Some examples of the hypothesis testing for Poisson processes

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Frequency modulation

We observed a Poisson process $\mathbf{X}^T$ with the intensity function $\lambda(\vartheta, t) = \lambda(\vartheta t)$, $0 \le t \le T$, which satisfies regularity conditions (but the derivative w.r.t ϑ $\dot\lambda(\vartheta, t)$ is not periodic). Let us fix some value $\vartheta_1 \in (\alpha, \beta)$ and consider the problem of testing the following two hypotheses
$\mathscr{H}_1 : \vartheta = \vartheta_1$ and $\mathscr{H}_2 : \vartheta > \vartheta_1$.
The change of variables $\vartheta = \vartheta_1 + u\varphi_T$; $\varphi_T = T^{-3/2}$ defines the rate of the convergence, and reduces the problem to
$\mathscr{H}_1 : u = 0$ and $\mathscr{H}_2 : u > 0$.
We also define the following class of tests

$$\mathcal{K}_\varepsilon = \left\{\bar\psi_T : \lim_{T\to\infty} \mathbf{E}_{\vartheta_1} = \varepsilon\right\}; \quad \varepsilon \in (0,1).$$

The power function of the test $\bar\psi_T$ is defined by the equality
$\beta\left(\bar\psi_T, u\right) = \mathbf{E}_{\vartheta_1 + u\varphi_T}\bar\psi_T\left(X^T\right)$.
The likelihood ratio function in this problem is

$$L(\vartheta, X^T) = \exp\Big\{\int_0^T \ln\lambda(\vartheta,t)\,dX_t - \int_0^T (\lambda(\vartheta,t) - 1)\,dt\Big\}.$$

Under hypothesis $\mathscr{H}_1$, $Z_T(u) = L\left(\vartheta_1 + u\varphi_T, X^T\right)/L\left(\vartheta_1, X^T\right)$ can be written as

$$Z_T(u) = \exp\left\{u\tilde\Delta_T\left(\vartheta_1, X^T\right) - \frac{u^2}{2}\sigma^2(\vartheta_1) + r_T\right\},$$

where

$$\sigma^2(\vartheta_1) = \lim_{T\to+\infty} \varphi_T^2 \int_0^T \frac{\dot\lambda^2(\vartheta_1,t)}{\lambda(\vartheta_1,t)}\,dt > 0; \qquad r_T = r_T\left(\vartheta_1, u, X^T\right) \to 0$$

and

$$\tilde\Delta_T\left(\vartheta_1, X^T\right) = \varphi_T \int_0^T \frac{\dot\lambda(\vartheta_1,t)}{\lambda(\vartheta_1,t)}\left[dX_t - \lambda(\vartheta_1,t)\,dt\right] \Rightarrow \tilde\Delta \sim \mathcal{N}\left(0, \sigma^2(\vartheta_1)\right).$$

Score function test and GLRT

Let us introduce *score function test* $\hat\psi_T\left(X^T\right) = \mathbb{1}_{\left\{\tilde\Delta_T(\vartheta_1, X^T) > z_\varepsilon\sigma(\vartheta_1)\right\}}$,
where z_ε is the $(1-\varepsilon)$-quantile of the standard normal distribution $\mathcal{N}(0,1)$.
Under alternative, $\beta_T(u) \longrightarrow \beta^*(u) = P(\zeta > z_\varepsilon - u\sigma(\vartheta_1))$, $\zeta \sim \mathcal{N}(0,1)$.
We define the general likelihood ratio test (GLRT)

$$\hat\phi_T\left(X^T\right) = \mathbb{1}_{\{Q(X^T) \ge r_\varepsilon\}}, \qquad Q\left(X^T\right) = \sup_{\vartheta > \vartheta_1} \frac{L\left(\vartheta, X^T\right)}{L\left(\vartheta_1, X^T\right)}.$$

The threshold r_ε we chose from

$$P_{\vartheta_1}\left\{\sup_{v>0} Z(v) \ge r_\varepsilon\right\} = P_{\vartheta_1}\left\{\frac{\tilde\Delta^2}{2\sigma^2} \ge \ln r_\varepsilon\right\} = P_{\vartheta_1}\left\{\zeta \ge \sqrt{2\ln r_\varepsilon}\right\},$$

