Sharp large deviations for the non-stationary Ornstein-Uhlenbeck process

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Outline

1 Introduction
   - On the Cramer-Chernov theorem
   - On the Bahadur-Rao theorem

2 Discrete-time autoregressive process
   - Stable autoregressive process
   - Explosive autoregressive process

3 Continuous-time Ornstein-Uhlenbeck process
   - Stable Ornstein-Uhlenbeck process
   - Explosive Ornstein-Uhlenbeck process
   - Fractional Ornstein-Uhlenbeck process
The Gaussian example

Let \((X_n)\) be a sequence of iid \(\mathcal{N}(0, \sigma^2)\) random variables. If

\[
S_n = \sum_{k=1}^{n} X_k
\]

we clearly have \(S_n \sim \mathcal{N}(0, \sigma^2 n)\). Consequently, for all \(c > 0\)

\[
\mathbb{P}(S_n \geq nc) = \frac{\sigma}{c \sqrt{2\pi n}} \exp\left(-\frac{c^2 n}{2\sigma^2}\right) \left[1 + o(1)\right].
\]

Therefore,

\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq nc) = -\frac{c^2}{2\sigma^2}.
\]

Question

What about the non-Gaussian case?
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Large deviation principle

Let \((X_n)\) be a sequence of iid random variables with mean \(m\). The Fenchel-Legendre dual of the log-Laplace \(L\) of \((X_n)\) is

\[
l(c) = \sup_{t \in \mathbb{R}} \{ct - L(t)\}.
\]

**Theorem (Cramer-Chernov)**

The sequence \((S_n/n)\) satisfies an LDP with rate function \(l\)

- **Upper bound:** for any closed set \(F \subset \mathbb{R}\)
  \[
  \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \frac{S_n}{n} \in F \right) \leq -\inf_F l,
  \]

- **Lower bound:** for any open set \(G \subset \mathbb{R}\)
  \[
  \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \frac{S_n}{n} \in G \right) \geq -\inf_G l.
  \]
Introduction

Discrete-time autoregressive process
Continuous-time Ornstein-Uhlenbeck process

On the Cramer-Chernov theorem
On the Bahadur-Rao theorem

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Theorem (Bahadur-Rao, AMS 1960)

Assume that $L$ is finite on all $\mathbb{R}$ and that the law of $(X_n)$ is absolutely continuous. Then, for all $c > m$,

$$
\mathbb{P}(S_n \geq nc) = \frac{\exp(-nl(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + o(1) \right]
$$

where $t_c$ is given by $L'(t_c) = c$ and $\sigma_c^2 = L''(t_c)$.

Remark. The proof relies on Berry-Esseen’s theorem.
On the Bahadur-Rao theorem

**Theorem (Bahadur-Rao, AMS 1960)**

The sequence \((S_n/n)\) satisfies an SLDP. For all \(c > m\), it exists \((d_{c,k})\) such that for any \(p \geq 1\) and \(n\) large enough

\[
P(S_n \geq nc) = \frac{\exp(-nl(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + O \left( \frac{1}{n^{p+1}} \right) \right].
\]

**Remark.** The coefficients \((d_{c,k})\) may be explicitly calculated as functions of the derivatives \(l_k = L^{(k)}(t_c)\). For ex, \(\sigma_c^2 = l_2\),

\[
d_{c,1} = \frac{1}{\sigma_c^2} \left( \frac{l_4}{8\sigma_c^2} - \frac{5l_3^2}{24\sigma_c^4} - \frac{l_3}{2t_c\sigma_c^2} - \frac{1}{t_c^2} \right).
\]
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Consider the stable autoregressive process

\[ X_{n+1} = \theta X_n + \varepsilon_{n+1}, \quad |\theta| < 1 \]

where \((\varepsilon_n)\) is a sequence of iid \(\mathcal{N}(0, \sigma^2)\) random variables. If \(X_0\) is independent of \((\varepsilon_n)\) with \(\mathcal{N}(0, \sigma^2/(1 - \theta^2))\) distribution, \((X_n)\) is a centered stationary Gaussian process with spectral density given, for all \(x \in \mathbb{T}\), by

\[ g(x) = \frac{\sigma^2}{2\pi(1 + \theta^2 - 2\theta \cos x)}. \]

