Parameter Estimation for the Square-root Diffusions : Ergodic and Nonergodic Cases.

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Outline of The Talk

Outline

The CIR Model

Let us consider the Cox-Ingersoll-Ross (CIR) model

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma|X_t|}dW_t, \tag{1}$$

where $X_0 = x > 0$, $a > 0$, $b \in \mathbb{R}$, $\sigma > 0$ and $(W_t)_{t \geq 0}$ is a standard Brownian motion.

- This SDE has a unique strong solution $(X_t)_{t \geq 0}$ and from the comparison theorem for one-dimensional diffusion process $X_t \geq 0$.
- In the case $b = 0$ and $\sigma = 2$, we recover the square of a a-dimensional Bessel process starting at x, denoted by BESQ_x^a

Main Properties of The CIR

Let

$$\tau_0 := \inf\{t \geq 0 | X_t = 0\},$$

with the convention $\inf \emptyset = \infty$.

- $a \geq \sigma$, $X_t > 0$ and $\mathbb{P}_x(\tau_0 = \infty) = 1$.
- $a < \sigma$ and $b \geq 0$, $\mathbb{P}_x(\tau_0 < \infty) = 1$.
- $a < \sigma$ and $b < 0$, $\mathbb{P}_x(\tau_0 < \infty) \in]0, 1[$.
- $a < \sigma$, the state 0 is instantaneously reflecting.
- $b > 0$, the CIR process is ergodic with stationary distribution $\pi \rightsquigarrow \Gamma(a/\sigma, \sigma/b)$ and $\forall h \in L^1(\pi)$,

$$\frac{1}{t}\int_0^t h(X_s)ds \longrightarrow \int_{\mathbb{R}} h(x)\pi(dx)$$

almost surely.

Outline

Parameter Estimation

- We suppose that we observe $(X_t)_{t\in[0,T]}$, $T > 0$ and we want to estimate the model parameters.
- This theory is well-summarized in books of **Ibragimov & Has'minskii** or **Kutoyants** or **Lipster & Shiryayev**.
- Concerning the CIR model, it was only studied in the particular case $b > 0$ and $a > \sigma$:

  ✓ **Fournié & Talay.** Application de la statistique des diffusions à un modèle de taux d'intérêt. *Finance*, 12(2):79-111, **1991**.

  ✓ **Overbeck.** Estimation for continuous branching processes. Scand. J. Statist., 25(1):111-126, **1998**.

- The aim of our work is to investigate the MLE of the drift parameters in the CIR model for a range of values (a, b, σ) covering ergodic and nonergodic situations.

Main Idea

- The MLE error has the form $(\langle M \rangle_t)^{-1} M_t$, where $(M_t)_{t \geq 0}$ is a Brownian martingale with quadratic variation $\langle M \rangle_t$.
- The quadratic variation $\langle M \rangle_t$ involves the quantities $\int_0^t X_s ds$ and $\int_0^t \frac{ds}{X_s}$.
- If $b > 0$ and $a > \sigma$, the asymptotic normality of the estimators is obtained using the classical martingale central limit theorem.
- Otherwise, this argument is no more valid,

$\Rightarrow$ To overcome this difficulty, we study the asymptotic behavior of the couple $(M_t, \langle M \rangle_t)$.

Outline

Asymptotic of $\int_0^t X_s ds$

We consider the CIR process $(X_t)_{t\geq 0}$ with $b = 0$, $(X_t)_{t\geq 0}$ satisfies the SDE

$$dX_t = adt + \sqrt{2\sigma X_t} dW_t. \tag{2}$$

Proposition

Let $(X_t)_{t\geq 0}$ be a CIR process solution to (2), we have

$$\left(\frac{X_t}{t}, \frac{1}{t^2}\int_0^t X_s ds\right) \xrightarrow{law} (R_1, I_1),$$

where $(R_t)_{t\geq 0}$ is the CIR process starting from 0, solution to (2) and $I_t = \int_0^t R_s ds$.

