Effective discretization of stochastic differential equations

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The aim of study

Find an effective method for generating a path of the solution of a stochastic differential equation

$$\mathrm{d}X_t^i = f_j^i(X_t)\mathrm{d}W_t^j = \sum_{j=0}^{d} f_j^i(X_t)\mathrm{d}W_t^j,$$

where $1 \leq i \leq d_X$, $0 \leq j \leq d$, $W_t^0 = t$,

$$W = (W^1, \ldots, W^d)$$

is a d-dimensional Brownian motion and

$$f = \{f_j^i\} : \mathbb{R}^{d_X} \to \mathbb{R}^{d_X} \otimes \mathbb{R}^{d+1}$$

has uniformly continuous and bounded first derivatives.

Euler-Maruyama scheme

It is popular because it is simple and generic...

- strong convergence under mild regularity conditions
- weak convergence for irregular, path-dependent functionals
- easy coding, no differentiation, intuitive construction

Could we do better ?

Euler-Maruyama revisited

Let τ_m^n be a sequence of increasing stopping times with $\tau_0^n = 0$.
Define $X^n = (X^{n,1}, \dots, X^{n,d_X})$ by $X_0^n = X_0$ and

$$X_t^{n,i} = X_{\tau_m^n}^{n,i} + f_j^i(X_{\tau_m^n}^n)(W_t^j - W_{\tau_m^n}^j), \quad t \in (\tau_m^n, \tau_{m+1}^n].$$

As usual, if we take $\tau_m^n = m/n$, then

$$W_t - W_{\tau_m^n} \sim \mathcal{N}_d(0, (t - \tau_m^n) I_d).$$

- n controls computational efforts
- as $n \to \infty$, $X^n \to X$ in various senses.

Convergence of dicretization error

Kurtz and Protter (1991) : Put $U_t^{n,i} = \sqrt{n}(X_t^{n,i} - X_t^i)$ and

$$Z_t^{n,l,j} = \sqrt{n} \sum_{m=0}^{\infty} \int_{t \wedge \tau_m^n}^{t \wedge \tau_{m+1}^n} (W_s^l - W_{\tau_m^n}^l) \mathrm{d} W_s^j.$$

If $\{Z^{n,l,j}\}$ is a "good" sequence, converging to a process $\{Z^{l,j}\}$ as $n \to \infty$, then $\{U^{n,i}\}$ converges to the solution $\{U^i\}$ of

$$\mathrm{d} U_t^i = \partial_k f_j^i(X_t) U_t^k \mathrm{d} W_t^j - \partial_k f_j^i(X_t) f_l^k(X_t) \mathrm{d} Z_t^{l,j}.$$

In particular if $\langle Z^{l,j}, W^i \rangle = 0$ for all l, j, i, then

$$U_t = Y_t \int_0^t Y_s^{-1} \mathrm{d} H_s, \quad \mathrm{d} H_s^i = -\partial_k f_j^i(X_s) f_l^k(X_s) \mathrm{d} Z_s^{l,j},$$

where $Y = [Y^{p,q}]$ is the solution of

$$\mathrm{d} Y_t^{p,q} = \partial_k f_j^p(X_t) Y_t^{k,q} \mathrm{d} W_t^j, \quad Y_0^{p,q} = \delta^{pq}.$$

Limit distribution

The limit U^i is of the form

$$U^i_t = Y^{i,k}_t (F_{k,l,j} \cdot Z^{l,j}_t).$$

Recall that $Z^{l,j}$ is the limit of

$$Z^{n,l,j}_t = \sqrt{n} \sum_{m=0}^{\infty} \int_{t \wedge \tau^n_m}^{t \wedge \tau^n_{m+1}} (W^l_s - W^l_{\tau^n_m}) \mathrm{d} W^j_s .$$

Note that in case of $l = j$,

$$Z^{n,l,l}_t = \frac{\sqrt{n}}{2} \left\{ \sum_{m=0}^{\infty} (W^l_{\tau^n_{m+1} \wedge t} - W^l_{\tau^n_m \wedge t})^2 - t \right\} .$$

If the limit $Z^{l,j}$ is a conditionally Gaussian martingale, then U_t is conditionally Gaussian with mean 0 and covariance

$$Y^{i,p}_t Y^{j,q}_t ((F_{p,\alpha,\beta} F_{q,\gamma,\delta}) \cdot \langle Z^{\alpha,\beta}, Z^{\gamma,\delta} \rangle_t).$$

Equidistant case

If $\tau_m^n = m/n$, then $Z^{0,j} = Z^{j,0} = 0$ for all $0 \leq j \leq d$ and

$$Z^{l,j} = \frac{1}{\sqrt{2}} \hat{W}^{l,j}, \quad 1 \leq l, j \leq d,$$

where $\hat{W}$ is a d^2-dim. Brownian motion independent of W.

