Essential Supremum and Essential Maximum with Respect to a Random Preference Relation with Financial Applications [1]

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Basic Notions, 1

$\succeq$ is a *preference relation* or *preorder* in X if it reflexive ($x \succeq x$) and transitive (if $x \succeq y$ and $y \succeq z$ then $x \succeq z$). A preorder is *partial order* if it is antisymmetric (if $x \succeq y$ and $y \succeq x$ then $x = y$).

Order intervals : $[x, y] := \{z \in X : \ y \succeq z \succeq x\}$,

$$]-\infty, x] := \{z \in X : \ x \succeq z\}, \qquad [x, \infty[:= \{z \in X : \ z \succeq x\}.$$

$\Gamma \succeq x$ means that $y \succeq x$ for all $y \in \Gamma \subseteq X$.
$\Gamma_1 \succeq \Gamma$ means that $x \succeq y$ for all $x \in \Gamma_1$ and $y \in \Gamma$;

$$[\Gamma, \infty[:= \cap_{z \in \Gamma}\{z \in X : \ z \succeq x\},$$

When X is a topological space : a preorder is *upper semi-continuous* (resp., *lower semi-continuous*) if $[x, \infty[$ (resp., $]-\infty, x]$) is closed for any $x \in X$ and *semi-continuous* if it is both upper and lower semi-continuous. It is called *continuous* if its graph $\{(x, y) : \ y \succeq x\}$ is a closed subset of $X \times X$.

Basic Notions, 2

A set $\mathcal{U}$ of real functions on X *represents* the preorder $\succeq$ if for any $x, y \in X$,

$$x \succeq y \;\Leftrightarrow\; u(x) \geq u(y) \quad \forall u \in \mathcal{U}.$$

This set $\mathcal{U}$ is called *multi-utility representation* of the preorder. If its elements are continuous functions, we say that $\mathcal{U}$ is a *continuous multi-utility representation* of the preorder.
Any preorder can be represented by the family $\mathcal{U} = \{I_{[x,\infty[}, \, x \in X\}$.
Evren and Ok (2011) :
1) any upper (lower) semicontinuous preorder admits an upper (lower) semicontinuous multi-utility representation ;
2) any continuous preorder on a locally and σ-compact topological space X admits a continuous multi-utility representation.
An arbitrary family $\mathcal{U}$ defines a preorder. It is a partial order if the equalities $u(x) = u(y)$ for all $u \in \mathcal{U}$ imply that $x = y$.
Notation : $x \sim y$ if $x \succeq y$ and $y \succeq x$; $[x]$ is the class containing x.

Supremum as a Set

Definition

Let $\Gamma \subseteq X$. *Then* $\mathrm{Sup}\,\Gamma$ *is the* **largest** *subset* $\hat{\Gamma}$ *of* X *such that :*

(a_0) $\hat{\Gamma} \succeq \Gamma$;

(b_0) *if* $x \succeq \Gamma$, *then there is* $\hat{x} \in \hat{\Gamma}$ *such that* $\hat{x} \preceq x$;

(c_0) *if* $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$, *then* $\hat{x}_1 \succeq \hat{x}_2$ *implies* $\hat{x}_1 \sim \hat{x}_2$.

If $\succeq$ is a partial order, the equivalence $\hat{x}_1 \sim \hat{x}_2$ means that $\hat{x}_1 = \hat{x}_2$. For a partial order the set $\hat{\Gamma}$ satisfying (a_0), (b_0), (c_0) is unique but for a general preorder this is not true. Note also that $\mathrm{Sup}\,\Gamma$ may not exist. Clearly, $\mathrm{Sup}\,\Gamma$ is the union of all subset $\hat{\Gamma}$ satisfying (a_0), (b_0), (c_0).

When $\mathrm{Sup}\,\Gamma \neq \emptyset$? 1

Theorem

Let $\succeq$ be a partial order represented by a countable family $\mathcal{U} = \{u_j\}$ of lower semicontinuous functions be such that all order intervals $[x, y]$, $y \succeq x$, are compacts. If Γ is bounded from above, i.e. $\bar{x} \succeq \Gamma$ for some $\bar{x}$, then $\mathrm{Sup}\,\Gamma$ exists.

