Asymptotic properties of drift parameter estimator based on discrete observations of stochastic differential equation driven by fractional Brownian motion

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1 Introduction

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Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\left\{B_t^H, t \geq 0\right\}$ with the covariance $\mathrm{E}\left[\, B_t^H B_s^H \,\right] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$. Stochastic differential equations driven by fBm has been an active research are for the last two decades. Main reason is that such equations seem to be one of the most suitable tools to model long-range dependence in many applied areas, such as physics, finance, biology, network studies etc.

In modeling, the problems of statistical estimation of model parameters are of a particular importance, so the growing number of papers devoted to statistical methods for equations with fractional noise is not surprising. We will cite only few of them, further references can be found in [Bishwal (2008), Prakasa Rao (2010), M (2008)]. In [Kleptsyna and Le Breton (2002)], the authors proposed and studied maximum likelihood estimators for fractional Ornstein–Uhlenbeck process. Related results were obtained in [Prakasa Rao (2003)], where a more general model was considered. In [Hu and Nualart (2010)] the authors proposed a least squares estimator for fractional Ornstein–Uhlenbeck process and proved its asymptotic normality.

The estimators constructed in these papers have the same disadvantage: they are based on the whole trajectory of solution to stochastic differential equations, so are not directly implementable. In view of this, estimators based on discrete observations of solutions were proposed in [Hu et al (2011), Bertin, Torres, Tudor (2011), Tudor and Viens (2007), Xiao, Zhang, Xu (2011)]. It is worth to mention that papers [Hu et al (2011), Tudor and Viens (2007)] deal with the whole range of Hurst parameter $H \in (0, 1)$, while other papers cited here consider only the case $H > 1/2$ (which corresponds to long-range dependence). As to the model considered, in the papers [Bertin, Torres, Tudor (2011), Hu et al (2011)] a simple linear model was considered.

The paper [Xiao, Zhang, Xu (2011)] deals with the problem of estimating the parameters for fractional OrnsteinUhlenbeck processes from discrete observations when the Hurst parameter H is known. Both the drift and the diffusion coefficient estimators of discrete form are obtained based on approximating integrals via Riemann sums with Hurst parameter $H \in (1/2, 3/4)$. In the paper [Tudor and Viens (2007)] the model which is linear in fractional Brownian motion was considered. The authors mentioned, and it is really so, that the parameter estimation based on the discretization of maximum likelihood function is very complicated technically in the fractional Brownian case. Remind that in the case $H = 1/2$ we have a classical diffusion, and there is a huge literature devoted to it, we refer to the books [Ibragimov and Khasminski (1981), Liptser and Shiryaev (2001), Kutoyants (1984)] for the review of the topic. Finally, we mention papers
[Xiao (2011), Kozachenko, Melnikov, M (2011)], which deal with parameter estimation in so-called mixed models, involving standard Wiener process along with fBm.

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Preliminaries on fBm

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\left\{B_t^H, t \geq 0\right\}$ on a complete probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$ with the covariance $\mathrm{E}\left[\, B_t^H B_s^H \,\right] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$. It is well known that B^H has a modification with almost surely continuous paths (even Hölder continuous of any order up to H), and further we will assume that it is continuous itself.

In what follows we assume that the Hurst parameter $H \in (1/2, 1)$ is fixed. In this case, the integral with respect to the fBm B^H will be understood as the generalized Lebesgue–Stieltjes integral. Its construction uses the fractional derivatives, defined for $a < b$ and $\alpha \in (0,1)$ as

$$\big(D_{a+}^{\alpha} f\big)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}} du\right),$$

$$\big(D_{b-}^{1-\alpha} g\big)(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x)-g(u)}{(u-x)^{2-\alpha}} du\right).$$

Preliminaries on fBm

Provided that $D^{\alpha}_{a+} f \in L_1[a,b]$, $D^{1-\alpha}_{b-} g_{b-} \in L_\infty[a,b]$, where $g_{b-}(x) = g(x) - g(b)$, the generalized Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) = e^{i\pi\alpha} \int_a^b \big(D^{\alpha}_{a+} f\big)(x)\big(D^{1-\alpha}_{b-} g_{b-}\big)(x)dx. \qquad (1)$$

It follows from Hölder continuity of B^H that for $\alpha \in (1-H, 1)$ $D^{1-\alpha}_{b-} B^H_{b-} \in L_\infty[a,b]$ a.s. (we will prove this result in a stronger form further). Then for a function f with $D^{\alpha}_{a+} f \in L_1[a,b]$ we can define integral with respect to B^H through (1):

$$\int_a^b f(x)\, dB^H(x) := e^{i\pi\alpha} \int_a^b (D^{\alpha}_{a+} f)(x)(D^{1-\alpha}_{b-} B^H_{b-})(x)\, dx. \qquad (2)$$

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Consider stochastic differential equation driven by fBm

$$X_t = X_0 + \theta \int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s^H \qquad (3)$$

with an unknown parameter $\theta \in \mathbb{R}$.