This is essentially due to

$$\begin{aligned}\langle Z^{n,l,j} \rangle_t &= n \sum_{m=0}^{\infty} \int_{\tau_m^n \wedge t}^{\tau_{m+1}^n \wedge t} (W_t^l - W_{\tau_m^n}^l)^2 \mathrm{d}t \\ &\approx \frac{n}{6} \sum_{m=0}^{\infty} (W_{\tau_{m+1}^n \wedge t}^l - W_{\tau_m^n \wedge t}^l)^4\end{aligned}$$

and $W_{\tau_{m+1}^n} - W_{\tau_m^n} \sim \mathcal{N}(0, 1/n)$ if $\tau_m^n = m/n$, and so,

$$\mathbb{E}[(W_{\tau_{m+1}^n}^l - W_{\tau_m^n}^l)^4 | \mathcal{F}_{\tau_m^n}] = 3n^{-2}.$$

One-dimensional case

Let $d_X = d = 1$. If $\tau^n_m = m/n$, then

$$U_t = \frac{1}{\sqrt{2}} Y_t \int_0^t Y_u^{-1} f'(X_u) f(X_u) \mathrm{d}\hat{W}_u,$$
$$Y_t = \exp\left\{ \int_0^t f'(X_u) \mathrm{d}W_u - \frac{1}{2}\int_0^t f'(X_u)^2 \mathrm{d}u \right\}.$$

On the other hand, if

$$\tau^n_{m+1} = \inf\{t > \tau^n_m; |W_t - W_{\tau^n_m}|^2 = 1/n\}, \quad \tau^n_0 = 0,$$

then, F.(2011) showed that $\max\{m; \tau^n_m \le t\}/n \to t$ and

$$U_t = \frac{1}{\sqrt{6}} Y_t \int_0^t Y_u^{-1} f'(X_u) f(X_u) \mathrm{d}\hat{W}_u.$$

Class of stationary and symmetric increments

Denote by $\Delta_m^n W$ the increments

$$\Delta_m^n W^i = W^i_{\tau^n_{m+1}} - W^i_{\tau^n_m}, \quad 0 \leq i \leq d.$$

Denote by $\mathcal{T}$ the set of the sequence τ_m^n with

- stationarity: $\{\Delta_m^n W\}_{m=0}^{\infty}$ is IID for each $n \in \mathbb{N}$
- symmetry: $\mathbb{E}[|W^1_{\tau^n_1}|^4] = \cdots = \mathbb{E}[|W^d_{\tau^n_1}|^4]$ and

$$\mathbb{E}[\int_{\tau^n_m}^{\tau^n_{m+1}} (W^j_t - W^j_{\tau^n_m})(W^l_t - W^l_{\tau^n_m}) \mathrm{d}t | \mathcal{F}_{\tau^n_m}] = 0$$

$$\mathbb{E}[\int_{\tau^n_m}^{\tau^n_{m+1}} (W^j_t - W^j_{\tau^n_m}) \mathrm{d}t | \mathcal{F}_{\tau^n_m}] = 0$$

  for $1 \leq j \neq l \leq d$ and $m, n \in \mathbb{N}$
- normalization: $\mathbb{E}[\tau^n_1] = 1/n$ for $n \in \mathbb{N}$

Result of stationary and symmetric increments

Theorem : For any $\{\tau_m^n\} \in \mathcal{T}$, we have

$$n^2\mathbb{E}[|W^1_{\tau_1^n}|^4] \geq \frac{3d}{d+2}$$

and if $\alpha := \lim n^2\mathbb{E}[|W^1_{\tau_1^n}|^4]$ exists, then $U^{n,i} = \sqrt{n}(X^{n,i} - X^i)$ converges in law in $C[0,\infty)$ as $n \to \infty$ to the solution U^i of

$$\mathrm{d}U_t^i = \partial_k f_j^i(X_t)U_t^k \mathrm{d}W_t^j - \sqrt{\frac{\alpha}{6}}\partial_k f_j^i(X_t) f_l^k(X_t)\mathrm{d}\hat{W}_t^{l,j}$$

where $\hat{W}$ is a d^2 dim. Brownian motion independent of W.
The lower bound is attained by

$$\tau_0^n = 0, \quad \tau_{m+1}^n = \inf\{t > \tau_m^n; \|W_t - W_{\tau_m^n}\|^2 = d/n\}.$$