The process \((X_n)\) is positive recurrent.
Let $\hat{\theta}_n$ be the least squares estimator of the parameter $\theta$

$$
\hat{\theta}_n = \frac{\sum_{k=1}^{n} X_k X_{k-1}}{\frac{1}{n} \sum_{k=1}^{n} X^2_{k-1}}.
$$

We have $\hat{\theta}_n \to \theta$ a.s. and $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{L} \mathcal{N}(0, 1 - \theta^2)$. One can also estimate $\theta$ by the Yule-Walker estimator

$$
\tilde{\theta}_n = \frac{\sum_{k=1}^{n} X_k X_{k-1}}{\frac{1}{n} \sum_{k=0}^{n} X^2_k}.
$$
\[ a = \frac{\theta - \sqrt{\theta^2 + 8}}{4} \quad \text{and} \quad b = \frac{\theta + \sqrt{\theta^2 + 8}}{4}. \]

**Theorem (Bercu-Gamboa-Rouault, SPA 1997)**

- \( \hat{\theta}_n \) satisfies an \textbf{LDP} with rate function

\[
J(c) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in [a, b], \\
\log |\theta - 2c| & \text{otherwise.}
\end{cases}
\]

- \( \tilde{\theta}_n \) satisfies an \textbf{LDP} with rate function

\[
l(c) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in ]-1, 1[, \\
+\infty & \text{otherwise.}
\end{cases}
\]
a = \frac{\theta - \sqrt{\theta^2 + 8}}{4} \quad \text{and} \quad b = \frac{\theta + \sqrt{\theta^2 + 8}}{4}.

Theorem (Bercu-Gamboa-Rouault, SPA 1997)

\( \hat{\theta}_n \) satisfies an LDP with rate function

\[
J(c) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in [a, b], \\
\log |\theta - 2c| & \text{otherwise.}
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\( \tilde{\theta}_n \) satisfies an LDP with rate function

\[
l(c) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right) & \text{if } c \in ]-1, 1[, \\
+\infty & \text{otherwise.}
\end{cases}
\]
Least squares and Yule-Walker
The sequence \( (\tilde{\theta}_n) \) satisfies an SLDP. For all \( c \in \mathbb{R} \) with \( c > \theta \) and \( |c| < 1 \), it exists a sequence \( (d_{c,k}) \) such that for any \( p \geq 1 \) and \( n \) large enough

\[
P(\tilde{\theta}_n \geq c) = \frac{\exp(-nl(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + O \left( \frac{1}{n^{p+1}} \right) \right]
\]

\[
t_c = \frac{c(1 + \theta^2) - \theta(1 - c^2)}{1 - c^2},
\]

\[
\sigma_c^2 = \frac{1 - c^2}{(1 + \theta^2 - 2\theta c)^2},
\]

\[
H(c) = -\frac{1}{2} \log \left( \frac{(1 - c\theta)^4}{(1 - \theta)^2(1 + \theta^2 - 2\theta c)(1 - c^2)^2} \right).
\]
Comparison with Bahadur-Rao

All the coefficients \((d_{c,k})\) may be given as functions of the derivatives \(\ell_k = L^{(k)}(t_c)\) and \(h_k = H^{(k)}(t_c)\). For example,

\[
\sigma_c^2 = \ell_2
\]

and the first coefficient

\[
d_{c,1} = \frac{1}{\sigma_c^2} \left( -\frac{h_2}{2} - \frac{h_1^2}{2} + \frac{\ell_3 h_1}{2\sigma_c^2} + \frac{h_1}{t_c} + \frac{\ell_4}{8\sigma_c^2} - \frac{5\ell_3^2}{24\sigma_c^4} - \frac{\ell_3}{2t_c\sigma_c^2} - \frac{1}{t_c^2} \right).
\]
Yule-Walker, Stable autoregressive process
Yule-Walker, Stable autoregressive process

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- Explosive autoregressive process

Graph showing Yule-Walker stable case.
Consider the explosive autoregressive process

\[ X_{n+1} = \theta X_n + \varepsilon_{n+1}, \quad |\theta| > 1 \]

where \((\varepsilon_n)\) is a sequence of iid \(N(0, \sigma^2)\) random variables. The process \((X_n)\) is transient.