Asymptotics of the triplet $(X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$

Proposition

Under the above notations, we have

1. $\mathbb{P}_x\left(\int_0^t \frac{ds}{X_s} < \infty\right) = 1$ *if and only if* $a \geq \sigma$.
2. *If* $a > \sigma$ *then*

$$(\frac{X_t}{t}, \frac{1}{t^2}\int_0^t X_s ds, \frac{1}{\log t}\int_0^t \frac{ds}{X_s}) \xrightarrow{law} (R_1, I_1, \frac{1}{a-\sigma})$$

3. *If* $a = \sigma$ *then*

$$(\frac{X_t}{t}, \frac{1}{t^2}\int_0^t X_s ds, \frac{1}{(\log t)^2}\int_0^t \frac{ds}{X_s}) \xrightarrow{law} (R_1, I_1, \tau_1).$$

where $\tau_1 := \inf\{t > 0 \ : \ W_t = \frac{1}{\sqrt{2\sigma}}\}$ *independent of* (R_1, I_1).

Sketch of The Proof

In order to prove these propositions, we choose to compute the Laplace transform of the triplet $(X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$. We prove the following lemma using recent results by

- ✓ **Craddock & Lennox.** The calculation of expectations for classes of diffusion processes by Lie symmetry methods. *The Annals of Applied Probability*, 19(1):127-157, **2009**.

Lemma

For $\rho \geq 0, \lambda \geq 0,\ \mu > 0$ and $\eta \in]-k-\frac{\nu}{2}-\frac{1}{2}, +\infty[$, we have

$$\mathbb{E}_x\left(X_t^\eta e^{-\rho X_t - \lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}}\right) = \frac{\Gamma(\eta + k + \frac{\nu}{2} + \frac{1}{2})}{\Gamma(\nu+1)} x^\eta \left(z(t)\right)^{-\frac{\nu}{2}-\frac{1}{2}-k-\eta}$$
$$\times \left(\frac{\sqrt{\sigma\lambda}x}{\sigma\sinh(\sqrt{\sigma\lambda}t)}\right)^{\frac{\nu}{2}+\frac{1}{2}-k-\eta} \exp\left(-\frac{\sqrt{\sigma\lambda}x}{\sigma}\coth(\sqrt{\sigma\lambda}t)\right)$$
$$\times\ {}_1F_1\left(\eta + k + \frac{\nu}{2} + \frac{1}{2}, \nu + 1, \frac{\sqrt{\sigma\lambda}x}{\sigma\sinh(\sqrt{\sigma\lambda}t)z(t)}\right)$$

where $z(t) = \left((\sqrt{\sigma}\rho/\sqrt{\lambda})\sinh(\sqrt{\sigma\lambda}t) + \cosh(\sqrt{\sigma\lambda}t)\right),\ k = \dfrac{a}{2\sigma}$ and $\nu = \dfrac{1}{\sigma}\sqrt{(a-\sigma)^2 + 4\mu\sigma}$.

Outline

Asymptotics of $\int_0^t X_s ds$

Let us resume the general model of the CIR given by relation (1) with $b \neq 0$, namely

$$dX_t = (a - bX_t)dt + \sqrt{2\sigma X_t}dW_t, \tag{3}$$

Proposition

Let $(X_t)_{t\geq 0}$ be a CIR process solution to (3), we have

1. *If $b > 0$ then* $\frac{1}{t}\int_0^t X_s ds \xrightarrow{\mathbb{P}} \frac{a}{b}$.
2. *If $b < 0$ then* $\left(e^{bt}X_t, e^{bt}\int_0^t X_s ds\right) \xrightarrow{law} (R_{t_0}, t_0 R_{t_0})$, *where $t_0 = -1/b$ and $(R_t)_{t\geq 0}$ is the CIR process, starting from x, solution to (2),*

$$dR_t = adt + \sqrt{2\sigma R_t}dW_t,$$

Asymptotics of the triplet $(X_t, \int_0^t X_s ds, \int_0^t \frac{ds}{X_s})$

Proposition

Under the above notations, we have

1. $\mathbb{P}_x \left(\int_0^t \frac{ds}{X_s} < \infty \right) = 1$ *if and only if* $a \geq \sigma$.
2. *If* $b > 0$ *and* $a > \sigma$ *then* $\left(\frac{1}{t} \int_0^t X_s ds, \frac{1}{t} \int_0^t \frac{ds}{X_s} \right) \xrightarrow{\mathbb{P}} \left(\frac{a}{b}, \frac{b}{a - \sigma} \right)$.
3. *If* $b > 0$ *and* $a = \sigma$ *then* $\left(\frac{1}{t} \int_0^t X_s ds, \frac{1}{t^2} \int_0^t \frac{ds}{X_s} \right) \xrightarrow{law} \left(\frac{a}{b}, \tau_2 \right)$, *with* $\tau_2 := \inf\{t > 0 \ : \ W_t = \frac{b}{\sqrt{2\sigma}}\}$.
4. *If* $b < 0$ *and* $a \geq \sigma$ *then* $\left(e^{bt} X_t, e^{bt} \int_0^t X_s ds, \int_0^t \frac{ds}{X_s} \right) \xrightarrow{law} (R_{t_0}, t_0 R_{t_0}, I_{t_0})$.

