The Euler-Maruyama approximations for the CEV model

V. Abramov, F.Klebater, and R.Liptser

Monash and Tel Aviv Universities

1. Constant Elasticity of Variance Model (CEV)

We consider the Constant Elasticity of Variance (CEV) model (e.g. Cox) defined by the Itô equation with respect to Brownian motion W_t:

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma(X_s \vee 0)^p dW_s, \quad t \wedge \tau$$

where μ, σ, p are constants and X_0 an initial condition: on the time interval $[0, \tau]$, where

τ the first time of hitting zero $\tau = \inf\{t : X_t = 0\}$. It is known (Shiga, Watanabe) that zero is an absorbing state and that $P(\tau < \infty) > 0$.
$\mu \in \mathbb{R}$; $\sigma > 0$; $p \in \left[\frac{1}{2}, 1\right)$; $X_0 > 0$.
CEV model possesses a unique strong nonnegative solution (see Delbaen, Deelstra)

2. Approximation by Euler-Maruyama scheme

Our aim is to approximate a solution of CEV Itô's equation by the which can be applied in finance models, as well as for evaluation of the ruin probability $\mathsf{P}(\tau \leq T)$ by simulations.

The parameter $p = \frac{1}{2}$ is related to population model and is known as the diffusion with represents the size of a population and is Feller's branching diffusion.

In many setting theoretical expressions exists but its numerical presentation cannot be easily obtained in view of a nonstandard setting: diffusion coefficients is

singular and non-Lipschitz

what makes the analysis non-standard

3. Euler-Maruyama scheme

Taking for simplicity equidistant partitions of $[0, T]$, $0 \equiv t_0^n < t_1^n < \ldots < t_{t_{n-1}}^n < t_n^n \equiv T$, $t_k^n - t_{k-1}^n \equiv \frac{T}{n}$, it is defined by the following recursion

$$\begin{aligned} X_{t_0^n}^n &= X_0, \\ X_{t_k^n}^n &= X_{t_{k-1}^n}^n + \mu \frac{T}{n} X_{t_{k-1}^n}^{n+} + \sigma (X_{t_{k-1}^n}^{n+})^p \sqrt{\frac{T}{n}} \xi_k, \\ X_t^n &= X_{t_{k-1}^n}^n, \quad t \in [t_{k-1}^n, t_k^n), \quad k = 1, \ldots n, \end{aligned} \tag{1}$$

where $(\xi_k)_{k \geq 1}$ is an i.i.d. sequence of $(0, 1)$-Gaussian random variables.

This model is not applicable in view of aforementioned property of diffusion coefficient (see also next slide)

4. Not applicable approaches.

For $p = 1/2$, the Fokker-Plank equation, related to CEV, obeys the unique solution

The Euler-Maruyama algorithm for different type of diffusion models can be found in Kloeden + Platten and Milstein+ Tretyakov, where only X_T is served.
A trajectory $0 \leq t \leq T$ are served for models with the drift and diffusion coefficients are Lipschitz and the diffusion coefficient is nonsingular, that is, the standard theory applies. (see Bally+Talay, Bossy + Diop, Gyöngy + Krylov, Halidias + Kloeden, etc.

5. Continuous of Euler-Maruyama model approximation

A continuous approximation $\widetilde{X}_t^n$ used in the proof of weak convergence

$$\widetilde{X}_t^n = X_0 + \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t \wedge t_k^n} \mu \widetilde{X}_{t_{k-1}^n}^{n+} ds + \sum_{k=1}^{n} \int_{t_{k-1}^n}^{t \wedge t_k^n} \sigma(\widetilde{X}_{t_{k-1}^n}^{n+})^p d\widetilde{W}_s,$$

where $\widetilde{W}_t$ is a Brownian motion such that $\widetilde{W}_{t_k^n} - \widetilde{W}_{t_{k-1}^n} \equiv \sqrt{\frac{T}{n}}\xi_k$.

Also $X = (X_t)_{t \in [0,T]}$ is nonnegative, we have to use a stopping time

$$\tau^n = \inf\{t \leq T : \widetilde{X}_t^n \leq 0\}.$$

6. Euler-Maruyama, Continuous Euler-Maruyama and CEV models: $X_t^n, \widetilde{X}_{t^n}^n, X_t$

The paths of the process $X^n = (X_t^n)_{t\in[0,T]}$ are right continuous piece-wise constant functions with left limits belonging to the Skorokhod space $\mathbb{D} = \mathbb{D}_{[0,T]}$.

Consider a metric space $(\mathbb{D}, d_0)$ endowed with the Skorokhod metric d_0: if $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}$, then

$$d_0(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\varphi\in\Phi} \left\{ \sup_{t\in[0,T]} |\boldsymbol{x}_t - \boldsymbol{y}_{\varphi(t)}| + \sup_{0\le s<t\le 1} \left| \log \frac{\varphi(t)-\varphi(s)}{t-s} \right| \right\},$$

where $\varphi = (\varphi(t))_{0\le t\le T}$ is a set of strictly increasing continuous functions with $\varphi(0) = 0$, $\varphi(T) = 1$.

Denote
- Q^n the distribution of $X^n = (X^n_t)_{t\in[0,T]}$, (the measure on $\mathbb{D}$)

- Q the distribution of $X = (X_t)_{t\in[0,T]}$.

Since X is a continuous process, $\mathsf{Q}(\mathbb{C}) = 1$, where $\mathbb{C}$ denotes the subspace of $\mathbb{D}$.