In [Nualart and Rășcanu (2002)], it is shown that this equation has a unique solution under the following assumptions: there exist constants $\delta \in (\frac{1}{H} - 1, 1]$, $K > 0$, $L > 0$ and for every $N \geq 1$ there exists $R_N > 0$ such that

(A) $|a(x)| + |b(x)| \leq K \quad$ for all $x, y \in \mathbb{R}$,

(B) $|a(x) - a(y)| + |b(x) - b(y)| \leq L\,|x - y| \quad$ for all $x, y \in \mathbb{R}$,

(C) $|b'(x) - b'(y)| \leq R_N\,|x - y|^{\delta} \quad$ for all $x \in [-N, N], y \in [-N, N]$.

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Our main problem is the following: to construct an estimator for θ based on discrete observations of X.

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Auxiliary estimate for fBm

Let $\alpha \in (1-H, 1/2)$ be fixed. Denote for $t_1 < t_2$

$$Z(t_1,t_2) = \left(D^{1-\alpha}_{t_2-} B^H_{t_2-}\right)(t_1)$$
$$= \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)} \left(\frac{B^H_{t_1} - B^H_{t_2}}{(t_2-t_1)^{1-\alpha}} + (1-\alpha)\int_{t_1}^{t_2} \frac{B^H_{t_1} - B^H_u}{(u-t_1)^{2-\alpha}} du \right).$$

Theorem 1

For any $\gamma > 1/2$,

$$\xi_{H,\alpha,\gamma} := \sup_{0 \le t_1 < t_2 \le t_1+1} \frac{|Z(t_1,t_2)|}{(t_2-t_1)^{H+\alpha-1}\left(|\log(t_2-t_1)|^{1/2}+1\right)(\log(t_2+3))^{\gamma}}$$

is finite almost surely. Moreover, there exists $c_{H,\alpha,\gamma} > 0$ such that $\mathrm{E}\left[\exp\left\{x\xi^2_{H,\alpha,\gamma}\right\}\right] < \infty$ for $x < c_{H,\alpha,\gamma}$.

Proof I

Denote for $s > 0$ $h(s) = s^{H+\alpha-1}\left(|\log s|^{1/2} + 1\right)$ and define for $T > 0$

$$M_T = \sup_{0 \le t_1 < t_2 \le t_1 + 1 \le T} \frac{|Z(t_1, t_2)|}{h(t_2 - t_1)}.$$

We will first prove that M_T is finite almost surely. Since $\mathrm{E}\left[\left(B_t^H - B_s^H\right)^2\right] = (t-s)^{2H}$, it follows from [Marcus (1968), Theorem 4] that there exists a random variable ξ_T such that almost surely for all t_1, t_2 with $0 \le t_1 < t_2 \le t_1 + 1$

$$\left|B_{t_1}^H - B_{t_2}^H\right| \le \xi_T (t_2 - t_1)^H \left(|\log(t_2 - t_1)|^{1/2} + 1\right).$$

Then

$$|Z(t_1, t_2)| \le \frac{\xi_T}{\Gamma(\alpha)} (t_2 - t_1)^{H+\alpha-1}\left(|\log(t_2 - t_1)|^{1/2} + 1\right) + I,$$

Proof II

where

$$I = \left| \int_{t_1}^{t_2} \frac{B_u^H - B_{t_1}^H}{(u-t_1)^{2-\alpha}} du \right| \leq \frac{\xi_T}{\Gamma(\alpha)} \int_{t_1}^{t_2} \frac{\left(|\log(u-t_1)|^{1/2} + 1 \right)}{(u-t_1)^{2-H-\alpha}} du$$

$$\leq \frac{\xi_T}{\Gamma(\alpha)} (t_2 - t_1)^{H+\alpha-1} \int_0^1 z^{H+\alpha-2} \left(|\log z|^{1/2} + |\log(t_2-t_1)|^{1/2} + 1 \right) dz$$

$$\leq C\xi_T (t_2-t_1)^{H+\alpha-1} \left(|\log(t_2-t_1)|^{1/2} + 1 \right),$$

whence finiteness of M_T follows. Since M_T is a supremum of Gaussian family, Fernique's theorem implies that $\mathrm{E}\left[e^{\varepsilon M_T^2} \right] < \infty$ for some $\varepsilon > 0$, in particular, all moments of M_T are finite.

