Asymptotic properties of the MLE for the autoregressive process coefficients under stationary Gaussian noise

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Statement of the problem

We consider the AR(p) process $(X_n,\ n \geq 1)$ defined as

- $X_n = \sum_{i=1}^{p} \vartheta_i X_{n-i} + \xi_n, \quad n \geq 1, \quad X_r = 0,$ $r = 0, -1, \cdots, -(p-1)$
- $E\xi_m\xi_n = c(m,n) = \rho(|\,n-m|), \quad \rho(0) = 1.$
- $c(.,.)$ is positive definite function.

Our goal is to study the large asymptotical properties of the Maximum Likelihood Estimator (MLE) $\hat{\vartheta}_N$ of $\vartheta = (\vartheta_1, \cdots, \vartheta_p)'$ based on the observation model $X^{(N)} = (X_1, \ldots, X_N)$

$$\hat{\vartheta}_N = \sup_{\vartheta \in \mathbb{R}^p} \mathcal{L}(\vartheta, X^{(N)}).$$

Assumption

- Let$f_\xi(\lambda)$ be the spectral density of ξ, we suppose that

$$|\int_{-\pi}^{\pi} \ln f_\xi(\lambda) d\lambda| < \infty \qquad (1)$$

or the sufficient condition

$$|\rho(n)| \leq Cn^{-\alpha}$$

Examples: The condition (1) is valid for any ARMA(p,q) process, fractional Gaussian noises(fGn) and mixed fGn.

result

- Let $p \geq 1$ and A_0 be the companion matrix, $r(\vartheta)$ be the spectral radius of A_0 and the parameter set be:

$$\Theta = \{\vartheta \in \mathbb{R}^p \,|\, r(\vartheta) < 1\}.$$

The MLE $\hat{\vartheta}_N$ is asymptotically normal, *i.e.*, for any $\vartheta \in \Theta$

$$\sqrt{N}\left(\hat{\vartheta}_N - \vartheta\right) \overset{law}{\Rightarrow} \mathcal{N}(0, \mathcal{I}^{-1}(\vartheta)),$$

where $\mathcal{I}(\vartheta)$ is the unique solution of Lyapunov equation:

$$\mathcal{I}(\vartheta) = A_0 \mathcal{I}(\vartheta) A_0^* + bb^*, \ b = \begin{pmatrix} 1 & 0_{1\times(p-1)} \end{pmatrix}^*$$

- Let $p = 1$, then $\hat{\vartheta}_N$ is uniformly consistent on compacts $\mathbb{K} \subset (-1, 1)$, *i.e.* for any $\nu > 0$,

$$\lim_{N\to\infty} \sup_{\vartheta \in \mathbb{K}} P_\vartheta^N \left\{ \left| \hat{\vartheta}_N - \vartheta \right| > \nu \right\} = 0,$$

and for any $q > 0$:

$$\lim_{N\to\infty} \sup_{\vartheta \in \mathbb{K}} \left| E_\vartheta \left| \sqrt{N}\left(\hat{\vartheta}_N - \vartheta\right) \right|^q - E\,|\eta|^q \right| = 0,$$

where $\eta \sim \mathcal{N}(0, 1 - \vartheta^2)$.

- let $p = 1$, the MLE $\hat{\vartheta}_N$ is strongly consistent *i.e.* for any $\vartheta \in \mathbb{R}$

$$\lim_{N\to\infty} \hat{\vartheta}_N = \vartheta.$$

Sketch of proof: Stationary Gaussian sequence (1)

For the Gaussian sequence $(\xi_n)_{n\geq 1}$, we will denote the innovation sequences of ξ defined as

$$\sigma_n\varepsilon_n = \xi_n - E(\xi_n \,|\, \xi_1, \ldots, \xi_{n-1}), \quad n \geq 1,$$

where $\varepsilon_n \sim \mathcal{N}(0,1)$ are independent. So there exist the deterministic kernel denoted $k(n,m)$ such that

$$\sigma_n\varepsilon_n = \sum_{m=1}^{n} k(n,m)\xi_m, \quad k(n,n) = 1.$$

Sketch of proof: Stationary Gaussian sequence (2)