**Theorem (White, AMS 1958)**

The Yule-Walker estimator satisfies

\[ \hat{\theta}_n \xrightarrow{a.s.} \frac{1}{\theta} \]

In addition,

\[ |\theta|^n(\hat{\theta}_n - \frac{1}{\theta}) \xrightarrow{\mathcal{L}} \frac{(\theta^2 - 1)}{\theta^2} \mathcal{C} \]

where \(\mathcal{C}\) stands for the Cauchy distribution.
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Theorem (Bercu, B 2001)

The sequence \((\tilde{\theta}_n)\) satisfies an LDP with rate function

\[
I(c) = \begin{cases} 
\frac{1}{2} \log\left(\frac{1 + \theta^2 - 2\theta c}{1 - c^2}\right) & \text{if } c \in ]-1, 1[, \ c \neq 1/\theta, \\
0 & \text{if } c = 1/\theta, \\
+\infty & \text{otherwise.}
\end{cases}
\]
Discontinuity point

EXPLOSIVE AUTOREGRESSIVE PROCESS

Bernard Bercu
Non stationary Ornstein-Uhlenbeck process
The sequence \( (\tilde{\theta}_n) \) satisfies an SLDP. For all \( c \in \mathbb{R} \) with \( c > 1/\theta \) and \( |c| < 1 \), it exists a sequence \((d_{c,k})\) such that for any \( p \geq 1 \) and \( n \) large enough

\[
P(\tilde{\theta}_n \geq c) = \frac{\exp(-nl(c) + H(c))}{\sigma_c t_c \sqrt{2\pi n}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{n^k} + O \left( \frac{1}{n^{p+1}} \right) \right]
\]

\[
t_c = \frac{(\theta c - 1)(\theta - c)}{1 - c^2},
\]

\[
\sigma_c^2 = \frac{1 - c^2}{(1 + \theta^2 - 2\theta c)^2},
\]

\[
H(c) = -\frac{1}{2} \log \left( \frac{(\theta c - 1)^2}{(1 + \theta^2 - 2\theta c)(1 - c^2)} \right).
\]
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Consider the stable Ornstein-Uhlenbeck process

\[ dX_t = \theta X_t \, dt + dB_t, \quad \theta < 0 \]

where \((B_t)\) is a standard Brownian motion and the initial state \(X_0 = 0\). The process \((X_T)\) is **positive recurrent**. We study the SLDP for the energy

\[ S_T = \int_0^T X_t^2 \, dt \]

and for the maximum likelihood estimator of \(\theta\)

\[ \hat{\theta}_T = \frac{\int_0^T X_t \, dX_t}{\int_0^T X_t^2 \, dt} = \frac{X_T^2 - T}{2 \int_0^T X_t^2 \, dt}. \]
**Theorem**

*We have the SLLN* \( S_T / T \to -1 / 2\theta \) a.s. *Moreover, we have the CLT*

\[
\sqrt{T} \left( \frac{S_T}{T} + \frac{1}{2\theta} \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, -\frac{1}{2\theta^3} \right).
\]

---

**Theorem**

*We have the SLLN* \( \hat{\theta}_T \to \theta \) a.s. *Moreover, we have the CLT*

\[
\sqrt{T} \left( \hat{\theta}_T - \theta \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, -2\theta \right).
\]
Stable Ornstein-Uhlenbeck process

Theorem (Bryc-Dembo, JTP 1997)

The sequence \((S_T / T)\) satisfies an LDP with rate function

\[
l(c) = \begin{cases} 
\frac{(2\theta c + 1)^2}{8c} & \text{if } c > 0, \\
+\infty & \text{otherwise.}
\end{cases}
\]

Theorem (Florens-Pham, JAP 1999)

The sequence \((\hat{\theta}_T)\) satisfies an LDP with rate function

\[
l(c) = \begin{cases} 
-\frac{(c - \theta)^2}{4c} & \text{if } c < \theta/3, \\
2c - \theta & \text{otherwise.}
\end{cases}
\]
Theorem (Bercu-Rouault, TPA 2002)

The sequence \( (S_T / T) \) satisfies an \textbf{SLDP}. For all \( c > -1/2\theta \), it exists a sequence \( (b_{c,k}) \) such that, for any \( p \geq 1 \) and \( T \) large enough

\[
P(S_T \geq cT) = \frac{\exp(-Tl(c) + H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{b_{c,k}}{T^k} + \mathcal{O} \left( \frac{1}{T^{p+1}} \right) \right]
\]

\[
t_c = \frac{4\theta^2 c^2 - 1}{8c^2}, \quad H(c) = -\frac{1}{2} \log \left( \frac{1}{2} (1 - 2\theta c) \right)
\]

\[
\sigma_c^2 = 4c^3.
\]
Theorem (Bercu-Rouault, TPA 2002)