Now observe that from H-selfsimilarity of B^H it follows that for any $a > 0$

$$\{Z(at_1, at_2), 0 \leq t_1 < t_2\} \stackrel{d}{=} \left\{ a^{H+\alpha-1} Z(t_1, t_2), 0 \leq t_1 < t_2 \right\}.$$

Proof III

Therefore, for any $k \geq 1$

$$M_1 \overset{d}{=} \sup_{0 \leq t_1 < t_2 \leq 1} \frac{2^{-k(H+\alpha-1)} \left| Z(2^k t_1, 2^k t_2) \right|}{|t_2 - t_1|^{H+\alpha-1} \left(|\log(t_2 - t_1)|^{1/2} + 1 \right)}$$

$$= \sup_{0 \leq s_1 < s_2 \leq 2^k} \frac{|Z(s_1, s_2)|}{|s_2 - s_1|^{H+\alpha-1} \left(|\log(s_2 - s_1) - k \log 2|^{1/2} + 1 \right)}$$

$$\geq \sup_{0 \leq s_1 < s_2 \leq s_2 + 1 \leq 2^k} \frac{|Z(s_1, s_2)|}{|s_2 - s_1|^{H+\alpha-1} \left(|\log(s_2 - s_1) - k \log 2|^{1/2} + 1 \right)}$$

$$\geq \sup_{0 \leq s_1 < s_2 \leq s_2 + 1 \leq 2^k} \frac{|Z(s_1, s_2)|}{|s_2 - s_1|^{H+\alpha-1} \left(|\log(s_2 - s_1)|^{1/2} + (k \log 2)^{1/2} + 1 \right)}$$

$$\geq \frac{M_{2^k}}{(k \log 2)^{1/2} + 1}.$$

Proof IV

Consequently, for any $q \geq 1$

$$\mathrm{E}\left[\, M_{2^k}^q \,\right] \leq \mathrm{E}\left[\, M_1^q \,\right] \left((k \log 2)^{1/2} + 1\right)^q .$$

This implies that for any $p > q/2 + 1$

$$\mathrm{E}\left[\sum_{k=1}^{\infty} \frac{M_{2^k}^q}{k^p} \right] = \sum_{k=1}^{\infty} \frac{\mathrm{E}\left[\, M_{2^k} \,\right]^q}{k^p} \leq C\mathrm{E}\left[\, M_1^q \,\right] \sum_{k=1}^{\infty} k^{q/2-p} < \infty.$$

In particular, the sum $\sum_{k=1}^{\infty} |M_{2^k}|^q k^{-p}$ is finite almost surely, so $M_{2^k} = o(k^{p/q})$, $k \to \infty$, a.s. If we choose some $q > (\gamma - 1/2)^{-1}$, then $q/2 + 1 < \gamma q$. Hence, we can take some $p \in (q/2 + 1, \gamma q)$ and arrive at $M_{2^k} = o(k^{\gamma})$, $k \to \infty$, a.s. Consequently, the random variable $\zeta = \sup_k M_{2^k} k^{-\gamma}$ is finite almost surely.

Obviously, for $t_2 \leq 2$

$$\frac{|Z(t_1, t_2)|}{h(t_2 - t_1) \log(t_2 + 3)} \leq \frac{M_2}{\log 3}.$$

Proof V

Now let $t_2 \in (2^{k-1}, 2^k]$ for some $k \geq 2$. Then we have for any $t_1 \in [t_2 - 1, t_2)$

$$|Z(t_1, t_2)| \leq M_{2^k} h(t_2 - t_1) \leq \zeta k^\gamma h(t_2 - t_1) \leq \zeta \left(\frac{\log t_2}{\log 2} + 1 \right)^\gamma h(t_2 - t_1)$$
$$\leq 2^\gamma \zeta (\log t_2)^\gamma \zeta h(t_2 - t_1) < 2^\gamma \zeta (\log(t_2 + 3))^\gamma h(t_2 - t_1).$$

Consequently, $\xi_{H,\alpha,\gamma} \leq \max \{ M_2 / \log 3, 2^\gamma \zeta \} < \infty$ a.s.
The second statement follows from Fernique's theorem, since $\xi_{H,\alpha,\gamma}$ is a supremum of absolute values of a centered Gaussian family.

