On Statistical Inference for some Nonlinear SPDEs

Igor Cialenco

Department of Applied Mathematics
Illinois Institute of Technology
igor@math.iit.edu

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Outline of the talk

- The problem: estimating the drift for SPDEs
- Motivation and General/Abstract Scheme
- Stochastic (fractional) Burgers
  - additive noise
  - spectral multiplicative noise
  - geometric multiplicative noise
- Simulations

$$dU(t) = \theta A U(t)dt + F(U)dt + \sigma dW(t), \quad U(0) = U_0$$

- given stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$
- assume that $U(\omega, t)$ belongs to some "suitable" Hilbert space $\mathcal{H}$; in particular $U = U(\omega, t, x)$
- $(-A)$ a linear, selfadjoint, positive-defined (think (-Laplace)$^\beta$) in $\mathcal{H}$ with eigenfunctions $\{\Phi_k\}_{k\geq 1}$ CONS in $\mathcal{H}$
- $\sigma dW(t) = \sum_{k\geq 1} \sigma_k \Phi_k dW_k(t)$, $W_k, k \in \mathbb{N}$ ind. Brownian Motions
- F maybe nonlinear; σ **known**
- U observed for all $t \in [0, T]$ - **continuous observations**

Goal:

Find estimators $\hat{\theta}(\omega)$, $\omega \in \Omega$, for parameters θ by **observing a single outcome** $u = u(\omega, t) \in \mathcal{H}$ over a finite time horizon $t \in [0, T]$.

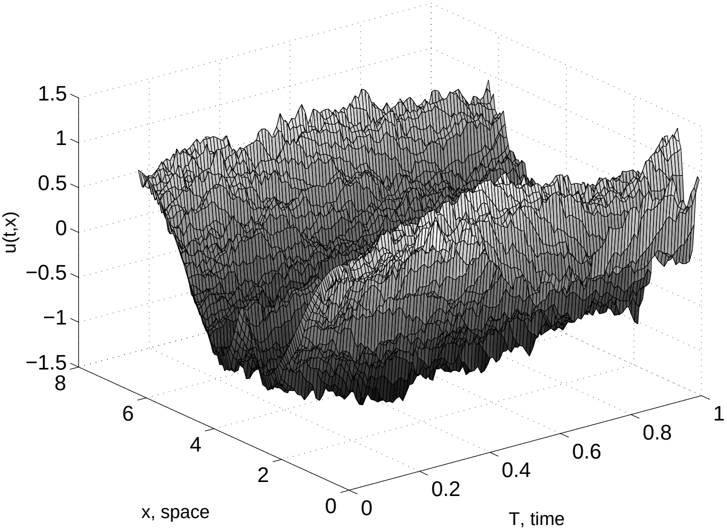

Figure: *Additive noise. Burgers Equation.*

$\nu = 0.01,\ \beta = 1,\ \gamma = 1,\ U(0) = \cos(2\pi x) + 1/\cosh(4(2\pi x - \pi))$

Formal Procedure to Derive an Estimator

- Project the full system down to N dimensions $P_N(\mathcal{H}) = \mathcal{H}_N \simeq \mathbb{R}^N$

$$dU^N = (\theta A U^N + \Psi_N)dt + P_N \sigma dW, \quad U(0) = U_0$$

- Let $\mathbb{P}_\theta^{N,T}(\cdot) = \mathbb{P}(U^N \in \cdot)$ be the measure on $C([0,T];\mathbb{R}^N)$ generated by U^N;
  $\mathbb{P}_\theta^T$ be the measure generated by U on $C([0,T];\mathcal{H})$.
- Usually (at least in linear case), we can prove that $\mathbb{P}_{\theta_1}^{N,T} \sim \mathbb{P}_{\theta_2}^{N,T}$
  Hence, get MLE type estimators $\widehat{\theta}_{N,T}$.
- Usually (at least in linear case) $\mathbb{P}_{\theta_1}^T \perp \mathbb{P}_{\theta_2}^T$;
  An indication that the true parameter θ can be found exactly.