The sequence \((\hat{\theta}_T)\) satisfies an SLDP. For all \(\theta < c < \theta/3\), it exists a sequence \((d_{c,k})\) such that, for any \(p \geq 1\) and \(T\) large enough

\[
P(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + H(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]
\]

\[
t_c = \frac{c^2 - \theta^2}{2c}, \quad H(c) = -\frac{1}{2} \log \left( \frac{(c + \theta)(3c - \theta)}{4c^2} \right)
\]

\[
\sigma_c^2 = -1/2c. \text{ Similar expansion holds for } c > \theta/3 \text{ with } c \neq 0.
\]
Theorem (Bercu-Rouault, TPA 2002)

For \( c = 0 \), it exists a sequence \( (b_k) \) such that, for any \( p \geq 1 \) and \( T \) large enough

\[
P(\hat{\theta}_T \geq 0) = \frac{\exp(\theta T)}{\sqrt{\pi T} \sqrt{-\theta}} \left[ 1 + \sum_{k=1}^{p} \frac{b_k}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right].
\]

For \( c = \theta / 3 \), it exists a sequence \( (d_k) \) such that, for any \( p \geq 1 \) and \( T \) large enough

\[
P(\hat{\theta}_T \geq c) = \frac{\exp(-Tl(c))}{4\pi T^{1/4} \tau_\theta} \left[ 1 + \sum_{k=1}^{2p} \frac{d_k}{(\sqrt{T})^k} + O \left( \frac{1}{T^{p+\sqrt{T}}} \right) \right]
\]

where \( \tau_\theta = (-\theta / 3)^{1/4} / \Gamma(1/4) \).
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Consider the explosive Ornstein-Uhlenbeck process

\[ dX_t = \theta X_t \, dt + dB_t, \quad \theta > 0 \]

where \((B_t)\) is a standard Brownian motion and \(X_0 = 0\). The process \((X_T)\) is transient.

**Theorem (Feigin, AAP 1976)**

We have

\[ \widehat{\theta}_T \overset{\text{a.s.}}{\to} \theta \]

In addition,

\[ \exp(\theta T) \left( \widehat{\theta}_T - \theta \right) \overset{\mathcal{L}}{\to} 2\theta C \]

where \(C\) stands for the Cauchy distribution.
The maximum likelihood estimator, Explosive case

**Theorem (Bercu-Coutin-Savy, SPA 2012)**

The sequence \((\hat{\theta}_T)\) satisfies an LDP with rate function

\[
l(c) = \begin{cases} 
-\frac{(c - \theta)^2}{4c} & \text{if } c \leq -\theta, \\
\theta & \text{if } |c| < \theta, \\
0 & \text{if } c = \theta, \\
2c - \theta & \text{if } c > \theta.
\end{cases}
\]

**Remark.** The size of the jump is precisely \(\theta\).
Discontinuity point

EXPLOSIVE ORNSTEIN–UHLENBECK PROCESS

DISCONTINUITY POINT

-6 -5 -4 -3 -2 -1 0 1 2 3 4

0
0.5
1
1.5
2
2.5
3
3.5
4

e

EXPLOSIVE ORNSTEIN–UHLENBECK PROCESS
DISCONTINUITY POINT

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Non stationary Ornstein-Uhlenbeck process
The maximum likelihood estimator, Explosive case

Theorem (Bercu-Coutin-Savy, SPA 2012)

The sequence \( (\hat{\theta}_T) \) satisfies an SLDP. For all \( c > \theta \), it exists a sequence \( (d_{c,k}) \) such that, for any \( p \geq 1 \) and \( T \) large enough

\[
\mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + K(c))}{\sigma_c t_c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right]
\]

\[
\sigma_c^2 = \frac{c^2}{2(2c - \theta)^3}, \quad t_c = 2(c - \theta),
\]

\[
K(c) = -\frac{1}{2} \log \left( \frac{(c - \theta)(3c - \theta)}{4c^2} \right).
\]