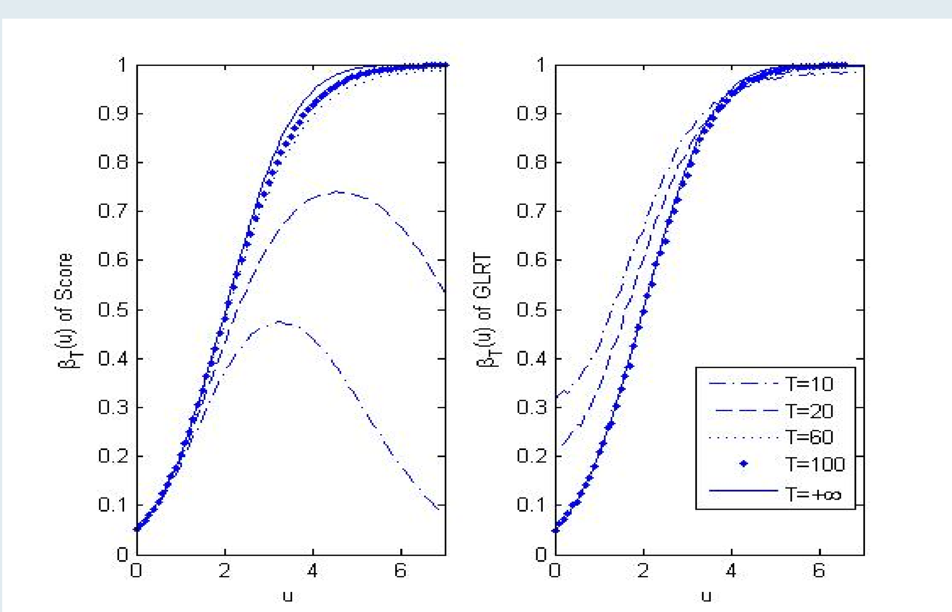
where $z_\varepsilon = \sqrt{2\ln r_\varepsilon}$. And also, under alternative, $\beta_T(u) \longrightarrow \beta^*(u)$.
And the both tests belong to the class $\mathcal{K}_\varepsilon$.

Numeric example

We suppose here $\lambda(\vartheta, t) = 3\cos^2(\vartheta t) + 1$, $0 \le t \le T$, $\vartheta \in \Theta = (0, 7)$.
The value $\vartheta_1 = 3$ and the size of the test $\varepsilon = 0.05$. In this example, $\sigma^2(\vartheta_1) = 2/3$.
We remark that, in the score function test, we can write

$$\begin{aligned}\tilde\Delta_T\left(3, X^T\right) &= -3T^{-3/2}\int_0^T \frac{t\sin(6t)}{3\cos^2(3t)+1}\left[dX_t - \left(3\cos^2\left(\left(3 + \frac{u}{T^{3/2}}\right)t\right) + 1\right)dt\right] \\ &+ 9T^{-3/2}\int_0^T \frac{t\sin(6t)}{3\cos^2(3t)+1}\left[\cos^2(3t) - \cos^2\left(\left(3 + \frac{u}{T^{3/2}}\right)t\right)\right]dt.\end{aligned}$$

When $T = 10$, the second integral begins to decrease at $u = 3.8$ and reach to 0 at $u = 7.0$, which explain, in the plot why the value of the power decreases from around $u = 3.8$ and to around 0.05 when $u = 7.0$.



Cusp type singularity

Suppose that we have n independent observations $X^n = (X_1, \ldots, X_n)$ of inhomogeneous Poisson process, where $X_j = \{X_j(t), 0 \le t \le T\}$ of intensity function $\lambda(\vartheta, t) = a\left|t - \vartheta\right|^\kappa + h(t)$, $0 \le t \le \tau$,
$\vartheta \in \Theta = (\alpha, \beta)$ with $\kappa \in (0, 1/2)$, $\alpha > 0, \beta < \tau$ and $h(\cdot)$ is a known positive bounded function.
We consider the problem of testing the following two hypotheses,
$\mathscr{H}_1 : \vartheta = \vartheta_1$ and $\mathscr{H}_2 : \vartheta > \vartheta_1$.
Putting the rate $\varphi_n = n^{-\frac{1}{2\kappa+1}}$, the change of variables $\vartheta = \vartheta_1 + \varphi_n u$ reduces the problem to
$\mathscr{H}_1 : u = 0$ and $\mathscr{H}_2 : u > 0$,
We also define the following class of tests

$$\mathcal{K}_\varepsilon = \left\{\bar\psi_n : \lim_{n\to\infty} \mathbf{E}_{\vartheta_1}\bar\psi_n = \varepsilon\right\}.$$