Proof. Assuming wlg that all $|u_j| \le 1$, define the function $u := \sum_j 2^{-j} u_j$. Then u is l.s.c., hence, for every $x \succeq \Gamma$ it attains minimum on the compact set $[\Gamma, x]$. Put $\Lambda(x) := \mathrm{argmin}_{y \in [\Gamma, x]} u(y)$. Then $\hat{\Gamma} := \bigcup_{x \succeq \Gamma} \Lambda(x)$ obviously satisfies (a_0) and (b_0). Let $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$, $\hat{x}_1 \succeq \hat{x}_2$, i.e. $u_j(\hat{x}_1) \ge u_j(\hat{x}_2)$ for all j and $u(\hat{x}_1) \ge u(\hat{x}_2)$. There is $x_1 \succeq \Gamma$ such that $\hat{x}_1 \in \Lambda(x_1)$. Since $\hat{x}_2 \in [\Gamma, \hat{x}_1] \subseteq [\Gamma, x_1]$, we have that $u(\hat{x}_2) \ge u(\hat{x}_1)$. So, $u(\hat{x}_2) = u(\hat{x}_1)$ and this is possible only if $u_j(\hat{x}_1) = u_j(\hat{x}_2)$ for all j. Therefore, $\hat{x}_1 = \hat{x}_2$ and (c_0) holds.

When $\mathrm{Sup}\,\Gamma \neq \emptyset$? 2

Notation : $\tilde{X} := X/\sim$ and $q : x \mapsto [x]$ is the quotient mapping.

Theorem

Let $\succeq$ be a preorder represented by a countable family of l.s.c. functions such that all order intervals $[[x],[y]]$, $[y] \succeq [x]$, are compact subsets of $\tilde{X}$. If $\bar{x} \succeq \Gamma$ for some $\bar{x}$, then $\mathrm{Sup}\,\Gamma \neq \emptyset$.

Proof. The partial order in $\tilde{X}$ is given by the family of l.s.c. functions $\{u(f(.)) : \ u \in \mathcal{U}\}$ where $f : \tilde{X} \to X$ is an arbitrary function associating with $[x] \in \tilde{x}$ a point $f(x) \in [x]$. In the previous theorem $\mathrm{Sup}\, q(\Gamma)$ is not empty and so is the set $\mathrm{Sup}\,\Gamma = q^{-1}(\mathrm{Sup}\, q(\Gamma))$.

Proposition

Let X be a σ-compact metric space. Suppose that a family $\mathcal{U}$ of continuous functions defines a preorder on X. Then this preorder can be defined by a countable subfamily of $\mathcal{U}$.

Preorder in a Hilbert Space X Defined by a Cone, 1

A closed convex cone $G \subseteq X$ defines a preorder : $x \succeq 0$ means that $x \in G$ and $y \succeq x$ means that $y - x \succeq 0$, i.e. $y \in x + G$.

If $y \succeq x$ then $\lambda y \succeq \lambda x$ for any $\lambda \geq 0$.

If $y \succeq x$, $v \succeq u$, then $x + v \succeq y + u$.

The graph $\{(x, y) : \ y - x \in G\}$ is a closed subset of $X \times X$.

A cone G defines a partial order iff G is proper, i.e. if $G^0 = \{0\}$ where $G^0 = G \cap (-G)$. In this case all intervals $[x, y]$ are compact.

The cone G is the intersection of all closed half-spaces $L = \{x \in X : \ lx \geq 0\}$ containing G. Its complement G^c is the union of the open half-spaces L^c. In X any covering of an open set contains a countable subcovering. Hence, there is a countable family of vectors l_j such that $G = \cap_j \{x \in X : \ l_j x \geq 0\}$. That is the countable family of linear functions $u_j(x) = l_j x$ represents the preorder defined by G. So, the preorder defined by $G \subseteq X$ can be generated by a countable family of linear functions. Clearly, the converse is true.

Preorder in a Hilbert Space X Defined by a Cone, 2

For the preorder given by a cone the properties defining the set $\hat{\Gamma} = \mathrm{Sup}\, \Gamma$ can be reformulated in geometric terms as follows :

(a'_0) $\hat{\Gamma} - \Gamma \subseteq G$;

(b'_0) if $x - \Gamma \subseteq G$, then there is $\hat{x} \in \hat{\Gamma}$ such that $x - \hat{x} \in G$;

(c'_0) if $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$, then $\hat{x}_1 - \hat{x}_2 \notin G \setminus G^0$.