Main Idea of The Proof

For $\rho \geq 0, \lambda \geq 0$ and $\mu > 0$, we have

$$\mathbb{E}_x\left(e^{-\rho X_t - \lambda \int_0^t X_s ds - \mu \int_0^t \frac{ds}{X_s}}\right) = \frac{\Gamma(k + \frac{\nu}{2} + \frac{1}{2})}{\Gamma(\nu + 1)} \left(\frac{Ax}{2\sigma \sinh(At/2)}\right)^{\frac{\nu}{2} + \frac{1}{2} - k}$$

$$\times \exp\left(\frac{b}{2\sigma}(at + x) - \frac{Ax}{2\sigma}\coth(At/2)\right)(z(t))^{-\frac{\nu}{2} - \frac{1}{2} - k}$$

$$\times\ {}_1F_1\left(k + \frac{\nu}{2} + \frac{1}{2}, \nu + 1, \frac{Ax}{2\sigma \sinh(At/2)\,(z(t))}\right)$$

where $z(t) = \dfrac{2\sigma\rho + b}{A}\sinh(At/2) + \cosh(At/2)$,
$k = \dfrac{a}{2\sigma}$, $A = \sqrt{b^2 + 4\sigma\lambda}$ and $\nu = \dfrac{1}{\sigma}\sqrt{(a - \sigma)^2 + 4\mu\sigma}$.

Outline

The MLE of b

The appropriate likelihood ratio is given by

$$L_T(b) := L_T^{\theta,\theta_0} = \exp\left\{\frac{b}{2\sigma}(x - X_T) + \frac{1}{4\sigma}\int_0^T (2ab - b^2 X_s)ds\right\}.$$

The MLE $\hat{b}_T$ of b is

$$\hat{b}_T = \frac{aT + x - X_T}{\int_0^T X_s ds}.$$

Hence, the error is given by

$$\hat{b}_T - b = -\sqrt{2\sigma}\frac{\int_0^T \sqrt{X_s} dW_s}{\int_0^T X_s ds}.$$

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Hence, the error is given by

$$\hat{b}_T - b = \frac{aT + x - X_T - b\int_0^T X_s ds}{\int_0^T X_s ds}.$$

Asymptotics of $\hat{b}_T$

Theorem

The MLE of b satisfies

1. *Case* $b > 0$: $\mathcal{L}_b\left\{\sqrt{T}(\hat{b}_T - b)\right\} \Rightarrow \mathcal{N}(0, 2\sigma\frac{b}{a})$.
2. *Case* $b = 0$: $\mathcal{L}_b\left\{T(\hat{b}_T - b)\right\} \Rightarrow \frac{a - R_1}{I_1}$, *where* (R_t) *is the CIR process, starting from* 0*, solution to (2) and* $I_t = \int_0^t R_s ds$.
3. *Case* $b < 0$: $\mathcal{L}_b\left\{e^{-bT/2}(\hat{b}_T - b)\right\} \Rightarrow \frac{G}{R}$, *where*

$$\mathbb{E}\left(e^{\lambda G - \mu R}\right) = \left(\frac{b}{\mu\sigma/b + b}\right)^{\frac{a}{\sigma}} \exp\left(x\frac{\sigma\lambda^2/b + \mu}{\mu\sigma/b + b}\right),$$

for $\lambda \in \mathbb{R}$ *and* $\mu \geq 0$.