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Reasonable ansatz:

$$\widehat{\theta}_{N,T} \xrightarrow[N\to\infty]{} \theta$$

Formal Procedure to Derive an Estimator in Nonlinear Case

- Formally treat $\Psi_N = P_N F(U)$ as an external and known quantity (independent of θ)
- Assume that $P_N \sigma$ is invertible on H_N
- Take $G := P_N \sigma(U)(P_N \sigma(U))^*$ and assume it commutes with A
- For a reference values θ_0, apply (formally) Girsanov Theorem and get the 'Likelihood Ratio' (Radon-Nikodym derivative) $d\mathbb{P}_\theta^{N,T}/d\mathbb{P}_{\theta_0}^{N,T}$
- Maximize the Log-Likelihood Ratio
$\widetilde{\theta}_{N,T}(\omega) := \arg\max_\theta d\mathbb{P}_\theta^{N,T}/d\mathbb{P}_{\theta_0}^{N,T}(\omega)$

$$\frac{d\mathbb{P}_{\theta}^{N,T}}{d\mathbb{P}_{\theta_0}^{N,T}} = \exp\Big[- \int_0^T (\theta-\theta_0)\langle AU^N, GdU^N(t)\rangle$$

$$- \frac{1}{2}\int_0^T (\theta^2-\theta_0^2)\langle AU^N, GAU^N dt\rangle$$

$$- \int_0^T (\theta-\theta_0)\langle AU^N, G\psi^N dt\rangle\Big],$$

$$\widetilde{\theta}_N = \frac{\int_0^T \langle AU^N, GdU^N\rangle + \int_0^T \langle AU^N, GP_N\mathbf{F}(\mathbf{U})\rangle dt}{\int_0^T \langle AU_N, GAU^N\rangle dt}$$

$$\frac{d\mathbb{P}_{\theta}^{N,T}}{d\mathbb{P}_{\theta_0}^{N,T}} = \exp\Big[-\int_0^T (\theta-\theta_0)\langle AU^N, GdU^N(t)\rangle$$
$$-\frac{1}{2}\int_0^T (\theta^2-\theta_0^2)\langle AU^N, GAU^N dt\rangle$$
$$-\int_0^T (\theta-\theta_0)\langle AU^N, G\psi^N dt\rangle\Big],$$

$$\widetilde{\theta}_N = \frac{\int_0^T \langle AU^N, GdU^N\rangle + \int_0^T \langle AU^N, GP_N\mathbf{F}(\mathbf{U})\rangle dt}{\int_0^T \langle AU_N, GAU^N\rangle dt}$$

Main Idea #1: Modified MLE

$$\widetilde{\theta}_N = -\frac{\int_0^T A^{1+\rho_1}U_N G_N^{\rho_2} dU_N + \int_0^T A^{1+\rho_1}U_N G_N^{\rho_2} P_N F(U))dt}{\int_0^T A^{1+\rho_1}U_N G_N^{\rho_2} AU_N dt}$$

for some ρ_1, ρ_2.

Motivated by MLE type estimator

$$\hat{\theta}_{1,N} = -\frac{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} dU_N + \int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} P_N B(U))dt}{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} A U_N dt},$$

$$\hat{\theta}_{2,N} = -\frac{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} dU_N + \int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} P_N B(U_N))dt}{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} A U_N dt},$$

$$\hat{\theta}_{3,N} = -\frac{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} dU_N}{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} A U_N dt}.$$

Choose ρ_1, ρ_2 such that we can prove

$$\widehat{\theta}_{i,N} \longrightarrow \theta, \qquad \text{as } N \to \infty,$$

for $i = 1, 2, 3$.