Lemma

For the partial order defined by a closed proper cone G in $\mathbb{R}^d$ the order intervals $[x, y]$ are bounded.

Proof. Suppose that $z_n \in [x, y]$ and $|z_n| \to \infty$. We may assume wlg that $z_n/|z_n| \to z_\infty$ with $|z_\infty| = 1$. For any linear function $u(x)$ from $\mathcal{U}$ we have $u(x)/|z_n| \le u(z_n/|z_n|) \le u(y/|z_n|)$. It follows that $u(z_\infty) = 0$ for all $u \in \mathcal{U}$. That is $z_\infty = 0$. A contradiction.

Partial Order in a Hilbert Space X Defined by a Cone

Thus, for the case of the partial order on $\mathbb{R}^d$ given by a cone the hypotheses of the general existence theorem are fulfilled. It is also clear that the arguments in the proof of the above lemma does not work for infinite-dimensional Hilbert space : though one can always find a weakly convergent subsequence of $z_n/|z_n|$, the norm of the limit might be well equal to zero. But the convexity of the order intervals combined with the property that the balls in a Hilbert space are weakly compact leads to the following result.

Theorem

Let $\succeq$ be the partial order generated by a proper closed convex cone G in a Hilbert space X. If $\Gamma \subseteq X$ is such that $\bar{x} \succeq \Gamma$ (i.e. $\bar{x} - \Gamma \subseteq G$) for some $\bar{x} \in X$ and for every $x \succeq \Gamma$ the order interval $[\Gamma, x]$ is bounded, then $\mathrm{Sup}\,\Gamma \neq \emptyset$.

Maximum as a Subset of Γ, 1

Definition

Let Γ be a non-empty subset of X. We put

$$\mathrm{Max}\,\Gamma = \{x \in \bar{\Gamma} : \ \bar{\Gamma} \cap [x, \infty[= [x, x]\}.$$

Note that $[x, x] = [x]$. In the case of partial order $[x, x] = \{x\}$.

Definition

Let Γ be a non-empty subset of X. We denote by $\mathrm{Max}_1\Gamma$ the maximal subset $\hat{\Gamma} \subseteq \bar{\Gamma}$ (possibly empty) such that :

(α) *if $x \in \bar{\Gamma}$, then there is $\hat{x} \in \hat{\Gamma}$ such that $\hat{x} \succeq x$;*

(β) *if $\hat{x}_1, \hat{x}_2 \in \hat{\Gamma}$, then $\hat{x}_1 \succeq \hat{x}_2$ implies $\hat{x}_1 \sim \hat{x}_2$.*

Obviously, $\mathrm{Max}\,\Gamma = \mathrm{Max}\,\bar{\Gamma} = (\mathrm{Max}\,\Gamma)^{\sim} \cap \bar{\Gamma} = q^{-1}(q(\mathrm{Max}\,\Gamma)) \cap \bar{\Gamma}$,
$\mathrm{Max}_1\Gamma = \mathrm{Max}_1\bar{\Gamma} = (\mathrm{Max}_1\Gamma)^{\sim} \cap \bar{\Gamma} = q^{-1}(q(\mathrm{Max}_1\Gamma)) \cap \bar{\Gamma}$.

Maximum as a Subset of Γ, 2

Lemma

Let $\succeq$ be a preorder on X. Let Γ be a non-empty subset of X such that $\overline{q(\Gamma)} = q(\bar{\Gamma})$. Then $q(\mathrm{Max}\,\Gamma) = \mathrm{Max}\,q(\Gamma)$.

Lemma

Let $\succeq$ be a preorder on X. Let Γ be a non-empty subset of X such that $\overline{q(\Gamma)} = q(\bar{\Gamma})$. Then $q(\mathrm{Max}_1\Gamma) = \mathrm{Max}_1 q(\Gamma)$.

Proposition

Let $\succeq$ be a partial order represented by a countable family of upper semicontinuous functions and such that all order intervals $[x, y]$, $y \succeq x$, are compacts. Suppose that there exists $\bar{x}$ such that $\bar{x} \succeq \Gamma$. Then $\mathrm{Max}\,\Gamma$ and $\mathrm{Max}_1\Gamma$ are non-empty sets and $\mathrm{Max}\,\Gamma = \mathrm{Max}_1\Gamma$.

The hypotheses holds when $\succeq$ is generated by a proper cone in $\mathbb{R}^d$.