Estimates for solution of SDE with fBm

Fix some $\beta \in (1/2, H)$. Denote for $t_1 < t_2$

$$\Lambda_\beta(t_1, t_2) = 1 \vee \sup_{t_1 \le u < v \le t_2} \frac{|Z(u, v)|}{(v - u)^{\beta+\alpha-1}}.$$

Let also

$$\|f\|_{a,b,\beta} = \sup_{a \le s < t \le b} \frac{|f(t) - f(s)|}{|t - s|^\beta}.$$

Theorem 2

There exists a constant $M_{\alpha,\beta,\theta}$ depending on α, β, θ, K, and L such that for any $t_1 \ge 0, t_2 \in (t_1, t_1 + 1]$

$$|X_{t_2} - X_{t_1}| \le M_{\alpha,\beta,\theta} \left(\Lambda_\beta(t_1, t_2)(t_2 - t_1)^\beta + \Lambda_\beta(t_1, t_2)^{1/\beta}(t_2 - t_1) \right).$$

Proof I

Fix $t_1 \geq 0$ and $t_2 \in (t_1, t_1 + 1]$. Abbreviate $\Lambda = \Lambda_\beta(t_1, t_2)$. Take any s, t such that $t_1 \leq s < t \leq t_2$. Write

$$|X_t - X_s| \leq \int_s^t |a(X_u)|\, ds + \left| \int_s^t b(X_s) dB_u^H \right| \leq K(t-s) + \left| \int_s^t b(X_s) dB_u^H \right|.$$

Estimate

$$\begin{aligned} \left| \int_s^t b(X_s) dB_u^H \right| &\leq \int_s^t \left| \left(D_{s+}^{\alpha} b(X) \right)(u) \right| \left| \left(D_{t-}^{1-\alpha} B_{t-}^H \right)(u) \right| du \\ &\leq \Lambda \int_s^t \left| \left(D_{s+}^{\alpha} b(X) \right)(u) \right| (u-s)^{\beta+\alpha-1} du. \end{aligned} \tag{4}$$

Proof II

Now

$$\left|\left(D^{\alpha}_{s+}b(X)\right)(u)\right| \le \left(\frac{|b(X_u)|}{(u-s)^{\alpha}} + \int_s^u \frac{|b(X_u)-b(X_v)|}{(u-v)^{1+\alpha}}dv\right)$$
$$\le K(u-s)^{-\alpha} + L\|X\|_{s,t,\beta}\int_s^u (u-v)^{\beta-\alpha-1}dv$$
$$\le C_{\alpha,\beta}\left((u-s)^{-\alpha} + \|X\|_{s,t,\beta}(u-s)^{\beta-\alpha}\right).$$

Hence,

$$\left|\int_s^t b(X_s)dB^H_u\right| \le C_{\alpha,\beta}\Lambda\left((t-s)^{\beta} + \|X\|_{s,t,\beta}(t-s)^{2\beta}\right)$$

and

$$\|X\|_{s,t,\beta} \le K_{\alpha,\beta}\Lambda\left(1 + \|X\|_{s,t,\beta}(t-s)^{\beta}\right)$$

with a constant $K_{\alpha,\beta}$ depending only on α, β, K, and L. Setting $\Delta = (2K_{\alpha,\beta}\Lambda)^{-1/\beta}$, we obtain $\|X\|_{s,t,\beta} \le 2K_{\alpha,\beta}\Lambda$ whenever $t-s<\Delta$.

Proof III

Now, if $t_2 - t_1 \le \Delta$, then

$$|X_{t_2} - X_{t_1}| \le \|X\|_{t_1,t_2,\beta}(t_2 - t_1)^\beta \le 2K_{\alpha,\beta}\Lambda(t_2 - t_1)^\beta.$$

On the other hand, if $t_2 - t_1 > \Delta$, then, partitioning the interval $[t_1, t_2]$ into $k = [(t_2 - t_1)/\Delta]$ parts of length Δ and, possibly, an extra smaller part, we obtain

$$\begin{aligned}|X_{t_2} - X_{t_1}| \le |X_{t_1+\Delta} - X_{t_1}| + \cdots + \left|X_{t_1+k\Delta} - X_{t_1+(k-1)\Delta}\right| + |X_{t_2} - X_{t_1+k\Delta}| \\ \le (k+1)\,2K_{\alpha,\beta}\Lambda\Delta^\beta \le 4kK_{\alpha,\beta,\theta}\Delta^\beta \le 4K_{\alpha,\beta,\theta}\Lambda(t_2 - t_1)\Delta^{\beta-1} \\ = 2(2K_{\alpha,\beta}\Lambda)^{1/\beta}(t_2 - t_1).\end{aligned}$$

The proof is now complete.