Let us denote the partial correlation coefficient $-k(n,1) = \beta_{n-1},\ n \geq 1$, we have some relationships when $n \geq 1$

$$\sigma_n^2 = \prod_{m=1}^{n-1} (1 - \beta_m^2),$$

$$\sum_{m=1}^{n} k(n,m)\rho(m) = \beta_n \sigma_n^2,$$

$$k(n+1, n+1-m) = k(n, n-m) - \beta_n k(n,m).$$

Transformation of the observation

Let $Y_n = (X_n, X_{n-1}, \cdots, X_{n-p+1})^*$, then Y_n satisfies the first order autoregressive equation:

$$Y_n = A_0 Y_{n-1} + b\xi_n, \quad n \geq 1, \quad Y_0 = 0_{p\times 1}.$$

Let us introduce $(Z_n)_{n\geq 1}$ such that

$$Z_n = \sum_{m=1}^{n} k(n,m) Y_m,\ n \geq 1,$$

then the process $\zeta = (\zeta_n,\ n \geq 1)$ defined by $\zeta_n = \begin{pmatrix} Z_n \\ \sum_{r=1}^{n-1} \beta_r Z_r \end{pmatrix}$ is a 2p-dimansional Markovian process which satisfies the following equation:

$$\zeta_n = A_{n-1}\zeta_{n-1} + \ell\sigma_n\varepsilon_n,\ n \geq 1,\ \zeta_0 = 0_{2p\times 1},$$

where

$$A_n = \begin{pmatrix} A_0 & \vartheta\beta_n \\ \beta_n Id_{p\times p} & Id_{p\times p} \end{pmatrix}, \quad \ell = b$$

Likelihood function and MLE

The log-likelihood function is

$$\ln \mathcal{L}(\vartheta, X^{(N)}) = -\frac{1}{2}\sum_{n=1}^{N} \left(\frac{\ell^*(\zeta_n - A_{n-1}\zeta_{n-1})}{\sigma_n} \right)^2 - \frac{N}{2}\ln 2\pi - \frac{1}{2}\sum_{n=1}^{N} \ln \sigma_n^2$$

and the maximum likelihood estimator $\hat{\vartheta}_N$ is

$$\hat{\vartheta}_N = \left(\sum_{n=1}^{N} \frac{a_{n-1}^* \zeta_{n-1}\zeta_{n-1}^* a_{n-1}}{\sigma_n^2} \right)^{-1} \cdot \left(\sum_{n=1}^{N} \frac{a_{n-1}^* \zeta_{n-1} \ell^* \zeta_n}{\sigma_n^2} \right),$$

where $a_n = \begin{pmatrix} Id_{p\times p} \\ \beta_n Id_{p\times p} \end{pmatrix}.$

Laplace Transform Computation

Actually,$\hat{\vartheta}_N - \vartheta = (\langle M \rangle_N)^{-1} \cdot M_N$, where

$$M_N = \sum_{n=1}^{N} \frac{a_{n-1}^*\zeta_{n-1}}{\sigma_n}\varepsilon_n, \qquad \langle M \rangle_N = \sum_{n=1}^{N} \frac{a_{n-1}^*\zeta_{n-1}\zeta_{n-1}^* a_{n-1}}{\sigma_n^2}.$$

Note that $(M_N, n \geq 1)$ is a martingale and $(\langle M \rangle_N, N \geq 1)$ is its bracket process. Let us define the Laplace transform:

$$L_N^\vartheta(\mu) = E_\vartheta \exp\left(-\frac{\mu}{2}\alpha^* \langle M \rangle_N \alpha \right),$$

for arbitrary $\alpha \in \mathbb{R}^p$ and a positive real number μ. The proof is based on the following property:

$$\lim_{N\to\infty} L_N^\vartheta\left(\frac{1}{N}\right) = \exp\left(-\frac{1}{2}\alpha^* \mathcal{I}(\vartheta)\alpha \right),$$

where $\mathcal{I}(\vartheta)$ is the unique solution of Lyapunov equation.