$$\hat{\theta}_{2,N} = \theta + \frac{\int_0^T \langle A^{1+\rho_1} U^N, G^{\rho_2} \sum_{j=1}^N \sigma_j(U)\Phi_j dW_j(t)\rangle}{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} A U_N dt}$$
$$+ \frac{\int_0^T \langle A^{1+\rho_1} U^N, G^{\rho_2} (F^N(U) - F^N(U^N))\rangle dt}{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} A U_N dt}$$

$$\hat{\theta}_{3,N} = \theta + \frac{\int_0^T \langle A^{1+\rho_1} U^N, G^{\rho_2} \sum_{j=1}^N \sigma_j(U)\Phi_j dW_j(t)\rangle}{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} A U_N dt}$$
$$+ \frac{\int_0^T \langle A^{1+\rho_1} U^N, G^{\rho_2} F^N(U^N)\rangle dt}{\int_0^T A^{1+\rho_1} U_N G_N^{\rho_2} A U_N dt}$$

- Need to show that each of 'the excess term converge to zero'
- Successfully applied to:
  - Stochastic Heat Equation (or in general, linear parabolic SPDE)
  - Stochastic Navier Stokes Equations, 2D, additive noise (SPA 2011)
  - **Stochastic Fractional Burgers Equation, additive and multiplicative noise**

$$\mathrm{d}U + (\nu(-\Delta)^{\beta}U + UU_x)\ \mathrm{d}t = \sigma \mathrm{d}W, \quad U(0) = U_0,$$

on the unit circle, $\mathbb{S}^1$ (periodic boundary conditions).

- $\Lambda_s := (-\Delta)^{s/2}$. The eigenfunctions of $\Lambda_{1/2}$, $\Phi_k(x) = e^{2\pi i k x}$ and eigenvalues $\lambda_k = 2\pi|k|$
- The usual Sobolev spaces on the unit circle

$$\dot{\mathbb{H}}^s = D(\Lambda_s) = \left\{ f \in L^2(\mathbb{S}^1) :\ \|\Lambda_s f\| < \infty \text{ and } \int_{\mathbb{S}^1} f(x)\ \mathrm{d}x = 0 \right\}.$$

- Put $B(u,v) = uv_x$; P_N the projection of $L^2(\mathbb{S}^1)$ onto $\mathrm{Span}\,\{\Phi_{-N}, \Phi_{-N+1}, \ldots, \Phi_{N-1}, \Phi_N\}$; $U_N := P_N U$
- Toy model in geophysical modeling, turbulent flows etc.

Existence and Uniqueness of the Solution

Theorem (C., Glatt-Holtz, Kaligotla '12)

Assume that

$$1/2 < \beta \leq 1, \quad s > 3/2 - \beta,$$

U_0 *is an* $\dot{\mathbb{H}}^s$*-valued* $\mathcal{F}_0$*-measurable, and some additional assumptions on the noise* $\sigma \mathrm{d}W$*.*

Then, there exists a unique $\dot{\mathbb{H}}^s$ *valued, solution* U *of Burgers equation such that*

$$U \in C([0,\infty); \dot{\mathbb{H}}^s) \cap L^2_{loc}([0,\infty); \dot{\mathbb{H}}^{s+\beta}), \quad \textit{almost surely.}$$

Main idea #2: Splitting argument

Decompose $U = \bar{U} + R =$ linear + nonlinear

$$dR + (\nu(-\Delta)^{\beta} R - UU_x)dt = \sigma_R(R)dW, \quad R(0) = R_0,$$
$$d\bar{U} + \nu(-\Delta)^{\beta}\bar{U}dt = \sigma_{\bar{U}}(\bar{U})dW, \quad \bar{U}(0) = \bar{U}_0$$

and assume that $\sigma_R(R) + \sigma_{\bar{U}}(\bar{U}) = \sigma(R + \bar{U})$.

- Find explicit and exact rates for moments of the linear part
- Under 'correct/smart' splitting, R is 'more regular' than $\bar{U}$

$$\bar{U} \in C([0,T], \dot{\mathbb{H}}^{s}) \cap L^2([0,T]; \dot{\mathbb{H}}^{s+\beta}),$$
$$R \in C([0,T], \dot{\mathbb{H}}^{s+2\beta-1}) \cap L^2([0,T]; \dot{\mathbb{H}}^{s+3\beta-1})$$

Additive Noise

$$\begin{cases} \mathrm{d}U + (\nu(-\Delta)^\beta U + UU_x)\ \mathrm{d}t = \sum_{k\in\mathbb{Z}} \lambda_k^{-\gamma}\Phi_k\ \mathrm{d}W_k, & t\in[0,T] \\ U(0) = U_0. \end{cases}$$