Estimates for solution of SDE with fBm

Corollary 3

For any $\gamma > 1/2$*, there exist random variables* ξ *and* ζ *such that for all* $t_1 \geq 0,\ t_2 \in (t_1, t_1 + 1]$

$$|X_{t_2} - X_{t_1}| \leq \zeta (t_2 - t_1)^{\beta} \left(\log(t_2 + 3)\right)^{\kappa}, \quad \Lambda_{\beta}(t_1, t_2) \leq \xi (\log(t_2 + 3))^{\kappa\beta},$$

where $\kappa = \gamma/\beta$*. Moreover, there exists some* $c > 0$ *such that* $\mathrm{E}\left[\exp\left\{x\xi^2\right\}\right] < \infty$ *and* $\mathrm{E}\left[\exp\left\{x\zeta^{2\beta}\right\}\right] < \infty$ *for* $x < c$*. In particular, all moments of* ξ *and* ζ *are finite.*

Lemma 4

For any $n \geq 1$ *and any* $t_1, t_2 \in [0, 2^{2n}]$ *such that* $t_1 < t_2 \leq t_1 + 1$

$$|X_{t_2} - X_{t_1}| \leq \zeta n^{\kappa} (t_2 - t_1)^{\beta}, \quad \Lambda_{\beta}(t_1, t_2) \leq \xi n^{\gamma}.$$

Proof of Corollary 3

From Theorem 1 we have for all $u < v$

$$Z(u,v) \le \xi_{H,\alpha,\gamma}(v-u)^{H+\alpha-1}\left(|\log(v-u)|^{1/2}+1\right)(\log(v+3))^{\gamma}$$
$$\le C_{H,\beta}\xi_{H,\alpha,\gamma}(v-u)^{\beta+\alpha-1}(\log(v+3))^{\gamma},$$

Dividing by $(v-u)^{\beta+\alpha-1}$ and taking supremum over u, v such that $t_1 \le u < v \le t_2$, we get

$$\Lambda_\beta(t_1,t_2) \le 1 \vee \left(C_{H,\beta}\xi_{H,\alpha,\gamma}(\log(t_2+3))^{\gamma}\right) \le \left(1 \vee C_{H,\beta}\xi_{H,\alpha,\gamma}\right)(\log(t_2+3))^{\gamma}$$

Further, since $\Lambda_\beta(t_1,t_2) \ge 1$ and $t_2 - t_1 \le 1$, it follows from Theorem 2 that

$$|X_{t_2} - X_{t_1}| \le 2M_{\alpha,\beta}\Lambda_\beta(t_1,t_2)^{1/\beta}(t_2-t_1)^{\beta}.$$

Hence, the desired statement holds with $\xi = 1 \vee C_{H,\beta}\xi_{H,\alpha,\gamma}$ and $\zeta = 2M_{\alpha,\beta}\xi^{1/\beta}$.

Proof of Lemma 4

In this case

$$\log(t_2 + 3) \le \log(2^{2^n} + 3) \le \log 2^{2^{n+1}} = (2n+1)\log 2 \le n,$$

whence the statement follows.

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Now we turn back to the problem of drift parameter estimation in equation

$$X_t = X_0 + \theta \int_0^t a(X_s) ds + \int_0^t b(X_s) dB_s^H.$$

We will assume that for some $n \geq 1$ we observe values $X_{t_n^k}$ at the following uniform partition of $[0, 2^n]$: $t_k^n = k2^{-n}$, $k = 0, 1, \dots, 2^{2n}$.

In order to construct a consistent estimator, we need a lemma concerning discrete approximation of integrals in the SDE.

Lemma 5

For all $n \geq 1$ and $k = 1, 2, \dots, 2^{2n}$

$$\left| \int_{t_{k-1}^n}^{t_k^n} \Big(a(X_u) - a(X_{t_{k-1}^n}) \Big) \, du \right| \leq C \zeta n^{\kappa} 2^{-n(\beta+1)}$$

and

$$\left| \int_{t_{k-1}^n}^{t_k^n} \Big(b(X_u) - b(X_{t_{k-1}^n}) \Big) \, dB_u^H \right| \leq C \xi \zeta n^{\gamma+\kappa} 2^{-2n\beta}.$$

Proof I

Write

$$\left|\int_{t_{k-1}^n}^{t_k^n}\left(a(X_u)-a(X_{t_{k-1}^n})\right)du\right|\leq\int_{t_{k-1}^n}^{t_k^n}\left|a(X_u)-a(X_{t_{k-1}^n})\right|du$$

$$\leq K\zeta n^{\kappa}\int_{t_{k-1}^n}^{t_k^n}(u-t_{k-1}^n)^{\beta}du\leq C\zeta n^{\kappa}(t_k^n-t_{k-1}^n)^{\beta+1}=C\zeta n^{\kappa}2^{-n(\beta+1)}.$$

