Asymptotic properties of the MLE of the drift coefficcient of a bidimensional Ornstein-Uhlenbeck process with singular diffusion matrix

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Abstract

We consider a 2-dimension OU process for which the diffusion matrix is singular. This process is used as a model for the dynamic behaviour of vibrating engineering structures as bridges or suspended roofs. We study the problem of estimating the vibrating frequencies of the structure or, equivalently, the parameters of the SDE that governs the OU process. We apply existing theoretical results with the goal of clarifying the properties of the MLE estimator proposed by Koncz (1987).

In particular, we consider the stochastic differential equation

$$dX_t = AX_t dt + B^{\frac{1}{2}} dW_t$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \quad B^{\frac{1}{2}} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}$$

and W is a standard 2-dimension Wiener process.
This particular process appears in the engineering literature as a model for mechanical systems subjected to random vibrations.

The maximum likelihood estimator of the drift matrix derived by Koncz (1987)[7] is revisited, as well as the tools that are used in the literature to establish properties of the estimator. The local asymptotic normality of the estimator is analysed here in detail.

Since general regularity conditions do not hold in this case, usual theoretical results do not immediately apply (the diffusion matrix is singular). One can apply either rather general classical theorems (cf. Rao and Basawa (1980)[9]) or the Laplace transform (Kleptsyna and Brouste (2010)[4]) as an alternative approach. Only the first approach is detailed here.

Motivation: the engineering problem

Many real-life mechanical and structural systems respond dynamically to random loads, either

- environmental: such as wind, wave or earthquake forces
- operational: such as traffic

Examples:

- buildings vibrating due to turbulent wind
- bridges vibrating due to traffic
- surface wave action on ships

leading to stochastic inference problems.

The need to understand the response of the structure and problems involving different behaviors arise in the conception phase or when investigating the behavior of structures once they have been built, as they provide useful information for assessment of structural damage and failure. A prominent subject of research is the *Structural Health Monitoring of engineering structures*.

This leads to problems where the structure is modeled according to engineering/mechanical principles but it is considered to have unknown parameters which must be estimated based on the observations of the process.

More precisely, the structure is modeled by

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t)$$

Where

- $M, C, K \in \mathbb{R}^{n_2 \times n_2}$ mass, damping, stiffness matrices
- $\ddot{x}(t), \dot{x}(t), x(t)$ acceleration, velocity and displacement for each degree of freedom of de structure
- $f(t)$ column vector representing the forces applied to the system.

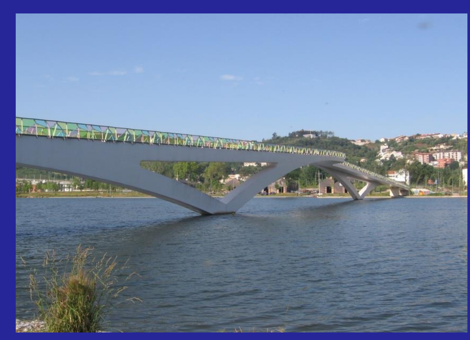
C and K are unknown matrices and engineers are particularly interested in estimating them or more currently estimating the eigenvalues of the matrix

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

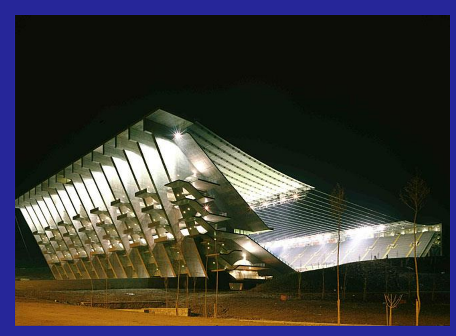
their real part (frequency of vibration) and, less commonly, their imaginary part (phase of vibration).

Examples of monitored structures

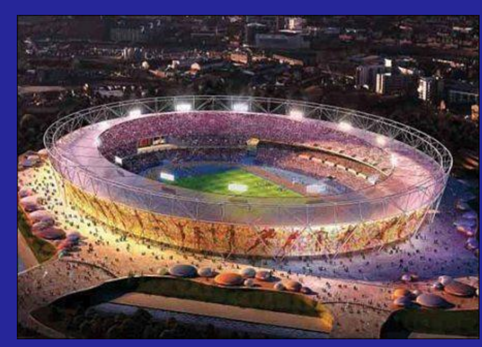
Millau viaduct, France



Pedro e Inês Footbridge, Coimbra



Braga Stadium, Portugal



London Olympic Stadium

Properties

- The MLE is consistent, unbiased and efficient (cf.[2]).
- Little is known about the limiting distribution of the estimator when the underlying process is not ergodic (cf.[7]);
- When the process is ergodic the estimator is assymptoticaly normal (cf.[6], [8]);
- The theoretical results in Arato(1982)[1] [Sec.4.6, Theorems 1 and 2] do not apply in the case "B is singular".
- To ensure the convergence of the covariance of the estimator to a finite invertible matrix and find this limit matrix is essencial (cf.[9]).

