On Approximation of FBSDE

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Abstract

We consider the approximation of the forward-backward stochastic differential equations. The observation is supposed to be a diffusion process satisfying some stochastic differential equation, where the trend coefficient depends on some unknown parameter. We try to construct a couple of processes such that the final value of one is a function of the final value of the given diffusion process. We show that when the diffusion coefficient is small, the couple of processes approximates well the solution of a backward stochastic differential equation. Moreover, we show that this approximation is asymptotically efficient.

Problem statement

Suppose that we observe a diffusion process $X^T = \{X_t, 0 \le t \le T\}$ satisfying

$$dX_t = S(X_t, \vartheta)\, dt + \varepsilon\sigma(X_t)\, dW_t, \quad X_0 = x_0, \quad 0 \le t \le T$$

and we are given functions $f(x, y) = k(x) + g(x)y$ and $\Phi(x)$. We have to construct a couple of processes (Y_t, Z_t) such that the solution of the equation

$$dY_t = (k(X_t) + g(X_t)Y_t)\ dt + Z_t\, dW_t, \quad 0 \le t \le T \tag{1}$$

has the final value $Y_T = \Phi(X_T)$.

Preliminaries

– FBSDE and PDE.

Suppose that $U(t, x, \vartheta)$ is the solution of the following equation

$$\frac{\partial U}{\partial t} + S(x)\frac{\partial U}{\partial x} + \frac{1}{2}\varepsilon^2\sigma(x)^2\frac{\partial^2 U}{\partial x^2} = k(x) + g(x)y, \tag{2}$$

with final value $U(T, x) = \Phi(x)$. The processes

$$Y_t = U(t, X_t, \vartheta), \quad Z_t = \varepsilon\sigma(X_t)\, U'(t, X_t, \vartheta)$$

give us the solution of the BSDE (1).

– Deterministic case.

We denote $x^t = (x_s)_{0 \le s \le t}$ the solution for the deterministic equation

$$dx_s = S(x_s, \vartheta)\ ds, \quad x_0, \quad 0 \le s \le t$$

and $u(t, x, \vartheta)$ the solution of the PDE (2) for $\varepsilon = 0$.

For any function $h(t, x, \vartheta)$, we define $h'(t, x, \vartheta)$ and $\dot{h}(t, x, \vartheta)$ the derivatives w.r.t. x and w.r.t. ϑ respectively.

Conditions $\mathcal{B}$

$\mathcal{B}_1$. *The functions $\sigma(x)$ and $S(\vartheta, x)$ are differentiable w.r.t. x, the function $S(\vartheta, x) \in \mathcal{C}^{(5)}_{\vartheta}$, and all these derivatives are continuous and bounded. In addition, there exists $\kappa > 0$ such that $\sigma(x)^2 > \kappa$, $x \in \mathbb{R}$.*

$\mathcal{B}_2$. *The function $\Phi(x)$ is bounded and continuous. The function $k(x)$ is bounded and has continuous bounded derivative $k'(x)$.*

$\mathcal{B}_3$. *For some $\delta > 0$, the Fisher information is positive:*

$$I(x^\delta, \vartheta) = \int_0^\delta \frac{\dot{S}(\vartheta, x_s)^2}{\sigma(x_s)^2} ds > 0,$$

and for any $\nu > 0$,

$$\inf_{|\theta - \vartheta| > \nu} \left\| \frac{S(\theta, x) - S(\vartheta, x)}{\sigma(x)} \right\|_\delta > 0.$$

Here $\|\cdot\|_\delta$ is the norm in the space of square integrable functions: $L_2(0,\ \delta)$.

MLE-Process

Let us introduce the maximum likelihood estimator process (MLE-process) $\hat{\vartheta}_{t,\varepsilon}$.

The likelihood ratio at time t is

$$L(X^t, \vartheta) = \exp\left\{ \frac{1}{\varepsilon^2}\int_0^t \frac{S(\vartheta, X_s)}{\sigma(X_s)^2} dX_s - \frac{1}{2\varepsilon^2}\int_0^t \frac{S(\vartheta, X_s)^2}{\sigma(X_s)^2} ds \right\}.$$

Then the MLE-process $\hat{\vartheta}_{t,\varepsilon}$ is defined as

$$\hat{\vartheta}_{t,\varepsilon} = \arg\max_{\theta \in \Theta} L(X^t, \theta), \quad t \in [\delta, T].$$

lemma

Let the condition $\mathcal{B}$ be fulfilled, then the MLE-process $\hat{\vartheta}_{t,\varepsilon}$ admits the following representation: for any $\nu > 0$

$$\sup_{\delta \le t \le T} \mathbf{P}_\vartheta \left\{ \left| \frac{\hat{\vartheta}_{t,\varepsilon} - \vartheta}{\varepsilon^2} - \frac{\xi_{t,1}(x^t, \vartheta)}{\varepsilon} - \xi_{t,2}(x^t, \vartheta) \right| > \nu \right\} \to 0.$$

Here

$$\xi_{t,1}(x^t, \vartheta) = I(x^t, \vartheta)^{-1} \int_0^t \frac{\dot{S}(x_s, \vartheta)}{\sigma(x_s)} dW_s$$

and $\xi_{t,2}(x^t, \vartheta)$ is also a random variable depending on x^t and on ϑ.