Similarly to (4),

$$\left|\int_{t_{k-1}^n}^{t_k^n}\left(b(X_u)-b(X_{t_{k-1}^n})\right)dB_u^H\right|$$

$$\leq\Lambda_{\beta}(t_{k-1}^n,t_k^n)\int_{t_{k-1}^n}^{t_k^n}\left|D_{t_{k-1}^n+}^{\alpha}(b(X)-b(X_{t_{k-1}^n}))(u)\right|(u-t_{k-1}^n)^{\beta+\alpha-1}du$$

$$\leq\xi n^{\gamma}\int_{t_{k-1}^n}^{t_k^n}\left|D_{t_{k-1}^n+}^{\alpha}(b(X)-b(X_{t_{k-1}^n}))(u)\right|(u-t_{k-1}^n)^{\beta+\alpha-1}du.$$

Proof II

and

$$\left|D^{\alpha}_{t^n_{k-1}+}(b(X)-b(X_{t^n_{k-1}}))(u)\right| \le \frac{\left|b(X_u)-b(X_{t^n_{k-1}})\right|}{(u-t^n_{k-1})^{\alpha}} + \int_{t^n_{k-1}}^{u} \frac{|b(X_u)-b(X_v)|}{(u-v)^{1+\alpha}}$$

$$\le K\zeta n^{\kappa}\left(u-t^n_{k-1}\right)^{\beta-\alpha} + K\zeta n^{\kappa}\int_{t^n_{k-1}}^{u}(u-v)^{\beta-\alpha-1}dv \le C\zeta\left(u-t^n_{k-1}\right)^{\beta-\alpha}.$$

Then

$$\left|\int_{t^n_{k-1}}^{t^n_k}\left(b(X_u)-b(X_{t^n_{k-1}})\right)dB^H_u\right| \le C\xi\zeta n^{\gamma+\kappa}\int_{t^n_{k-1}}^{t^n_k}(u-t^n_{k-1})^{2\beta-1}du$$

$$\le C\xi\zeta n^{\gamma+\kappa}\left(t^n_k-t^n_{k-1}\right)^{2\beta} = C\xi\zeta n^{\gamma+\kappa}2^{-2n\beta}.$$

Now we are ready to construct consistent estimators for θ. In order to proceed, we need another technical assumption, in addition to conditions (A)–(C):

(D) there exist a constant $M > 0$ such that for all $x \in \mathbb{R}$

$$|a(x)| \geq M, \qquad |b(x)| \geq M.$$

We now define the following estimator:

$$\hat{\theta}_n^{(1)} = \frac{\sum_{k=1}^{2^{2n}} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\sum_{k=1}^{2^{2n}} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right) \frac{1}{2^n}},$$

where $\lambda = 1/2 - H$.

Theorem 6

With probability one, $\hat{\theta}_n^{(1)} \to \theta$, $n \to \infty$. Moreover, there exists a random variable η with all finite moments such that $\left|\hat{\theta}_n^{(1)} - \theta\right| \leq \eta n^{\kappa+\gamma} 2^{-\rho n}$, where $\rho = (1-H) \wedge (2\beta - 1)$.

Proof I

It follows from (3) that

$$
\begin{aligned}
X_{t_k^n} - X_{t_{k-1}^n} &= \theta \int_{t_{k-1}^n}^{t_k^n} a(X_v)dv + \int_{t_{k-1}^n}^{t_k^n} b(X_v)dB_v^H \\
&= \theta \int_{t_{k-1}^n}^{t_k^n} a\left(X_{t_{k-1}^n}\right) dv + \theta \int_{t_{k-1}^n}^{t_k^n} \left(a(X_v) - a\left(X_{t_{k-1}^n}\right)\right) dv \\
&\quad + \int_{t_{k-1}^n}^{t_k^n} b\left(X_{t_{k-1}^n}\right) dB_v^H + \int_{t_{k-1}^n}^{t_k^n} \left(b(X_v) - b\left(X_{t_{k-1}^n}\right)\right) dB_v^H.
\end{aligned}
$$