Similar expansion holds for \( c < -\theta \) or \( |c| < \theta \) with \( c \neq 0 \).
The maximum likelihood estimator, Explosive case

**Theorem (Bercu-Coutin-Savy, SPA 2012)**

- For $c = 0$, it exists a sequence $(b_k)$ such that, for any $p \geq 1$ and $T$ large enough
  \[
  \mathbb{P}(\hat{\theta}_T \geq 0) = 2 \exp(-T\theta)\sqrt{\theta T} \frac{\sqrt{T}}{\sqrt{\pi}} \left[ 1 + \sum_{k=1}^{p} \frac{b_k}{(\nu_\theta(T))^k} + \mathcal{O}\left(\frac{1}{(\nu_\theta(T))^{p+1}}\right) \right]
  \]
  where $\nu_\theta(T) = \exp(2\theta T)/T$.

- For $c = -\theta$, it exists a sequence $(d_k)$ such that, for any $p \geq 1$ and $T$ large enough
  \[
  \mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-T\theta)}{\sqrt{2\pi} T^{1/4} \tau_\theta} \left[ 1 + \sum_{k=1}^{2p} \frac{d_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p \sqrt{T}}\right) \right]
  \]
  where $\tau_\theta = (\theta)^{-1/8}/\Gamma(1/4)$.
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Consider the fractional Ornstein-Uhlenbeck process

\[ dX_t = \theta X_t \, dt + dB^H_t, \quad \theta < 0 \]

where \((B^H_t)\) is a fractional Brownian motion with Hurst parameter \(0 < H < 1\), \((B^H_t)\) is a Gaussian process with continuous paths such that \(B^H_0 = 0\), \(\mathbb{E}[B^H_t] = 0\) and

\[ \mathbb{E}[B^H_t B^H_s] = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right). \]

The weighting function

\[ w(t, s) = w^{-1}_H s^{-H+1/2} (t - s)^{-H+1/2} \]

plays a crucial role for stochastic calculus associated with \((B^H_t)\).
A Gaussian martingale

For all $t > 0$ and $H > 1/2$, let

$$M_t = \int_0^t w(t, s) dB_s^H.$$  

Then, $(M_t)$ is a Gaussian martingale with quadratic variation

$$\langle M \rangle_t = \frac{t^{2-2H}}{\lambda_H}$$  

where

$$\lambda_H = \frac{8H(1-H)\Gamma(1-2H)\Gamma(H+1/2)}{\Gamma(1/2-H)}$$

and $\Gamma$ stands for the classical gamma function.
For all \( t > 0 \), let

\[
Y_t = \int_0^t w(t, s) \, dX_s
\]

\[
Q_t = \frac{\ell_H}{2} \left( t^{2H-1} Y_t + \int_0^t s^{2H-1} \, dY_s \right)
\]

where \( \ell_H = \lambda_H / (2(1 - H)) \). The energy is given by

\[
S_T = \int_0^T Q_t^2 \, d\langle M \rangle_t
\]

while the maximum likelihood estimator of \( \theta \) is

\[
\hat{\theta}_T = \frac{\int_0^T Q_t \, dY_t}{\int_0^T Q_t^2 \, d\langle M \rangle_t}.
\]
Strong laws and Central Limit Theorems

Theorem (Bercu-Coutin-Savy, TPA 2011)

We have the SLLN $S_T / T \longrightarrow -1/2\theta$ a.s. Moreover, we have the CLT

$$\sqrt{T} \left( \frac{S_T}{T} + \frac{1}{2\theta} \right) \xrightarrow{L} \mathcal{N} \left( 0, -\frac{1}{2\theta^3} \right).$$

Theorem (Brouste-Kleptsyna-Le Breton, SISP 2002 and 2010)

We have the SLLN $\hat{\theta}_T \longrightarrow \theta$ a.s. Moreover, we have the CLT

$$\sqrt{T} \left( \hat{\theta}_T - \theta \right) \xrightarrow{L} \mathcal{N} \left( 0, -2\theta \right).$$
The energy, Stable case

Theorem (Bercu-Coutin-Savy, TPA 2011)

The sequence \((S_T/T)\) satisfies an **LDP** with rate function

\[
I(c) = \begin{cases} 
\frac{(2\theta c + 1)^2}{8c} & \text{if } 0 < c \leq -\frac{1}{2\theta\delta_H}, \\
\frac{c\theta^2}{2} (1 - \delta_H^2) + \frac{\theta}{2} (1 - \delta_H) & \text{if } c \geq -\frac{1}{2\theta\delta_H}, \\
+\infty & \text{otherwise.}
\end{cases}
\]

where \(\delta_H = (1 - \sin(\pi H))/(1 + \sin(\pi H))\). In addition, the sequence \((S_T/T)\) satisfies an **SLDP**.