Let $U_0 \in \dot{\mathbb{H}}^{s_0}$, for some $s_0 \in \mathbb{R}$. Consider the following set of assumptions on β, γ, α and s_0:

A.1. $s_0 > \frac{3}{2} - \beta, \quad \gamma > \frac{1}{2} + s_0, \quad \frac{1}{2} < \beta \leq 1$

A.2. $\alpha > \gamma - \beta - \frac{1}{2}$

A.3. $\gamma < 3\beta + s_0$

A.2'. $\alpha > \gamma - \frac{\beta}{2} - \frac{1}{4}$

A.3'. $\gamma < 2\beta + s_0 - \frac{1}{2}$

Assumption A.1. guarantees existence and uniqueness;
A.1-A.3 are not contradictory; A.1, A.2' and A.3' are not contradictory;
We take in the general theory $\rho_1 = (\alpha - \gamma)/\beta$ and $\rho_2 = -1$.

Additive Noise: The Estimators

$$\hat{\nu}_{1,N} = -\frac{\int\limits_0^T \sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{2\beta+2\alpha} u_k \left(\mathrm{d}u_k + \widetilde{\psi}_k \mathrm{d}t\right)}{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{4\beta+2\alpha} \int_0^T u_k^2 \mathrm{d}t}$$

$$\hat{\nu}_{2,N} = -\frac{\int\limits_0^T \sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{2\beta+2\alpha} u_k \left(\mathrm{d}u_k + \psi_k \mathrm{d}t\right)}{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{4\beta+2\alpha} \int_0^T u_k^2 \mathrm{d}t}$$

$$\hat{\nu}_{3,N} = -\frac{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{2\beta+2\alpha} \int\limits_0^T u_k \mathrm{d}u_k}{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{4\beta+2\alpha} \int_0^T u_k^2 \mathrm{d}t}$$

$$= -\frac{\sum_{k\in\mathcal{Z}_N} \lambda_k^{2\beta+2\alpha} \left(u_k^2(T) - u_k^2(0) - T\lambda_k^{-2\gamma}\right)}{2\sum_{k\in\mathcal{Z}_N} \lambda_k^{4\beta+2\alpha} \int_0^T u_k^2(t)\mathrm{d}t},$$

where $\mathcal{Z}_N = \{-N, -N+1, \dots, N\}$, $[\widetilde{\psi}_k]_{k\in\mathcal{Z}_N} = P_N(B(U))$, $[\psi_k]_{k\in\mathcal{Z}_N} = P_N(B(U_{\lfloor N^p \rfloor}))$, and $\beta(p+1) - 1 + s_0 - \gamma > 0$.

Additive Noise: Main Result

Theorem (C., Glatt-Holtz, Kaligotla '12)

Assume that Assumption A.1 is satisfied. Then,

1. *(Consistency in number of Fourier modes) if Assumption A.2 and A.3 are satisfied, then*

$$\lim_{N\to\infty} \hat{\nu}_{1,N} = \lim_{N\to\infty} \hat{\nu}_{2,N} = \lim_{N\to\infty} \hat{\nu}_{3,N} = \nu \quad \textit{in probability.}$$

2. *(Asymptotic Normality) if Assumption A.2' and A.3' are fulfilled, then we also have that*

$$\lim_{N\to\infty} N^{\beta+\frac{1}{2}}(\hat{\nu}_{1,N} - \nu) \stackrel{d}{=} \lim_{N\to\infty} N^{\beta+\frac{1}{2}}(\hat{\nu}_{2,N} - \nu)$$
$$\stackrel{d}{=} \mathcal{N}\left(0, \frac{\nu}{T(2\pi)^{2\beta}} \frac{(2\beta+2\alpha-2\gamma+1)^2}{2\beta+4\alpha-4\gamma+1}\right).$$

Spectral Multiplicative Noise

$$\begin{cases} \mathrm{d}U + (\nu(-\Delta)^{\beta}U + UU_x)\ \mathrm{d}t = \sum_{k\in\mathbb{Z}} \lambda_k^{-\gamma} u_k \Phi_k \mathrm{d}W_k, \quad t \in [0,T] \\ U(0) = U_0, \end{cases}$$

where $U_0 \in L^2(\mathbb{S}^1)$, $\beta > 1/2$, and $u_k = \langle U, \Phi_k \rangle,\ k \in \mathbb{Z}$.