Approximation

We approximate the processes (Y, Z) in using the solution of the PDE $U(t, x, \vartheta)$, for ϑ replaced by the MLE-process $\hat{\vartheta}_{t,\varepsilon}$. Let us define

$$\hat{Y}_t = U(t, X_t, \hat{\vartheta}_{t,\varepsilon}), \quad \hat{Z}_t = \varepsilon\sigma(X_t)U'(t, Y_t, \hat{\vartheta}_{t,\varepsilon}), \quad t \in [\delta,\ T]. \tag{3}$$

Theorem

Under the regularity condition $\mathcal{B}$, the couple $\left(\hat{Y}_t, \hat{Z}_t\right)_{t \in [\delta, T]}$ admits the representation:

$$\hat{Y}_t = Y_t + \varepsilon\, \xi_{t,1}(x^t, \vartheta)\, \dot{U}(t, X_t, \vartheta) + \varepsilon^2 \xi_{t,2}(x^t, \vartheta)^2\, \dot{U}(t, X_t, \vartheta) + \frac{1}{2}\varepsilon^2 \xi_{t,1}(x^t, \vartheta)\, \ddot{U}(t, X_t, \vartheta) + O(\varepsilon^3)$$

$$\hat{Z}_t = Z_t + \varepsilon^2\, \xi_{t,1}(x^t, \vartheta)\, \sigma(X_t)\, \dot{U}'(t, X_t, \vartheta) + O(\varepsilon^3).$$

Here $\hat{Y} = Y + O(\varepsilon)$ (the same for $\hat{Z}$) means that for any $\nu > 0$,

$$\lim_{C \to \infty} \mathbf{P}(\varepsilon^{-1}\left|\hat{Y} - Y\right| > C) \longrightarrow 0.$$

Efficiency

Let us introduce the condition $\mathcal{C}$:

$\mathcal{C}_1$. Suppose that $\dot{u}(t, x, \vartheta)$ and $\dot{u}'(t, x, \vartheta_0)$ exist and that they are continuous.

$\mathcal{C}_2$. Suppose that $U(t, x, \vartheta)$, $\dot{U}(t, x, \vartheta)$, $U'(t, x, \vartheta)$, $\dot{U}'(t, x, \vartheta)$ are all of polynomial majorants w.r.t. x.

Theorem

For any estimator $(\bar{Y}_t,\ \bar{Z}_t)$ of $(Y_t,\ Z_t)$ and any t satisfying $\delta \le t \le T$,

$$\lim_{\nu \to 0}\lim_{\varepsilon \to 0}\sup_{|\vartheta - \vartheta_0| < \nu} \varepsilon^{-2}\mathbf{E}_\vartheta\left(\bar{Y}_t - Y_t\right)^2 \ge \frac{\dot{u}(t, x_t, \vartheta_0)^2}{I(x^t, \vartheta_0)},$$

and

$$\lim_{\nu \to 0}\lim_{\varepsilon \to 0}\sup_{|\vartheta - \vartheta_0| < \nu} \varepsilon^{-4}\mathbf{E}_\vartheta\left(\bar{Z}_t - Z_t\right)^2 \ge \sigma(x_t)^2\frac{\dot{u}'(t, x_t, \vartheta_0)^2}{I(x^t, \vartheta_0)}.$$

We say that an approximation $\bar{Y}$ or $\bar{Z}$ is asymptotically efficient, if for all $\vartheta_0 \in (\alpha, \beta)$ and $t \in [\delta,\ T]$ we have the equalities.

Theorem

Let the conditions $\mathcal{B}$ and $\mathcal{C}$ be fulfilled, then $\hat{Y}$ and $\hat{Z}$ defined in (3) are asymptotically efficient estimators of Y_t and Z_t.

Example

We consider the linear FBSDE

$$\begin{cases} dX_t = \vartheta dt + \varepsilon\sigma dW_t, \quad X_0 = x_0, \\ dY_t = -(\beta Y_t + \gamma Z_t)dt + Z_t dW_t, \quad Y_T = \Phi(X_T). \end{cases}$$

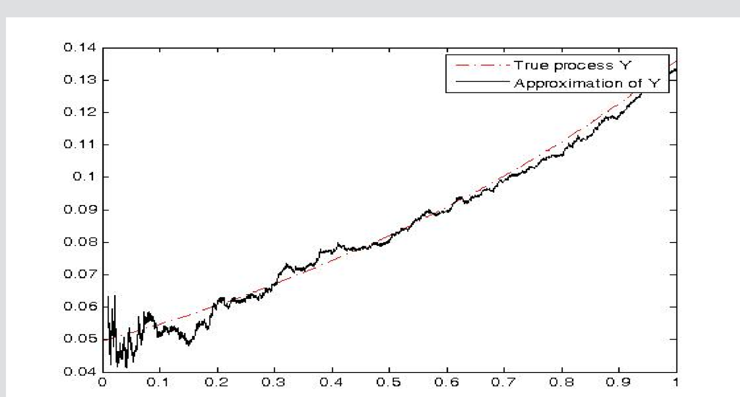
The corresponding PDE is

$$\frac{\partial U}{\partial t} + \frac{1}{2}\varepsilon^2\sigma^2\frac{\partial^2 U}{\partial x^2} + (\vartheta + \varepsilon\sigma\gamma)\frac{\partial U}{\partial x} + \beta U = 0,$$

with final value $U(T, x) = \Phi(x)$. This PDE can be solved explicitly, noting as $U(t, x, \vartheta)$. The MLE-process for ϑ is $\hat{\vartheta}_{t,\varepsilon} = \frac{X_t - x_0}{t}$. We have

$$\frac{\hat{\vartheta}_{t,\varepsilon} - \vartheta}{\varepsilon} = \frac{\sigma}{t}W_t, \quad \text{for } \delta < t \le T.$$

We construct the process $(\hat{Y}_t, \hat{Z}_t)_t$ as (3). The graphic on the right presents the numerical result for Y, with fixed value for parameters and for $\varepsilon = 0.1$.



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