The MLE of a

The likelihood ratio makes sense when $\mathbb{P}_a(\int_0^T \frac{ds}{X_s} < \infty) = 1$

$$L_T(a) := L_T^{\theta,\theta_0} = \exp\left\{ \frac{a}{2\sigma} \int_0^T \frac{dX_s}{X_s} - \frac{1}{4\sigma} \int_0^T \frac{a^2 - 2abX_s}{X_s} ds \right\}.$$

The MLE $\hat{a}_T$ of a is

$$\hat{a}_T = \frac{bT + \int_0^T \frac{dX_s}{X_s}}{\int_0^T \frac{ds}{X_s}}.$$

Hence,

$$\hat{a}_T - a = \sqrt{2\sigma} \frac{\int_0^T \frac{dW_s}{\sqrt{X_s}}}{\int_0^T \frac{ds}{X_s}}.$$

The MLE of a

The likelihood ratio makes sense when $\mathbb{P}_a(\int_0^T \frac{ds}{X_s} < \infty) = 1$

$$L_T(a) := L_T^{\theta,\theta_0} = \exp\left\{\frac{a}{2\sigma}\int_0^T \frac{dX_s}{X_s} - \frac{1}{4\sigma}\int_0^T \frac{a^2 - 2abX_s}{X_s} ds\right\}.$$

The MLE $\hat{a}_T$ of a is

$$\hat{a}_T = \frac{bT + \int_0^T \frac{dX_s}{X_s}}{\int_0^T \frac{ds}{X_s}}.$$

Hence,

$$\hat{a}_T - a = \frac{\log X_T - \log x + bT + (\sigma - a)\int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s}}.$$

Asymptotics of $\hat{a}_T$

Theorem

1. $b > 0$ *and* $a > \sigma$: $\mathcal{L}_a\left\{\sqrt{T}(\hat{a}_T - a)\right\} \Rightarrow \mathcal{N}(0, \dfrac{2\sigma(a-\sigma)}{b})$.
2. $b > 0$ *and* $a = \sigma$: $\mathcal{L}_a\left\{T(\hat{a}_T - a)\right\} \Rightarrow \dfrac{b}{\tau_2}$, *where* $\tau_2 = \inf\{t > 0 \,:\, W_t = \frac{b}{\sigma\sqrt{2}}\}$.
3. $b = 0$ *and* $a > \sigma$: $\mathcal{L}_a\left\{\sqrt{\log T}(\hat{a}_T - a)\right\} \Rightarrow \mathcal{N}(0, 2\sigma(a-\sigma))$.
4. $b = 0$ *and* $a = \sigma$: $\mathcal{L}_a\left\{(\log T)(\hat{a}_T - a)\right\} \Rightarrow \dfrac{1}{\tau_1}$, *where* $\tau_1 = \inf\{t > 0 \,:\, W_t = \frac{1}{\sqrt{2\sigma}}\}$.
5. $b < 0$ *and* $a \geq \sigma$: *the MLE estimator* $\hat{a}_T$ *is not consistent.*

Outline

The MLE of (a, b)

The likelihood ratio makes sense when $\mathbb{P}_{(a,b)}(\int_0^T \frac{ds}{X_s} < \infty) = 1$

$$L_T(\theta) = \exp\left\{ \frac{1}{2\sigma} \int_0^T \frac{a - bX_s}{X_s} dX_s - \frac{1}{4\sigma} \int_0^T \frac{(a - bX_s)^2}{X_s} ds \right\}.$$

the MLE $\hat{\theta}_T = (\hat{a}_T, \hat{b}_T)$ of $\theta = (a, b)$ is

$$\begin{cases} \hat{a}_T & = & \dfrac{\int_0^T X_s ds \int_0^T \frac{dX_s}{X_s} - T(X_T - x)}{\int_0^T \frac{ds}{X_s} \int_0^T X_s ds - T^2} \\ \\ \hat{b}_T & = & \dfrac{T \int_0^T \frac{dX_s}{X_s} - (X_T - x) \int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s} \int_0^T X_s ds - T^2} \end{cases}$$

The MLE of (a, b)

Hence, the error $\hat{\theta}_T - \theta$ is equal

$$= \begin{cases} \sqrt{2\sigma} \dfrac{\int_0^T X_s ds \int_0^T \frac{dW_s}{\sqrt{X_s}} - T \int_0^T \sqrt{X_s} dW_s}{\int_0^T \frac{ds}{X_s} \int_0^T X_s ds - T^2} \\ \\ \sqrt{2\sigma} \dfrac{T \int_0^T \frac{dW_s}{\sqrt{X_s}} - \int_0^T \frac{ds}{X_s} \int_0^T \sqrt{X_s} dW_s}{\int_0^T \frac{ds}{X_s} \int_0^T X_s ds - T^2} \end{cases}$$

The MLE of (a, b)