Essential Supremum in $L^0(X)$, 1

Let X be a separable metric space. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$. We define in $L^0(X)$ a preoder by a countable family of Carathéodory functions $\mathcal{U} = \{u_j\}$, i.e. such that

(i) $u_j(.,x) \in L^0(\mathbb{R}, \mathcal{F})$ for every $x \in X$;
(ii) $u_j(\omega, .)$ is continuous for almost all $\omega \in \Omega$.

So, $\gamma_2 \succeq \gamma_1$ means that $u_j(\gamma_2) \succeq u_j(\gamma_1)$ (a.s.) for all j.

Definition

Let Γ be a subset of $L^0(X, \mathcal{F})$. We denote by $\mathcal{H}$-$\mathrm{Esssup}\,\Gamma$ a subset $\hat{\Gamma}$ of $L^0(X, \mathcal{H})$ such that :

(a) $\hat{\Gamma} \succeq \Gamma$;

(b) if $\gamma \in L^0(X, \mathcal{H})$ and $\gamma \succeq \Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$, $\hat{\gamma} \preceq \gamma$;

(c) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

Essential Supremum in $L^0(X)$, 2

Theorem

Let $\succeq$ be a preference relation in $L^0(X,\mathcal{F})$ represented by a countable family of Carathéodory functions, i.e. satisfying (i), (ii), such that $|u_j| \le 1$. Let $\Gamma \neq \emptyset$ be such that $\bar\gamma \succeq \Gamma$ for some $\bar\gamma \in L^0(X,\mathcal{H})$. Suppose that for any $\gamma \in L^0([\Gamma,\infty[,\mathcal{H})$

$$\Lambda(\gamma) = \mathrm{argmin}_{\zeta \in L^0([\Gamma,\gamma],\mathcal{H})} Eu(\zeta) \neq \emptyset, \qquad (1)$$

where $u(\omega,z) = \sum 2^{-j} u_j(\omega,z)$. Then

$$\mathcal{H}\text{-Esssup}\,\Gamma = \cup_{\gamma \in L^0([\Gamma,\infty],\mathcal{H})} \Lambda(\gamma) \neq \emptyset.$$

This formulation is too technical... However, in the case $X = \mathbb{R}^d$ one can check the hypothesis (1) assuming that the ω-sections of the order intervals $[\gamma_1,\gamma_2]$ are compact.

Essential Maximum in $L^0(X)$, 1

The set $\Gamma \subseteq L^0(X,\mathcal{F})$ is *$\mathcal{H}$-decomposable* if for any $\gamma_1, \gamma_2 \in \Gamma$ and $A \in \mathcal{H}$ the random variable $\gamma_1 I_A + \gamma_2 I_{A^c} \in \Gamma$.
We denote by $\mathrm{env}_{\mathcal{H}}\Gamma$ the smallest $\mathcal{H}$-decomposable subset of $L^0(X,\mathcal{F})$ containing Γ and by $\mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma$ its closure in $L^0(X,\mathcal{F})$.
Then $\mathrm{env}_{\mathcal{H}}\Gamma$ is the set of r.v. $\sum \gamma_i I_{A_i}$ where $\gamma_i \in \Gamma$ and $\{A_i\}$ is a finite partition of Ω into $\mathcal{H}$-measurable subsets. It follows that the set $\mathcal{H}$-$\mathrm{cl}\,\mathrm{env}\,\Gamma$ is $\mathcal{H}$-decomposable.

Definition

Let Γ be a non-empty subset of $L^0(X,\mathcal{F})$. We put

$$\mathcal{H}\text{-Essmax}\,\Gamma = \{\gamma \in \mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma : \ \mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma \cap [\gamma, \infty[= [\gamma, \gamma]\}.$$

Essential Maximum in $L^0(X)$, 2

Definition

Let Γ be a non-empty subset of $L^0(X, \mathcal{F})$. We denote by $\mathcal{H}\text{-}\mathrm{Essmax}_1\Gamma$ the largest subset $\hat{\Gamma} \subseteq \mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma$ such that :
(i) if $\gamma \in \mathrm{cl}\,\mathrm{env}_{\mathcal{H}}\Gamma$, then there is $\hat{\gamma} \in \hat{\Gamma}$ such that $\hat{\gamma} \succeq \gamma$;
(ii) if $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, then $\hat{\gamma}_1 \succeq \hat{\gamma}_2$ implies $\hat{\gamma}_1 \sim \hat{\gamma}_2$.