Assumption B.1: Let $U_0 \in \dot{\mathbb{H}}^{s_0}$, for some $s_0 \in \mathbb{R}$ such that

$$s_0 > \frac{3}{2} - \beta, \quad \gamma > \frac{1}{2}, \quad \frac{1}{2} < \beta \leq 1, \tag{2.1}$$

Assumption B.2: Assume that $U_0 \notin \dot{\mathbb{H}}^{s'}$ and $U \in L^2((0,T);\dot{\mathbb{H}}^{s''})$, and such that

$$\alpha + \beta - \gamma/2 \leq s'' \leq s' \leq \alpha + \beta + \frac{1}{4}$$

Assumption B.3: Assume that

$$s_0 \geq \alpha - \gamma$$

Assumption B.4: Assume that

$$s' \leq \alpha + \beta$$

Assumption B.5: Assume that

$$s_0 > \alpha - \gamma/2 + \beta/2 - 1/4$$

We take in the general theory $\rho_1 = \frac{\alpha+\gamma(\delta-1)/2}{\beta}$ and $\rho_2 = \frac{\delta-1}{2}$

Estimators

$$\widetilde{\nu}_{1,N} = -\frac{\int\limits_0^T \sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{2\beta+2\alpha} u_k^\delta \left(\mathrm{d}u_k + \widetilde{\psi}_k \mathrm{d}t\right)}{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{4\beta+2\alpha} \int_0^T u_k^{\delta+1}\mathrm{d}t}$$

$$\widetilde{\nu}_{2,N} = -\frac{\int\limits_0^T \sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{2\beta+2\alpha} u_k^\delta \left(\mathrm{d}u_k + \psi_k \mathrm{d}t\right)}{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{4\beta+2\alpha} \int_0^T u_k^{\delta+1}\mathrm{d}t}$$

$$\widetilde{\nu}_{3,N} = -\frac{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{2\beta+2\alpha} \int\limits_0^T u_k^\delta \mathrm{d}u_k}{\sum\limits_{k\in\mathcal{Z}_N} \lambda_k^{4\beta+2\alpha} \int_0^T u_k^{\delta+1}\mathrm{d}t},$$

where $[\widetilde{\psi}_k]_{k=-N}^N = P_N(B(U))$, and $[\psi_k]_{k=-N}^N = P_N(B(U^N))$.
In general, one needs $\delta+1\in\mathbb{N}$. For technical reasons we take $\delta=1$.

Theorem (C., Glatt-Holtz, Kaligotla '12)

The following hold true:

1. *(Consistency) if Assumptions B.1-B.3 are satisfied, then*

$$\lim_{N\to\infty} \widetilde{\nu}_{1,N} = \nu \quad \textit{in probability.}$$

2. *(Consistency) if Assumptions B.1-B.4 are satisfied, then*

$$\lim_{N\to\infty} \widetilde{\nu}_{2,N} = \lim_{N\to\infty} \widetilde{\nu}_{3,N} = \nu \quad \textit{in probability.}$$

3. *(Asymptotic Normality) if Assumptions B.1-B.5 are fulfilled, then we also have that*

$$\lim_{N\to\infty} \phi_N(\widetilde{\nu}_{1,N} - \nu) \stackrel{d}{=} \xi,$$

where $\phi_N = \left\|\Lambda_{\alpha+\beta}\bar{U}_N(0)\right\|^2$*, and* ξ *is a Gaussian random variable with mean zero and variance* $\nu\sum_{k\in\mathbb{Z}_0}\lambda_k^{2\beta-2\gamma+4\alpha}\bar{u}_k^4(0)$.