Then

$$
\hat{\theta}_n^{(1)} = \theta + \frac{B_n + C_n + D_n}{A_n},
$$

Proof II

where

$$A_n = 2^{n(2H-3)} \sum_{k=1}^{2^{2n}} h_k^n a\left(X_{t_{k-1}^n}\right) b^{-1}\left(X_{t_{k-1}^n}\right),$$

$$B_n = 2^{2n(H-1)} \theta \sum_{k=1}^{2^{2n}} h_k^n b^{-1}\left(X_{t_{k-1}^n}\right) \int_{t_{k-1}^n}^{t_k^n} \left(a(X_v) - a\left(X_{t_{k-1}^n}\right)\right) dv,$$

$$C_n = 2^{2n(H-1)} \sum_{k=1}^{2^{2n}} h_k^n \left(B_{t_k^n}^H - B_{t_{k-1}^n}^H\right),$$

$$D_n = 2^{2n(H-1)} \sum_{k=1}^{2^{2n}} h_k^n b^{-1}\left(X_{t_{k-1}^n}\right) \int_{t_{k-1}^n}^{t_k^n} \left(b(X_v) - b\left(X_{t_{k-1}^n}\right)\right) dB_v^H,$$

$h_k^n = (t_k^n)^\lambda (2^n - t_k^n)^\lambda.$

Proof III

It is not hard to show that the sequence

$$\gamma_n := 2^{n(2H-3)} \sum_{k=1}^{2^{2n}} h_k^n = 2^{-2n} \sum_{k=1}^{2^{2n}} \left(\frac{k}{2^{2n}}\right)^{\lambda} \left(1 - \frac{k}{2^{2n}}\right)^{\lambda} \frac{1}{2^{2n}}$$

converges to $\int_0^1 x^{\lambda}(1-x)^{\lambda} dx = \mathrm{B}(1+\lambda, 1+\lambda)$, hence, is bounded and uniformly positive.

Indeed, $h(x) = x^{\lambda}(1-x)^{\lambda}$ is a decreasing function when $x \in \left(0, \frac{1}{2}\right]$, then

$$\int_0^{\frac{1}{2}} h(x)dx = \sum_{k=0}^{2^{2n-1}-1} \int_{\frac{k}{2^{2n}}}^{\frac{k+1}{2^{2n}}} h(x)dx < \int_0^{\frac{1}{2^{2n}}} h(x)dx + \sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right) \frac{1}{2^{2n}}.$$

On the other hand,

$$\int_0^{\frac{1}{2}} h(x)dx = \sum_{k=1}^{2^{2n-1}} \int_{\frac{k-1}{2^{2n}}}^{\frac{k}{2^{2n}}} h(x)dx > \sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right) \frac{1}{2^{2n}}.$$

Proof IV

So

$$0 < \int_0^{\frac{1}{2}} h(x)dx - \sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right)\frac{1}{2^{2n}} < \int_0^{\frac{1}{2^{2n}}} h(x)dx \to 0, \ n \to \infty.$$

Hence,

$$\sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right)\frac{1}{2^{2n}} \to \int_0^{\frac{1}{2}} h(x)dx, n \to \infty.$$

Similarly one can prove that

$$\sum_{k=2^{2n-1}+1}^{2^{2n}} h\left(\frac{k}{2^{2n}}\right)\frac{1}{2^{2n}} \to \int_{\frac{1}{2}}^{1} h(x)dx, \ n \to \infty.$$

Proof V

Note that $a(x)b^{-1}(x)$ is a uniformly positive continuous function. Therefore, it keeps its sign, and

$$\lim_{n\to\infty} |A_n| \geq MK^{-1} \lim_{n\to\infty} \gamma_n = MK^{-1}\mathrm{B}(1+\lambda, 1+\lambda) > 0.$$

So it sufficient to estimate B_n, E_n, and D_n.

By Lemma 5,

$$|B_n| \leq C\,|\theta|\,\zeta n^{\kappa} M^{-1} 2^{n(2H-\beta-3)} \sum_{k=1}^{2^{2n}} h_k^n \leq C_\theta \zeta n^{\kappa} 2^{-n\beta};$$

$$|D_n| \leq C\xi\zeta n^{\gamma+\kappa} M^{-1} 2^{n(2H-2-2\beta)} \sum_{k=1}^{2^{2n}} h_k^n \leq C\xi\zeta n^{\gamma+\kappa} 2^{n(1-2\beta)}.$$

Proof VI

Finally we estimate E_n. Start by writing

$$\mathrm{E}\left[\,E_n^2\,\right] = 2^{4n(H-1)}\mathrm{E}\left[\left(\sum_{k=1}^{2^{2n}}\int_{t_{k-1}^n}^{t_k^n} h_k^n dB_s^H\right)^2\right].$$