Hence, the error $\hat{\theta}_T - \theta$ is equal

$$= \begin{cases} \dfrac{\left(\log(X_T/x) + (\sigma - a)\int_0^T \frac{ds}{X_s}\right)\int_0^T X_s ds - T\left(X_T - x - aT\right)}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \\[2ex] \dfrac{T\left(\log(X_T/x) + bT + \sigma\int_0^T \frac{ds}{X_s}\right) - \left(X_T - x + b\int_0^T X_s ds\right)\int_0^T \frac{ds}{X_s}}{\int_0^T \frac{ds}{X_s}\int_0^T X_s ds - T^2} \end{cases}$$

Asymptotics of $(\hat{a}_T, \hat{b}_T)$

Theorem

1. *$b > 0$ and $a > \sigma$:*

$$\mathcal{L}_\theta \left\{ \sqrt{T}(\hat{\theta}_T - \theta) \right\} \Rightarrow \mathcal{N}_{\mathbb{R}^2}\left(0, 2\sigma C^{-1}\right),$$

with $C = \begin{pmatrix} \frac{b}{a-\sigma} & -1 \\ -1 & \frac{a}{b} \end{pmatrix}$.

2. *$b > 0$ and $a = \sigma$:*

$$\mathcal{L}_\theta \left\{ \mathrm{diag}\left(T, \sqrt{T}\right)(\hat{\theta}_T - \theta) \right\} \Rightarrow \left(\frac{b}{\tau_2}, \sqrt{2b}G \right)$$

where G is a gaussian $\amalg$ $\tau_2 = \inf\{t > 0 \,:\, W_t = \frac{b}{\sqrt{2\sigma}}\}$.

Asymptotics of $(\hat{a}_T, \hat{b}_T)$

Theorem

1. $b = 0$ *and* $a > \sigma$:

$$\mathcal{L}_\theta \left\{ \mathrm{diag}\,(\sqrt{\log T}, T)(\hat{\theta}_T - \theta) \right\} \Rightarrow \left(\sqrt{2\sigma(a-\sigma)}G, \frac{a - R_1}{I_1} \right)$$

where G *is a gaussian* $\amalg\ (R_1, I_1)$ *defined previously.*

2. $b = 0$ *and* $a = \sigma$:

$$\mathcal{L}_\theta \left\{ \mathrm{diag}\,(\log T, T)(\hat{\theta}_T - \theta) \right\} \Rightarrow \left(\frac{1}{\tau_1}, \frac{a - R_1}{I_1} \right)$$

where $\tau_1 := \inf\{t > 0 \ : \ W_t = \frac{1}{\sqrt{2\sigma}}\}$ *independent of* (R_1, I_1).

3. $b < 0$ *and* $a \geq \sigma$: *the MLE estimator* $\hat{\theta}_T$ *is not consistent.*

Outline

Reminder

We consider

- $(X_{t_k})_{0 \leq k \leq n}$ at instants $(t_k = k\Delta_n)_{0 \leq k \leq n}$,
- we suppose $\Delta_n \to 0$ and $n\Delta_n \to \infty$.

A common way to do that is to consider the contrast

$$L_{t_n}^{\Delta_n} = \frac{1}{2\sigma} \sum_{k=0}^{n-1} \frac{a - bX_{t_k}}{X_{t_k}} (X_{t_{k+1}} - X_{t_k}) - \frac{1}{4\sigma} \sum_{k=0}^{n-1} \Delta_n \frac{(a - bX_{t_k})^2}{X_{t_k}}$$

The discrete MLE of θ at time t_n maximizes $L_{t_n}^{\Delta_n}$

$$argmax(L_{t_n}^{\Delta_n})$$

Reminder

Our approach is slightly different since we discretize the continuous time MLE : $\hat{\theta}_{t_n}^{\Delta_n} = (\hat{a}_{t_n}^{\Delta_n}, \hat{b}_{t_n}^{\Delta_n}) =$

$$\begin{cases} \dfrac{\left(\log X_{t_n} - \log x + \sigma \sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}}\right) \sum_{k=0}^{n-1} \Delta_n X_{t_k} - t_n \left(X_{t_n} - x\right)}{\sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}} \sum_{k=0}^{n-1} \Delta_n X_{t_k} - t_n^2} \\[2em] \dfrac{t_n \left(\log X_{t_n} - \log x + \sigma \sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}}\right) - \left(X_{t_n} - x\right) \sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}}}{\sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}} \sum_{k=0}^{n-1} \Delta_n X_{t_k} - t_n^2}. \end{cases}$$