Geometric (multiplicative) Noise: Closed-form exact estimators

$$\mathrm{d}U + (\nu(-\Delta)^{\beta})U + B(U))\mathrm{d}t = U dW, \quad U(0) = U_0,$$

with $\beta \in (\frac{1}{2}, 1]$, and W **is a one dimensional Brownian motion**.

- Existence and uniqueness guaranteed if $U_0 \in \dot{\mathbb{H}}^{s^*},\ s^* > 3/2 - \beta$.
- We will derive a new class of estimators, called *closed-form exact estimators*, originally introduced and developed by [C., Lototsky, 2009].
- The next transformations are formal.
  Denote by $u_k := \langle U, \Phi_k \rangle,\ k \in \mathbb{Z}_0$, Then, the u_k Fourier coefficient satisfies

$$\mathrm{d}u_k + \nu\lambda_k^{2\beta} u_k \mathrm{d}t + \psi_k \mathrm{d}t = u_k dW(t), \quad u_k(0) = u_{k,0}, \quad k \in \mathbb{Z}_0,$$

where $\psi_k := \langle B(U), \Phi_k \rangle$, $P_N(B(U)) = (\psi_{-N}, \psi_{-N+1}, \dots, \psi_N)$.

By the Itô formula (applied formally), we deduce

$$\ln u_k = \ln u_k(0) - \nu\lambda_k^{2\beta}T - \int_0^T \frac{\psi_k}{u_k}\mathrm{d}t + W(T), \quad k \in \mathbb{Z}. \tag{2.2}$$

For any two distinct integers k, m, we consider the system of two equation generated by (2.2), and solve it for ν, eliminating $W(T)$. As a result, we get the following 'closed-form exact estimator' for the parameter ν

$$\check{\nu} = \frac{\ln \frac{u_k(0)u_m(T)}{u_m(0)u_k(T)} - \int_0^T \frac{\psi_k}{u_k}\mathrm{d}t + \int_0^T \frac{\psi_m}{u_m}\mathrm{d}t}{T(\lambda_k^{2\beta} - \lambda_m^{2\beta})}, \quad k \neq m, \ T > 0.$$

- We will assume that the above expression is well-defined.
- In linear case, $\check{\nu} = \nu$.
- We can not prove anything rigourously for these estimators
- However, the numerical simulations show that indeed these estimators are close to be 'exact'.

Geometric (multiplicative) Noise: MLE type estimators

$$\mathrm{d}U + (\nu(-\Delta)^{\beta})U + B(U))\mathrm{d}t = UdW,, \quad U(0) = U_0,$$

with $\beta \in (\frac{1}{2}, 1]$, and W **is a one dimensional Brownian motion**. Similarly as above, we can deduce MLE type estimators

$$\breve{\nu}_k = -\frac{\int_0^T u_k^{\delta}\mathrm{d}u_k + \int_0^T u_k^{\delta}\psi_k \mathrm{d}t}{\lambda_k^{2\beta}\int_0^T u_k^{\delta+1}\mathrm{d}t}, \quad k \in \mathbb{Z}_0,$$

for some fixed $\delta \geq -1$.

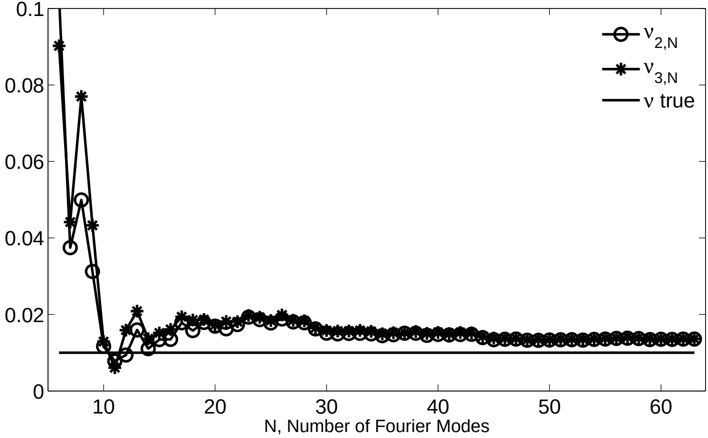

Figure: *Additive noise.*

$\nu = 0.01,\ \beta = 1,\ \gamma = 1,\ U(0) = \cos(2\pi x) + 1/\cosh(4(2\pi x - \pi))$