Problem formulation (dimension 2)

Structural model under random vibration:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Ito state space form:

$$\begin{cases} x^{(1)} = x \\ x^{(2)} = \dot{x} \end{cases}$$

$$\begin{cases} \dot{x}^{(1)} = x^{(2)} \\ m\dot{x}^{(2)} + cx^{(2)} + kx^{(1)} = f(t) \end{cases}$$

$$d\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix}dt + \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}\begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{bmatrix}$$

- W a 2-dimension standard Wiener process.
- $X_0 = 0$
- $X_t = [x_t^1, x_t^2]^*$ is observed in [0,t]
- k>0 and c>0 are unknown parameters
- m>0 is a known real value.

Problem: At time t, to obtain estimates of k and c based on the observations of the process $X_{\{s \le t\}}$

If the structure is damage the values for k and c will change. The aim is to estimate the drift matrix A with known diffusion singular matrix B

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \qquad B^{\frac{1}{2}} = \begin{bmatrix} 0 & 0 \\ 0 & \sigma \end{bmatrix}$$

The solution of this stochastic differential equation is the so called Ornstein-Uhlenbeck process,

$$X_t = e^{At}\left(X_0 + \int_0^t e^{-As} B^{\frac{1}{2}} dW_s \right)$$

is a stationarity gaussian process and if k,c > 0 it is ergodic (see[7]);

Maximum likelihood estimator

Radon-Nikodym derivative (cf.[1]) :

$$\frac{dP_X}{dP_W}(X(t)) = f(X_0)\exp\left(tr\left(B^{+}A\int_0^T X(t)dX^*(t) \right) - \frac{1}{2}tr\left(A^*B^{+}A\int_0^T X(t)X^*(t)dt \right) \right)$$

Likelihood function:

$$L(k,c) = \frac{-k}{m\sigma^2}\int_0^T X_1 dX_2 - \frac{c}{m\sigma^2}\int_0^T X_2 dX_2 - \frac{1}{2m^2\sigma^2}\left(k^2\int_0^T X_1^2 dt + 2ck\int_0^T X_1X_2 dt + c^2\int_0^T X_2^2 dt \right)$$

Maximum likelihood estimator:

$$\hat{\theta} = \begin{bmatrix} -\frac{\hat{k}}{m} \\ -\frac{\hat{c}}{m} \end{bmatrix} = \left(\int_0^T X(t)X^*(t)dt \right)^{-1} \int_0^T XdX_2$$

Remark : The MLE can easily be extended if the initial condition is gaussian.

Main Result

- $$\frac{1}{T}\int_0^T X_i(t)X_j(t)dt \underset{T\to+\infty}{\overset{P}{\to}} c_{ij} < +\infty \quad ,$$

where $C = (c_{ij})_{ij}$ is an invertible matrix.

Then

$$\sqrt{T}\left(\hat{\theta} - \theta \right) \underset{D}{\longrightarrow} N(0, C^{-1})$$

- **We show that**

$$C = \begin{bmatrix} \frac{m^2\sigma^2}{2kc} & 0 \\ 0 & \frac{m\sigma^2}{2c} \end{bmatrix}$$

Laplace transform based approach

- Analyse

$$\underset{T\to+\infty}{Lim}\ L_T(\mu) = \exp\left(-\frac{\mu}{2}a^* I(\theta) a \right) \quad \textbf{(conjecture)}$$

where

$$L_T(\mu) = E\exp\left(-\frac{\mu}{2}a^*\int_0^T X_t^* X_t dt a \right), a \in \mathbb{R}^2$$

Then

$$\sqrt{T}\left(\hat{\theta} - \theta \right) \underset{D}{\longrightarrow} N(0, I^{-1}(\theta))$$

- **The developments in this direction are being done in a collaboration work with Marina Klepstyna and are still under research. Results will be presented in a near future.**

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