The power function of the test $\bar\psi_n$ is defined by the equality
$\beta\left(\bar\psi_n, u\right) = \mathbf{E}_{\vartheta_1 + u\varphi_n}\bar\psi_n\left(X^n\right)$.
The likelihood ratio function is

$$L(\vartheta, X^n) = \exp\Big\{\sum_{j=1}^n \int_0^T \ln\lambda(\vartheta,t)\,dX_j(t) - n\int_0^T \left[\ln\lambda(\vartheta,t) - 1\right]dt\Big\}$$

and the normalized likelihood ratio
$Z_n(u) = L\left(\vartheta_1 + u\varphi_n, X^n\right)/L\left(\vartheta_1, X^n\right)$ has the limit

$$Z_n(u) \Rightarrow Z(u) = \exp\left\{\Gamma_{\vartheta_1} W^H(u) - \Gamma_{\vartheta_1}^2 \left|u\right|^{2H}/2\right\}, \quad u \in R_+,$$

where $W^H(\cdot)$ is a fractional Brownian motion (fBm), $H = \kappa + \frac{1}{2}$ is the *Hurst parameter* and the constant

$$\Gamma_{\vartheta_1}^2 = \frac{2a^2 B(\kappa+1, \kappa+1)}{h(\vartheta_1)}\left[\frac{1}{\cos(\pi\kappa)} - 1\right].$$

GLRT and Wald's test

We define the GLRT

$$\hat\phi_n\left(X^n\right) = \mathbb{1}_{\{Q(X^T) \ge C_\varepsilon\}}, \qquad Q\left(X^n\right) = \sup_{\vartheta > \vartheta_1} \frac{L\left(\vartheta, X^n\right)}{L\left(\vartheta_1, X^n\right)},$$

where the threshold $C_\varepsilon = C_\varepsilon(H)$ is solution of the equation

$$P\left\{\sup_{v>0} Z(v) > C_\varepsilon\right\} = P\left\{\sup_{s>0}\left[W^H(s) - s^{2H}/2\right] > \ln C_\varepsilon\right\} = \varepsilon.$$

Denoting $\gamma = \Gamma_{\vartheta_1}^{1/H}$, the test $\phi_n\left(X^n\right) \in \mathcal{K}_\epsilon$ and its power function has the limit

$$P\left\{\sup_{s>0}\left[W^H(u\gamma) + W^H(s - u\gamma) - \frac{\left|s - u\gamma\right|^{2H}}{2} + \frac{\left|u\gamma\right|^{2H}}{2}\right] > \ln C_\varepsilon\right\}.$$

The Wald's test is based on the MLE $\hat\vartheta_n$, which is one of the solutions of the equation, $L(\hat\vartheta_n, X^n) = \sup_{\vartheta > \vartheta_1} L(\vartheta, X^n)$. We know that $\varphi_n\left(\hat\vartheta_n - \vartheta_1\right) \Rightarrow \frac{\hat u}{\gamma}$, where the r.v. $\hat u$ is solution of the equation

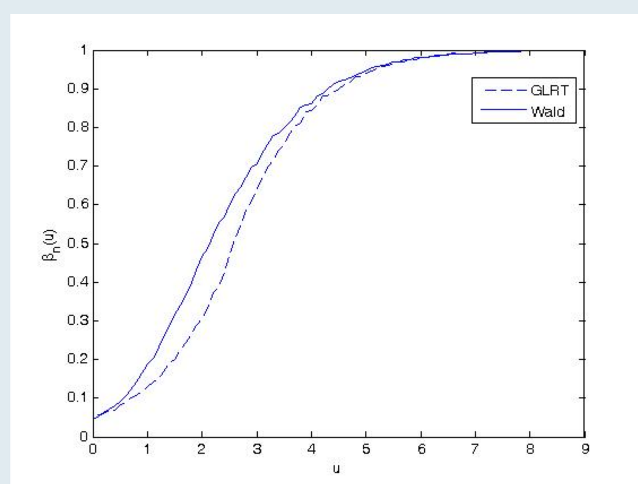
$$Z_0(\hat u) = \sup_{v>0} Z_0(v), \qquad Z_0(v) = e^{W^H(v) - \frac{v^{2H}}{2}}.$$

Therefore the test is $\hat\psi_n\left(X^n\right) = \mathbb{1}_{\left\{\varphi_n^{-1}\gamma\left(\hat\vartheta_n - \vartheta_1\right) > c_\varepsilon\right\}}$ where the threshold $c_\varepsilon = c_\varepsilon(H)$ is solution of the equation $P\{\hat u > c_\varepsilon\} = \varepsilon$. We show that the test $\hat\psi_n\left(X^n\right) \in \mathcal{K}_\epsilon$ and its power function

$$\beta\left(\hat\psi_n, u\right) \longrightarrow P\{\hat u > c_\varepsilon - u\gamma\}.$$

Numeric example

We suppose now
$\lambda(\vartheta, t) = 2 - \left|t - \vartheta\right|^{2/5}$, $\quad 0 \le t \le 2$, $\vartheta \in \Theta = (0, 2)$ where $\kappa = 0.4 \in (0, 1/2)$. In this example, $\Gamma_{\vartheta_1}^2 \approx 1.027$.



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