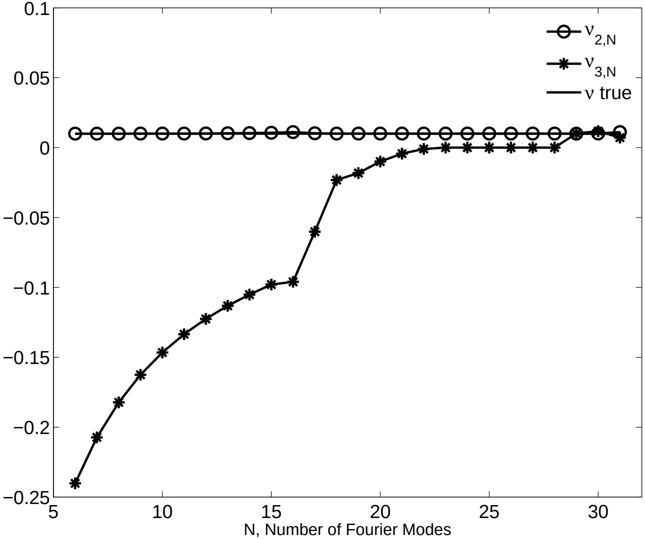

Figure: *Impact of the nonlinear correction. Additive noise.*
$\nu = 0.01,\ \beta = 1,\ \gamma = 50,\ U_{30}(0) = 0.1,\ \alpha = 50.$

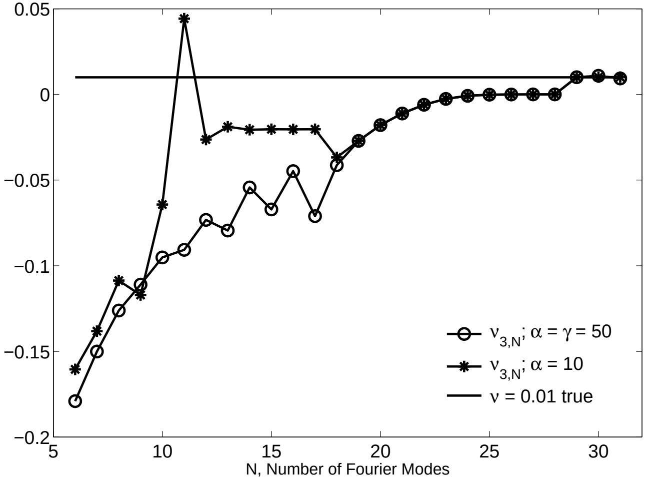

Figure: *Impact of the correction coefficient α. Additive noise.*
$\nu = 0.01,\ \beta = 1,\ \gamma = 50,\ U_{30}(0) = 0.1$

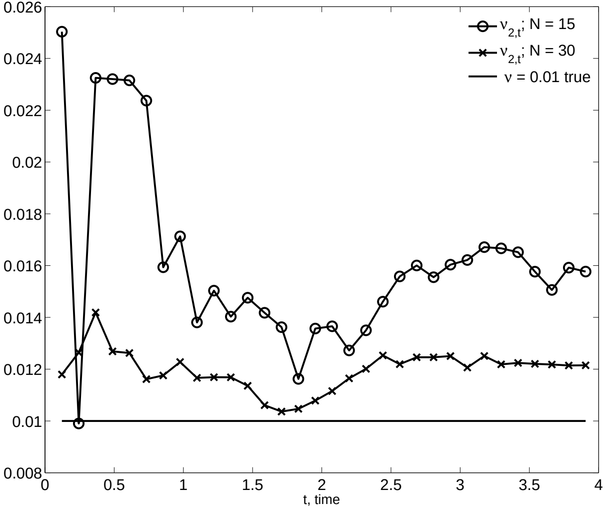

Figure: *Large time asymptotics. Additive noise.*
$\nu = 0.01,\ \beta = 1,\ \gamma = 1, \alpha = \gamma,\ U_k(0) = 1/k.$