Optimal scheme

Corollary : Let U^e be the limit distribution of the error process $U^n = (U^{n,1}, \dots, U^{n,d_X})$ for the equidistant scheme $\tau^n_m = m/n$. Let U be the limit distribution of the error process $U^n = (U^{n,1}, \dots, U^{n,d_X})$ for $\{\tau^n_m\} \in \mathcal{T}$ with

$$\alpha = \lim_{n\to\infty} n^2 \mathbb{E}[|W^1_{\tau^n_1}|^4].$$

Then,

$$U \sim \sqrt{\frac{\alpha}{3}} U^e.$$

Then optimal scheme in $\mathcal{T}$ is

$$\tau^n_0 = 0, \quad \tau^n_{m+1} = \inf\{t > \tau^n_m; \|W_t - W_{\tau^n_m}\|^2 = d/n\},$$

which results in

$$U \sim \sqrt{\frac{d}{d+2}} U^e.$$

Proof of the optimality

Let

$$Q_t = \sum_{j=1}^{d} |W_t^j|^4, \quad S_t = \sum_{j=1}^{d} |W_t^j|^2.$$

Then, by Itô's formula,

$$\mathbb{E}[Q_\tau] = 6\mathbb{E}[\int_0^\tau S_t \mathrm{d}t] = \frac{3}{d+2}\mathbb{E}[S_\tau^2]$$
$$\geq \frac{3}{d+2}\mathbb{E}[S_\tau]^2 = \frac{3d^2}{d+2}\mathbb{E}[\tau]^2.$$

The equality is attained if and only if S_τ is a constant.

Bessel hitting time distribution

For the optimal scheme, the Brownian increments are uniformly distributed on a sphere independent of time increments. No difficulty to sample.

Ciesielski and Taylor (1962):

$$\mathbb{P}[\tau_1^1 > t] = \frac{1}{2^{\nu-1}\Gamma(\nu+1)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu-1}}{J_{\nu+1}(j_{\nu,k})} \exp\{-\frac{j_{\nu,k}^2}{2} t\},$$

where $2\nu + 2 = d$, J_μ is the Bessel function of the first kind of order μ and $\{j_{\nu,k}\}_{k=1}^{\infty}$ is the increasing sequence of positive zeros of J_ν.

Alternative

Let $a = (1 + 2/d)^{1+d/2}/n$ and

$$\psi(t) = \sqrt{dt \log a/t}, \quad t \in [0, a].$$

Then the hitting time of a moving sphere

$$\tau = \inf\{t > 0; \|W_t\| > \psi(t)\}$$

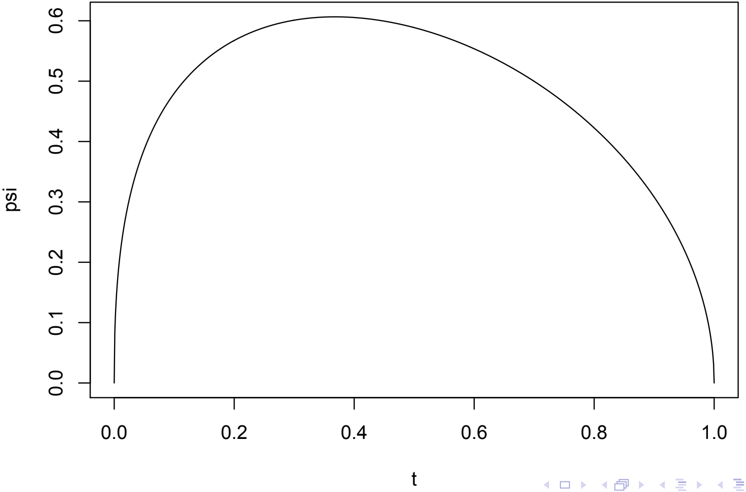
has the same distribution as

$$ae^{-Z},$$

where $Z \sim \Gamma(1 + d/2, 2/d)$ and $\mathbb{E}[\tau] = 1/n$.