Proposition

Let $\succeq$ be a partial order in $L^0(\mathbb{R}^d, \mathcal{F})$ represented by a countable family of functions satisfying (i), (ii) and such that all order intervals $[\gamma_1(\omega), \gamma_2(\omega)]$, $\gamma_2 \succeq \gamma_1$, are compacts a.s. Let Γ be a non-empty subset of $L^0(\mathbb{R}^d, \mathcal{H})$. Suppose that there exists $\bar{\gamma} \in L^0(\mathbb{R}^d, \mathcal{H})$ such that $\bar{\gamma} \succeq \Gamma$. Then $\mathcal{H}\text{-}\mathrm{Essmax}\,\Gamma = \mathcal{H}\text{-}\mathrm{Essmax}_1\Gamma \neq \emptyset$.

Random Cones, 1

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let X be a separable Hilbert space. Let $\omega \mapsto G(\omega) \subseteq X$ be a measurable set-valued mapping whose values are closed convex cones.
The measurability is understood as the measurability of the graph :

$$G := \{(\omega, x) \in \Omega \times X : \ x \in G(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(X).$$

The positive dual G^* of G is defined as the measurable mapping whose values are closed convex cones

$$G^*(\omega) := \{x \in X : \ xy \geq 0, \, \forall y \in G(\omega)\}.$$

Note that the measurable mapping G admits a Castaing representation : there is a countable set of measurable selectors ξ_i of G such that $G(\omega) = \overline{\{\xi_i(\omega) : \ i \geq 1\}}$ for all $\omega \in \Omega$. Thus,

$$G^* = \{(\omega, y) \in \Omega \times X : \ y\xi_i(\omega) \geq 0, \, \forall i \in \mathbf{N}\} \in \mathcal{F} \otimes \mathcal{B}(X), .$$

Random Cones, 2

Hence, G^* is a measurable mapping and admits a Castaing representation, i.e. there exists a countable set of $\mathcal{G}$-measurable selectors η_i of G^* such that $G^*(\omega) = \overline{\{\eta_i(\omega) : \; i \in \mathbf{N}\}}$ for all $\omega \in \Omega$.
Since $G = (G^*)^*$,

$$G(\omega) = \{(\omega, x) \in \Omega \times X : \; \eta_i(\omega)x \geq 0, \forall i \in \mathbf{N}\}. \qquad (2)$$

The relation $\gamma_2 - \gamma_1 \in G$ (a.s.) defines a preference relation $\gamma_2 \succeq \gamma_1$ in $L^0(X, \mathcal{F})$. Moreover, the countable family of functions $u_j(\omega, x) = \eta_j(\omega)x$ where η_j is a Castaing representation of G^*, represents the preference relation defined by G which is a partial order when the sections of G are proper cones.

Notation. Let $\mathcal{H}$ be a sub-σ-algebra of $\mathcal{F}$ and let $\Gamma \subseteq L^0(X, \mathcal{F})$. We shall use the notation $(\mathcal{H}, G)\text{-}\mathrm{Esssup}\,\Gamma$ instead of $\mathcal{H}\text{-}\mathrm{Esssup}\,\Gamma$ to indicate that the preorder is generated by the random cone G.

Random Cones, 3

Theorem

Let X be a separable Hilbert space and let $\succeq$ be a preorder in $L^0(X,\mathcal{F})$ defined by a random cone G. Suppose that the subspaces $(G^0(\omega))^\perp$ are finite-dimensional a.s. Let $\Gamma \neq \emptyset$ be such that $\bar{\gamma} \succeq \Gamma$ for some $\bar{\gamma} \in L^0(X,\mathcal{F})$. Then $\mathcal{F}\text{-}\mathrm{Esssup}\,\Gamma \neq \emptyset$.