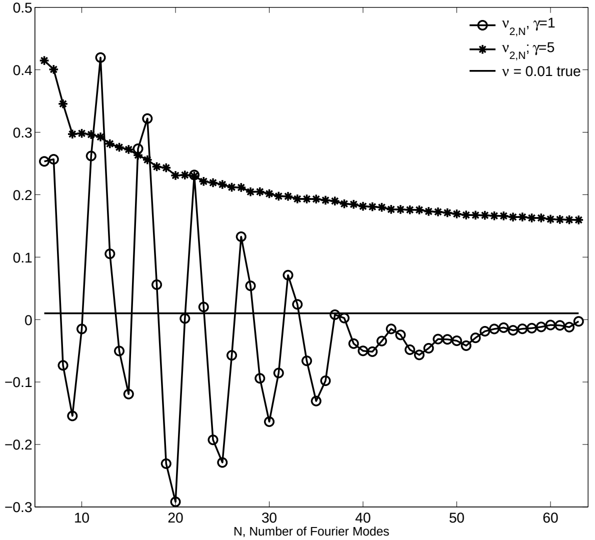

Figure: *Impact of parameter* γ*.* $\nu = 0.01,\ \beta = 0.6, \alpha = \gamma,\ \ U_1(0) = 1$

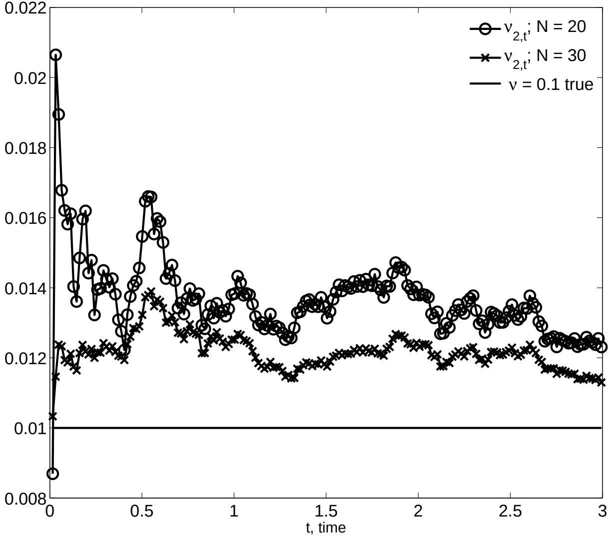

Figure: *Spectral multiplicative noise.*

$$\nu = 0.01, \quad \gamma = \alpha = 1, \ U_k(0) = 1/k, \delta = 1.$$

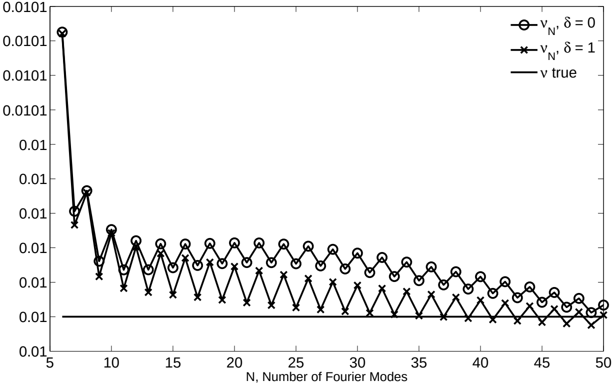

Figure: *Geometric noise. Multiplicative noise.*
$\nu = 0.01, \quad \gamma = 5, \ U_k(0) = 1/k.$

Closed-form exact estimators

k \ m	3	5	7	10
4	0.014893	0.012487	0.009115	0.009962
20	0.010053	0.010054	0.010056	0.009990
30	0.009954	0.009953	0.009951	0.009949

Table: *Burgers Equation, geometric noise.*
$\nu = 0.01,\ \gamma = 1,\ U_k(0) = 1/k,\ T = 0.1.$

Thank You !

The end of the talk ...
but not of the story