We need to control the errors

$$\int_0^{t_n} X_s ds - \sum_{k=0}^{n-1} \Delta_n X_{t_k} \text{ and } \int_0^{t_n} \frac{ds}{X_s} - \sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}}.$$

Outline

Case $b > 0$

Proposition

In the case $b > 0$, let $0 \leq s < t$ such that $0 < t - s < 1$ we have

1. *For all $q \geq 1$,*

$$\mathbb{E}_x |X_t - X_s|^q \leq C(t-s)^{\frac{q}{2}}.$$

2. *For all $a > 2\sigma$,*

$$\mathbb{E}_x \left| \frac{1}{X_t} - \frac{1}{X_s} \right| \leq C(t-s)^{\frac{1}{2}}.$$

Case $b = 0$

Proposition

In the case $b = 0$, let $0 \leq s < t$ such that $0 < t - s < 1$ we have

1. *For all $q \geq 2$,*

$$\mathbb{E}_x |X_t - X_s|^q \leq C(t-s)^{\frac{q}{2}} \sup_{s \leq u \leq t} \mathbb{E}_x(X_u^{\frac{q}{2}}).$$

2. *For all $1 \leq q < 2$,*

$$\mathbb{E}_x |X_t - X_s|^q \leq C(at + x)^{\frac{q}{2}} (t-s)^{\frac{q}{2}}.$$

3. *For all $a > 2\sigma$, there exists $q \geq 2$ and $2 < p < \frac{a}{\sigma}$, such that*

$$\mathbb{E}_x \left| \frac{1}{X_t} - \frac{1}{X_s} \right| \leq C(t-s)^{\frac{1}{2}} \sup_{s \leq u \leq t} \left(\mathbb{E}_x(X_u^{\frac{q}{2}}) \right)^{\frac{1}{q}} \left\| X_t^{-1} \right\|_p \left\| X_s^{-1} \right\|_p .$$

Outline

Ergodic case, $b > 0$

Theorem

For $b > 0$ and $a > 2\sigma$, if $n\Delta_n^2 \to 0$ then

$$\mathcal{L}_\theta \left\{ \sqrt{t_n}(\hat{\theta}_{t_n}^{\Delta_n} - \theta) \right\} \Rightarrow \mathcal{N}_{\mathbb{R}^2}\left(0, 2\sigma\Gamma^{-1}\right), \text{ with } \Gamma = \begin{pmatrix} \frac{b}{a-\sigma} & -1 \\ -1 & \frac{a}{b} \end{pmatrix}.$$

The proof is based on the following convergences

$$\sqrt{t_n}\left(\frac{1}{t_n}\int_0^{t_n} X_s ds - \frac{1}{t_n}\sum_{k=0}^{n-1} \Delta_n X_{t_k}\right) \xrightarrow{L^1} 0$$

and

$$\sqrt{t_n}\left(\frac{1}{t_n}\int_0^{t_n} \frac{1}{X_s} ds - \frac{1}{t_n}\sum_{k=0}^{n-1} \frac{\Delta_n}{X_{t_k}}\right) \xrightarrow{L^1} 0,$$

Nonergodic case, $b = 0$

Theorem

For $b = 0$ and $a > 2\sigma$, if $\max\left(n\Delta_n^2, \frac{n\Delta_n^{\frac{3}{2}}}{\log(n\Delta_n)}\right) \to 0$ *then*

$$\mathcal{L}_\theta\left\{\mathrm{diag}(\sqrt{\log t_n}, t_n)(\hat{\theta}_{t_n}^{\Delta_n} - \theta)\right\} \Rightarrow \left(\sqrt{2\sigma(a-\sigma)}G, \frac{a-R_1}{I_1}\right).$$

The proof is based on the following convergences :

$$\text{if } n\Delta_n^2 \to 0 \text{ then } t_n\left(\frac{1}{t_n^2}\int_0^{t_n} X_s ds - \frac{1}{t_n^2}\sum_{k=0}^{n-1}\Delta_n X_{t_k}\right) \xrightarrow{L^1} 0$$

$$\text{if } \frac{n\Delta_n^{\frac{3}{2}}}{\log(n\Delta_n)} \to 0 \text{ then } t_n\left(\frac{1}{\log t_n}\int_0^{t_n}\frac{1}{X_s}ds - \frac{1}{\log t_n}\sum_{k=0}^{n-1}\frac{\Delta_n}{X_{t_k}}\right) \xrightarrow{L^1} 0.$$

THANK YOU