Model of Financial Market with Transaction Costs, 1

We are given a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t=0,\dots,T}, P)$ with a d-dimensional adapted process $S = (S_t)$ with strictly positive components and an adapted process $K = (K_t)$ which values are closed convex cones $K_t \subset \mathbb{R}^d$ with interiors containing $\mathbb{R}^d_+ \setminus \{0\}$. Define the random diagonal operators

$$\phi_t : (x^1, \dots, x^d) \mapsto (x^1/S^1_t, \dots, x^d/S^d_t), \qquad t = 0, \dots, T,$$

and relate with them the random cones $\widehat{K}_t := \phi_t K$.
We consider the set $\widehat{\mathcal{V}}$ of $\mathbb{R}^d$-valued adapted processes $\widehat{V}$ such that the increments $\Delta \widehat{V}_t := \widehat{V}_t - \widehat{V}_{t-1} \in -\widehat{K}_t$ for all t and the set $\mathcal{V}$ which elements are the processes V with $V_t = \phi_t^{-1} \widehat{V}_t$, $\widehat{V} \in \widehat{\mathcal{V}}$.

Model of Financial Market with Transaction Costs, 2

In the context of the theory of markets with transaction costs, K_t are the solvency cone corresponding to the description of the model in terms of a numéraire, $\mathcal{V}$ is the set of value processes of self-financing portfolios. The notations with hat correspond to the description of the model in terms of "physical" units where the portfolio dynamics is much simpler because it does not depend on price movements. A typical example is the model of currency market defined via the adapted matrix-valued process of transaction costs coefficients $\Lambda = (\lambda_t^{ij})$. In this case

$$K_t = \mathrm{cone}\,\{(1+\lambda_t^{ij})e_i - e_j,\ e_i, 1 \le i,j \le d\}.$$

In this model European contingent claims are d-dimensional random vectors while American contingent claims are adapted d-dimensional random processes.

Minimal Portfolios for European Options under Friction

Assume that $\widehat{K}_t$ are proper, i.e. $\widehat{K}_t \cap (-\widehat{K}_t) = \{0\}$ for all t. We denote by $\succeq_{K_t}$ the partial order in the space $L^0(\mathbf{R}^d)$ generated by $\widehat{K}_t$, i.e. $\xi \succeq_{K_t} 0$ if $\xi \in L^0(\widehat{K}_t)$.
The value process $\widehat{V} \in \widehat{\mathcal{V}}$ (the set of all portfolio processes in physical units) is called *minimal* if $\widehat{V}_T = \widehat{C}$ and any process $W \in \widehat{\mathcal{V}}$ such that $\widehat{W_T} = \widehat{C}$ and $\widehat{W_t} \preceq_{\widehat{K}_t} \widehat{V}_t$ for all $t \leq T$ coincides with $\widehat{V}$. We denote $\widehat{\mathcal{V}}_{min}$ the set of all minimal processes.

Proposition

Suppose that $L^0(\widehat{K}_{t+1}, \mathcal{F}_t) \subseteq L^0(\widehat{K}_t, \mathcal{F}_t)$, $t \leq T-1$. Then $\widehat{\mathcal{V}}_{min} \neq \emptyset$. Moreover, $\widehat{\mathcal{V}}_{min}$ coincides with the set of solutions of backward inclusions

$$\widehat{V}_t \in (\mathcal{F}_t, \widehat{K}_{t+1})\text{-Esssup}\,\{\widehat{V}_{t+1}\}, \quad t \leq T-1, \quad \widehat{V}_T = \widehat{C}.$$

Minimal Portfolios for American Options under Friction

we shall use the notation $Y = (Y_t)$ when the American contingent claim is expressed in units of the numéraire and $\widehat{Y} = (\widehat{Y}_t)$ when it is expressed in physical units. The relation is obvious : $\widehat{Y}_t = \phi_t Y_t$. The value process $\widehat{V} \in \widehat{\mathcal{V}}$ is called *minimal* if $\widehat{V} \succeq_{\widehat{K}} \widehat{Y}$ and any process $\widehat{W} \in \widehat{\mathcal{V}}$ such that $\widehat{W} \preceq_{\widehat{K}} \widehat{V}$ coincides with $\widehat{V}$. The notation means that to compare values of the processes at time t one uses the partial order generated by the random cone $\widehat{K}_t$. We denote $\widehat{\mathcal{V}}_{min}$ the set of all minimal processes.

Proposition

The set $\mathcal{V}_{min}$ is non-empty and coincides with the set of solutions of backward inclusions

$$\widehat{V}_t \in (\mathcal{F}_t, \widehat{K}_t)\text{-}\mathrm{Essmin}_1 L^0((\widehat{Y}_t + \widehat{K}_t) \cap (\widehat{V}_{t+1} + \widehat{K}_{t+1}), \mathcal{F}_{t+1}), \ t \leq T-1.$$