**Remark.** In the particular case \(H = 1/2, \delta_H = 0\) and the **LDP** for \((S_T/T)\) is exactly the one established by Bryc and Dembo.
The maximum likelihood estimator, Stable case

Theorem (Bercu-Coutin-Savy, TPA 2011)

The sequence $(\hat{\theta}_T)$ satisfies an LDP with rate function

$$l(c) = \begin{cases} 
-\frac{(c - \theta)^2}{4c} & \text{if } c < \theta/3, \\
2c - \theta & \text{otherwise.}
\end{cases}$$

Remark. One can observe that $(\hat{\theta}_T)$ shares the same LDP than the one established by Florens-Landais and Pham for $H = 1/2$. 
Theorem (Bercu-Coutin-Savy, TPA 2011)

The sequence $(\hat{\theta}_T)$ satisfies an SLDP. For all $\theta < c < \theta/3$, it exists a sequence $(b_{c,k}^H)$ such that, for any $p > 0$ and $T$ large enough,

$$
\mathbb{P}(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + J(c) + K_H(c))}{\sigma_c t_c \sqrt{2\pi T}}
$$

$$
\left[ 1 + \sum_{k=1}^{p} \frac{b_{c,k}^H}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right]
$$

where $\sigma_c^2 = -1/2c$, $p_H = (1 - \sin(\pi H))/\sin(\pi H)$,

$$
t_c = \frac{c^2 - \theta^2}{2c}, \quad J(c) = -\frac{1}{2} \log \left( \frac{(c + \theta)(3c - \theta)}{4c^2} \right),
$$

$$
K_H(c) = -\frac{1}{2} \log \left( 1 + p_H \frac{(c - \theta)^2}{4c^2} \right).
$$
Theorem (Bercu-Coutin-Savy, TPA 2011)

For all \( c > \theta/3 \) with \( c \neq 0 \), it exists a sequence \( (d^H_{c,k}) \) such that, for any \( p > 0 \) and \( T \) large enough,

\[
P(\hat{\theta}_T \geq c) = \frac{\exp(-TI(c) + P(c)) \sqrt{\sin(\pi H)}}{\sigma^c t^c \sqrt{2\pi T}} \left[ 1 + \sum_{k=1}^{p} \frac{d^H_{c,k}}{T^k} + O \left( \frac{1}{T^{p+1}} \right) \right]
\]

where

\[
t^c = 2(c - \theta), \quad (\sigma^c)^2 = \frac{c^2}{2(2c - \theta)^3},
\]

\[
P(c) = -\frac{1}{2} \log \left( \frac{(c - \theta)(3c - \theta)}{4c^2} \right)
\]
Theorem (Bercu-Coutin-Savy, TPA 2011)

1. For $c = 0$, it exists a sequence $(b^H_k)$ such that, for any $p \geq 1$ and $T$ large enough

$$
\mathbb{P} \left( \hat{\theta}_T \geq 0 \right) = \frac{\exp(\theta T) \sqrt{\sin(\pi H)}}{\sqrt{\pi} T^{\sqrt{-\theta}}} \left[ 1 + \sum_{k=1}^{p} \frac{b^H_k}{T^k} + \mathcal{O} \left( \frac{1}{T^{p+1}} \right) \right].
$$

2. For $c = \theta/3$, it exists a sequence $(d^H_k)$ such that, for any $p \geq 1$ and $T$ large enough, and $\tau_\theta = (-\theta/3)^{1/4}/\Gamma(1/4)$,

$$
\mathbb{P} \left( \hat{\theta}_T \geq c \right) = \frac{\exp(-TI(c)) \sqrt{\sin(\pi H)}}{4\pi T^{1/4} \tau_\theta} \left[ 1 + \sum_{k=1}^{2p} \frac{d^H_k}{(\sqrt{T})^k} + \mathcal{O} \left( \frac{1}{T^{p+1} \sqrt{T}} \right) \right].
$$
Many thanks for your attention !!!!