Evaluations of functionals by simulations is justified by the convergence of measures in the Skorokhod space
$\mathsf{Q}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q}$, that is,

$$\lim_{n\to\infty}\int_{\mathbb{D}} f(\boldsymbol{x})d\mathsf{Q}^n = \int_{\mathbb{C}} f(\boldsymbol{x})d\mathsf{Q}$$

for any bounded and continuous in the metric d_0 function $f(\boldsymbol{x})$.

8. Diffusion approximation theorem in 3 steps

For any $T > 0$, the Euler-Maruyama approximation for CEV model converges weakly in the Skorokhod metric d_0 to the limit process $(X_t)_{t\in[0,T]}$.

step 1. $$\widetilde{Q}^n \xrightarrow[n\to\infty]{d_0} Q$$

step 2. $$\sup_{t\in[0,T]} |X_t^n - \widetilde{X}_t^n| \xrightarrow[n\to\infty]{\text{prob.}} 0.$$

step 3.

$$\left.\begin{array}{l} \sup_{t\in[0,T]} |X^n - \widetilde{X}^n| \xrightarrow[n\to\infty]{\text{prob.}} 0 \\ \widetilde{Q}^n \xrightarrow[n\to\infty]{d_0} Q \end{array}\right\} \Rightarrow Q^n \xrightarrow[n\to\infty]{d_0} Q.$$

9. Why 3 types method is effective?

Detailed presentation can be found in:
DISCRETE AND CONTINUOUS
doi:10.3934/dcdsb.2011.16.1
DYNAMICAL SYSTEMS SERIES B
Volume 16, Number 1, July 2011

Our focus lies in computer simulation and Monte-Carlo technique.

In computer realizations all numerical trajectories are independent objects. Therefore in practical Monte - Carlo simulations these trajectories can not be put on the same probability space.

10. Evaluation of ruin probability by simulations

In this section we evaluate numerically a ruin probability $\mathsf{P}(\tau \le T)$, for a finite positive T, where $\tau = \inf\{t : X_t = 0\}$ by Euler-Maruyama approximations.

The basis for analysis is an obvious formula $\mathsf{P}(\tau \le T) = \mathsf{P}(X_T = 0)$. It allows us to deal with the distribution function

$$F(x) := \mathsf{P}(X_T \le x)$$

of X_T instead of a harder to compute distribution function of

$$\mathsf{P}(\tau \le T).$$

Notice that $F(x) = \begin{cases} 0, & x < 0 \\ F(0) = \mathsf{P}(X_T = 0) > 0, & x = 0, \end{cases}$

11.

Note that $F(0) - F(0-) > 0$, i.e., distribution function $F(x)$ has an atom at the point 0.

The measure Q is supported on the space of continuous functions. So $Q^n \xrightarrow[n\to\infty]{d_0} Q$ implies weak convergence of finite marginals. That is,

$$\lim_{n\to\infty} F_n(x) = F(x)$$

at any point of continuity of F. Unfortunately 0, our point of interest, is an atom of F and we can not claim that

$$\lim_{n\to\infty} F_n(0) = P(\tau \leq T) = F(0).$$

12. Evaluation of ruin probability by simulations

Here we evaluate numerically a ruin probability $\mathsf{P}(\tau \le T)$, for a finite positive T, where $\tau = \inf\{t : X_t = 0\}$ by Euler-Maruyama approximations.

$$\mathsf{P}(\tau \le T) = \mathsf{P}(X_T = 0).$$

Set $F(x) := \mathsf{P}(X_T \le x)$ of X_T and note that

$$F(x) = \begin{cases} 0, & x < 0 \\ F(0) = \mathsf{P}(X_T = 0) > 0, & x = 0, \end{cases}$$

that is, $F(0) - F(0-) > 0$ and so the distribution function $F(x)$ has an atom at the point 0.

13. Monte Carlo simulation

We use Monte-Carlo simulations of 10^3 independent copies

of the process $(X^n_t)_{t\in[0,T]}$ generated by Euler- Maruyama algorithm with $t^n_k - t^n_{k-1} \equiv 10^{-2}$.

Numerical results are given below for $p = \frac{1}{2}$ and $p = \frac{3}{4}$. with $t^n_k - t^n_{k-1} \equiv 10^{-2}$.

For $p = \frac{1}{2}$, where theoretical formula is known, a good results hold in term of "lower bound" and "upper bound"

$$\mathsf{P}(X^n_T \le -\varepsilon) - \varepsilon \quad \text{and} \quad \mathsf{P}(X^n_T \le \varepsilon) + \varepsilon, \quad \varepsilon = 10^{-6},\ \varepsilon = 10^{-7}.$$

14. Numerical results 1.

p	μ	T	ε	lower bound	upper bound	theory
1/2	-1	3	10^{-7}	0.9730	0.9730	0.9830
1/2	-1	9	10^{-6}	0.9970	1.0000	0.9996
1/2	0	3	10^{-6}	0.8477	0.8480	0.8465
1/2	0	6	10^{-6}	0.9234	0.9253	0.9200
1/2	0	9	10^{-7}	0.9384	0.9388	0.9460
1/2	1	3	10^{-7}	0.5906	0.5912	0.5908
1/2	1	6	10^{-7}	0.5950	0.5962	0.6058
1/2	1	9	10^{-7}	0.5945	0.5958	0.6070

,

Таблица :

15. Numerical results. 2

For $p = 3/4$, where no theoretical formulae are available, we give results of simulations for a few values of parameters.

p	X_0	μ	T	ε	lower bound	upper bound
3/4	1/4	1	9	10^{-9}	0.3838	0.3864
3/4	1	1	9	10^{-9}	0.0782	0.0790
3/4	1/4	-1	3	10^{-9}	0.8757	0.8803

Таблица :