Deaconu and Herrmann (2012)
Chen, Cheng, Chadam and Sanders (2011)

Moving boundary



Idea of the proof

Let $\nu = d/2 - 1$ and

$$u(t,x) = p(0,0;t,x) - \frac{1}{2^\nu a^{\nu+1}\Gamma(\nu+1)} \int p(0,y;t,x) y^{2\nu+1} \mathrm{d}y$$
$$= \frac{1}{2^\nu \Gamma(\nu+1)} \left\{ \frac{1}{t^{\nu+1}} \exp\left\{ -\frac{x^2}{2t} \right\} - \frac{1}{a^{\nu+1}} \right\} x^{2\nu+1}.$$

Then $u(t,\psi(t)) = 0$ and $\max\{\mathcal{L}w, w - p\} = 0$, where

$$w(t,x) = \int_0^x u(t,y)\mathrm{d}y, \quad p(t) = w(t,\psi(t))$$
$$\mathcal{L}\phi = \frac{\partial \phi}{\partial t} - \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2} + \frac{d-1}{2x}\frac{\partial \phi}{\partial x}.$$

On the other hand, $\mathbb{P}[\tau \geq t, X_t < x]$ is the unique viscosity solution of the variational inequality.

Uniform improvement

Theorem : Let $\tau_0^n = 0$ and

$$\tau_{m+1}^n = \inf\{t > \tau_m^n; \|W_t - W_{\tau_m^n}\| > \psi(t - \tau_m^n)\}.$$

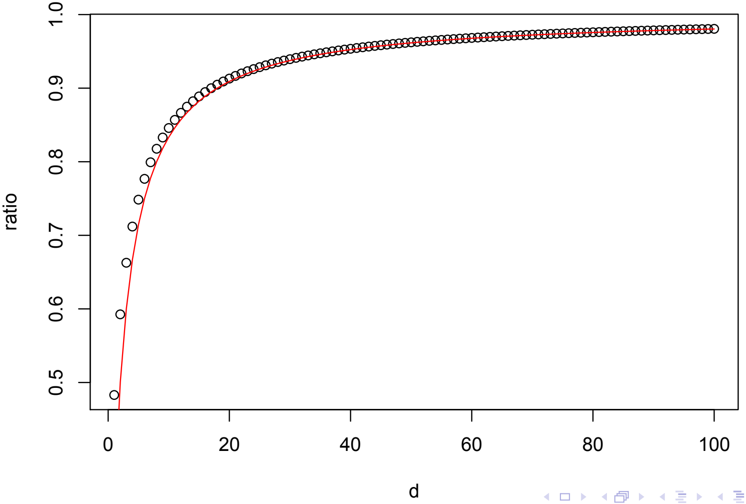
Then, $\{\tau_m^n\} \in \mathcal{T}$ with

$$\alpha = n^2 \mathbb{E}[|W^1_{\tau_1^n}|^4] = \frac{3(d+2)^{d+2}}{d^{d/2}(d+4)^{(d+4)/2}}.$$

Let U be the corresponding limit distribution of error. Then,

$$U \sim \sqrt{\frac{(d+2)^{d+2}}{d^{d/2}(d+4)^{(d+4)/2}}}\, U^e.$$

Reduction ratio



How to sample ?

$\Delta_m^n W$ is IID. So it suffices to give how to sample $\Delta_1^n W$.

Let $N \sim \mathcal{N}(0, I_d)$ and $E \sim \mathrm{Exp}(1)$.
Recall that $\|N\|^2 \sim \Gamma(d/2, 2)$ and so,

$$Z := \frac{1}{d}(\|N\|^2 + 2E) \sim \Gamma(1 + d/2, 2/d).$$

Therefore,

$$\tau_1^n \sim ae^{-Z}, \quad W_{\tau_1^n} \sim \psi(\tau_1^n)\frac{N}{\|N\|} = \sqrt{daZe^{-Z}}\frac{N}{\|N\|}.$$

Need just one additional $\mathrm{Exp}(1)$ other than $\mathcal{N}(0, I_d)$.

Comparison

To have for $t \in [0,1]$,

$$X_t^n - X_t \approx \frac{1}{\sqrt{n}} U_t^e,$$

- the equidistant scheme requires nd standard normal random variables.
- the moving boundary scheme does approximately

$$\frac{nd^2}{d+2} \text{ and } \frac{nd}{d+2}$$

standard normal and exponential random variables respectively.

Remarks

- The moving boundary scheme has an additional property that the both time and Brownian increments are bounded:

$$\tau_1^n \leq a, \quad \|W_{\tau_1^n}\| \leq \sqrt{\frac{da}{e}}, \quad a = \frac{1}{n}\left(1 + \frac{2}{d}\right)^{1+\frac{d}{2}}$$

- We can therefore control the size of increments (change n in an adapted manner) so that the discretized process X^n does not hit the natural boundary of X.