r_j , $0 < r_1 < \dots < r_\ell < 1$, and period 1; we define deterministic counting functions

$$N_j(s) := \sum_{k=1}^{\infty} 1_{[k+r_j, \infty)}(s) \quad , \quad s \geq 0, \quad 1 \leq j \leq \ell$$

fix $\vartheta > 0$ and put (written here for case $h < 0$, notation similar for $h > 0$)

$$\Phi_{\vartheta}^{h,n}(s) := \sum_{j=1}^{\ell} \rho_j \left[N_j\left(\frac{s}{\vartheta+h/n^2}\right) - N_j\left(\frac{s}{\vartheta}\right) \right] \quad , \quad 0 \leq s \leq n$$

structure in $0 \leq s \leq n$ provided n is large enough: for every j , $[\dots j \dots]$ is a sum of indicator functions for disjoint intervals of increasing length, and

$$\left[S_0\left(\frac{s}{\vartheta+h/n^2}\right) - S_0\left(\frac{s}{\vartheta}\right) \right] \quad \Phi_{\vartheta}^{h,n}(s) \quad , \quad 0 \leq s \leq n$$

rough structural approximation

closer look to summands contributing to $\Phi_{\vartheta}^{h,n}(s)$, $0 \leq s \leq n$:

$$\left[N_j\left(\frac{s}{\vartheta + h/n^2}\right) - N_j\left(\frac{s}{\vartheta}\right) \right] = \sum_{k=0}^{\infty} \underbrace{1_{\left((\vartheta + h/n^2)(k + r_j), \vartheta(k + r_j) \right)}}_{\text{length} \approx |h| \frac{k}{n^2}}(s)$$

in notation for $h < 0$ (analogous for $h > 0$)

in restriction to $s \in [0, n]$ when n large enough: intervals above are

- 'very small' and always located left of $\vartheta(k+r_j)$
- non-overlapping in $1 \leq k \neq k' \leq O(\frac{n}{\vartheta})$ in every $[\dots j, h \dots]$
- non-overlapping between $[\dots j, h \dots]$ and $[\dots j', h \dots]$ when $j \neq j'$
- when $-\infty < h_2 < h_1 < 0$, intervals in $[\dots j, h_1 \dots]$ are contained in the corresponding intervals in $[\dots j, h_2 \dots]$

case $h > 0$: analogous with intervals located right of $\vartheta(k+r_j)$

asymptotically as $n \rightarrow \infty$, the local parameter $h \in \mathbb{R}$ thus makes appear the covariance structure of two-sided Brownian motion $(W_h)_{h \in \mathbb{R}}$

$$K(h_1, h_2) = |h_1| \wedge |h_2| \quad \text{if } \text{sgn}(h_1) = \text{sgn}(h_2) \quad 1 \quad |h|$$

consider now any finite collection of points h_1, \dots, h_r in \mathbb{R}

convergence in P^ϑ -probability as $n \rightarrow \infty$

$$\int_0^{t \cdot n} \Phi_{\vartheta}^{h_1, n}(s) \Phi_{\vartheta}^{h_2, n}(s) \frac{1}{\sigma^2(\eta_s)} ds \longrightarrow K(h_1, h_2) t^2 J$$

allows to apply the martingale convergence theorem (martingales in time $0 \leq t \leq 1$) to prove weak convergence

$$\left(\begin{array}{c} \int_0^{\bullet n} \Phi_{\vartheta}^{h_1, n}(s) \frac{1}{\sigma^2(\eta_s)} dB_s^{(\vartheta)} \\ \dots \\ \int_0^{\bullet n} \Phi_{\vartheta}^{h_r, n}(s) \frac{1}{\sigma^2(\eta_s)} dB_s^{(\vartheta)} \end{array} \right) \longrightarrow \left(\begin{array}{c} \widehat{B}^{h_1} \\ \dots \\ \widehat{B}^{h_r} \end{array} \right)$$

to a high-dimensional limit process \widehat{B} , Gaussian in time t , independent of $B^{(\vartheta)}$ and thus of W , which in h_1, \dots, h_r has the covariance structure

$$E \left(\widehat{B}_t^{h_i}, \widehat{B}_t^{h_j} \right) = K(h_1, h_2) t^2 J$$

of two-sided Brownian motion $(\widetilde{W}(hJ))_{h \in \mathbb{R}}$ with scaling constant J

the following is our 'second Le Cam lemma' for convergence to the Ibragimov Khasminskii limit experiment:

theorem 4: (H-K 2011) in local models at ϑ we have as $n \rightarrow \infty$

i) a stochastic approximation valid for bounded sequences $(h_n)_n$ in \mathbb{R} :

$$\log L_n^{(\vartheta+h_n/n^2)/\vartheta} = \int_0^n \Phi_{\vartheta}^{h_n,n}(s) \frac{1}{\sigma(\eta_s)} dB_s^{(\vartheta)} - \frac{1}{2} \int_0^n [\Phi_{\vartheta}^{h_n,n}(s)]^2 \frac{1}{\sigma^2(\eta_s)} ds + o_{P^{(\vartheta)}}(1)$$

ii) finite dimensional convergence of

$$\left(\int_0^n \Phi_{\vartheta}^{h,n}(s) \frac{1}{\sigma(\eta_s)} dB_s^{(\vartheta)} - \frac{1}{2} \int_0^n [\Phi_{\vartheta}^{h,n}(s)]^2 \frac{1}{\sigma^2(\eta_s)} ds \right)_{h \in \Theta_{\vartheta,n}} \text{ under } P^{(\vartheta)}$$

as $n \rightarrow \infty$ to the likelihood function

$$\left(\widetilde{W}(hJ) - \frac{1}{2} |hJ| \right)_{h \in \mathbb{R}}$$

of the IK limit experiment, with scaling constant J from theorem 3

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