According to [M (2008), Corollary 1.9.4], for $f \in L_{\frac{1}{H}}[0,t]$ there exist a constant $C_H > 0$ such that

$$\mathrm{E}\left[\left(\int_0^t f(s)dB_s^H\right)^2\right] \le C_H\left(\int_0^t |f(s)|^{1/H}\,ds\right)^{2H}.$$

Proof VII

Hence,

$$\mathrm{E}\left[\, E_n^2 \,\right] \leq C2^{4n(H-1)} \left(\sum_{k=1}^{2^{2n}} \int_{t_{k-1}^n}^{t_k^n} (h_k^n)^{1/H} ds \right)^{2H}$$

$$= C2^{2n(H-1)} \left(\sum_{k=1}^{2^{2n}} \left(\frac{k}{2^{2n}} \right)^{\lambda/H} \left(1 - \frac{k}{2^{2n}} \right)^{\lambda/H} \frac{1}{2^{2n}} \right)^{2H} .$$

As above,

$$\sum_{k=1}^{2^{2n}} \left(\frac{k}{2^{2n}} \right)^{\lambda/H} \left(1 - \frac{k}{2^{2n}} \right)^{\lambda/H} \frac{1}{2^{2n}} \to \mathrm{B}\left(1 + \lambda/H, 1 + \lambda/H\right), \; n \to \infty,$$

which implies that $\mathrm{E}\left[\, E_n^2 \,\right] \leq C2^{2n(H-1)}$.

Proof VIII

Since E_n is Gaussian, $\mathrm{E}\left[\,|E_n|^p\,\right] \leq C_p 2^{pn(H-1)}$ for any $p \geq 1$. Therefore, for any $\nu > 1$

$$\mathrm{E}\left[\sum_{n=1}^{\infty} \frac{|E_n|^p}{n^{\nu} 2^{pn(H-1)}}\right] = \sum_{n=1}^{\infty} \frac{\mathrm{E}\left[\,|E_n|^p\,\right]}{n^{\nu} 2^{pn(H-1)}} \leq C_p \sum_{n=1}^{\infty} n^{-\nu} < \infty.$$

Consequently,

$$\xi' := \sup_{n\geq 1} \frac{|E_n|}{n^{\nu/p} 2^{n(H-1)}} < \infty$$

almost surely, moreover, by Fernique's theorem, all moments of ξ' are finite.

Proof IX

Let us summarize the estimates:

$$|B_n| \leq C_\theta \zeta n^\kappa 2^{-n\beta}, \ |D_n| \leq C\xi\zeta n^{\gamma+\kappa} 2^{n(1-2\beta)}, \ |E_n| \leq \xi' n^\delta 2^{n(H-1)},$$

where $\delta > 0$ can be taken arbitrarily small. We have $-\beta < -1/2 < H - 1$, $-\beta < 1 - 2\beta$, so $|B_n|$ is of the smallest order. Which of the remaining two estimates wins, depends on values of β and H: for H close to $1/2$, $1 - 2\beta$ is close to 0, while $H - 1$ is close to $-1/2$, while for β close to 1, $1 - 2\beta$ is close to -1, while $H - 1$ is close to 0. Thus, we arrive to

$$|B_n| + |E_n| + |D_n| \leq \eta n^{\gamma+\kappa} 2^{-\rho n},$$

where $\eta \leq C_\theta(\zeta + \xi\zeta + \xi')$, so all its moments are finite. The proof is now complete.

Consider a simpler estimator:

$$\hat{\theta}_n^{(2)} = \frac{\sum_{k=1}^{2^{2n}} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\frac{1}{2^n}\sum_{k=1}^{2^{2n}} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right)}.$$

Theorem 7

With probability one, $\hat{\theta}_n^{(2)} \to \theta$, $n \to \infty$. Moreover, there exists a random variable η' with all finite moments such that $\left|\theta_n^{(2)} - \theta\right| \le \eta' n^{\kappa+\gamma} 2^{-\rho n}$.

Remark 1

It can be shown with some extra technical work that

$$\left|\theta_n^{(i)} - \theta\right| \le \eta_1 n^{\mu} 2^{-\tau n},\ i = 1, 2,$$

where $\mu = 1/2 + \gamma(1 + 1/H)$, $\tau = (2H-1) \wedge (1-H)$; η_1 is a random variable, for which there exists some $c_\theta > 0$ such that $\mathrm{E}\left[\exp\left\{x\eta_1^{1+1/H}\right\}\right] < \infty$ for $x < c_\theta$. The constant c_θ can be computed explicitly in terms of H, K, L, θ.

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