Estimation problems and Hunt-Muckenhoupt-Wheeden condition

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Estimation of function in stationary noise

This talk is connected with nonparametric estimation of the function $s(t)$ as the observation process $Y(t)$ is given by

$$dY(t) = s(t)dt + dX(t), t \in [-T, T].$$

Here s is a locally square-integrable function, $s \in L^2_{loc}$, $X(t)$ is a Gaussian process with stationary increments and spectral density f.

We suppose that unknown function s belongs to a compact subspace $\mathscr{L}_*$ of the Banach space $\mathscr{L}$ with the norm $\|\cdot\|_{\mathcal{L}}$,

$$\|s\|^2_{\mathcal{L}} = \sup_x \int\limits_x^{x+1} |s\,(t)\,|^2\, dt < \infty, \tag{1}$$

and assume also that for large T and some positive c, C

$$c\,\|\varphi\|_T \leq \|\varphi\|^2_{\mathscr{L}} \leq C\,\|\varphi\|_T\,. \tag{2}$$

Here

$$\|\varphi\|_T^2 = \frac{1}{T} \int\limits_{-T}^{T} |\varphi(t)|^2 \, dt.$$

For an estimator $\widehat{s}_T$ of unknown function s we denote

$$R_T\left(\widehat{s}_T, f\right) = \sup_{s \in \mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|_{\mathscr{L}}^2 .$$

Let $R_T\left(f\right)$ be the minimax risk,

$$R_T\left(f\right) = \inf_{\widehat{s}_T} \sup_{s \in \mathscr{L}_*} \mathbf{E}_{s,f} \left\|\widehat{s}_T - s\right\|_{\mathscr{L}}^2 .$$

We plan construct an estimator $\widehat{s}_T$ such that for appropriate class $\mathscr{K}$ of spectral densities and sufficiently large T

$$\sup_{f \in \mathscr{K}} \frac{R_T\left(\widehat{s}_T, f\right)}{R_T\left(f\right)} \leq C\left(\mathscr{K}, \mathscr{L}_*\right), \tag{3}$$

where the constant $C\left(\mathscr{K}, \tau, \mathscr{L}_*\right)$ depends only on class $\mathscr{K}$.

For a locally square-integrable function h, we denote,

$$Y_{\mathrm{T}}[h] = \frac{1}{T} \int_{T}^{T} \overline{h(t)}\, dY(t) \tag{4}$$

So, our observations coincide with the collection of random variables

$$Y_{\mathrm{T}}[h] = s_{\mathrm{T}}[h] + X_{\mathrm{T}}[h],\ h \in L^2_{loc}.$$

Here

$$s_{\mathrm{T}}[h] = \frac{1}{T} \int_{T}^{T} \overline{h(t)}\, s(t)dt, \quad X_{\mathrm{T}}[h] = \frac{1}{T} \int_{T}^{T} \overline{h(t)}\, dX(t)$$

For fixed T random process $X_{\mathrm{T}}[h]$ is a Gaussian process with zero mean and the correlation function

$$\mathscr{R}_{\mathrm{T}}\,(\varphi, \psi) = \frac{1}{T^2} \int_{-\infty}^{\infty} \widehat{\psi}_{\mathrm{T}}(u)\, \overline{\widehat{\varphi}_{\mathrm{T}}(u)} f(u)\, du,$$

where $\widehat{\varphi}_{\mathrm{T}}$ is the Fourier transformation of function $\varphi(t)1\!\!\mathrm{I}_{[\text{-T, T}]}(t)$. So, we assume that the process $X_{\mathrm{T}}(h)$, $h \in L^2_{loc}$, has the spectral density f, which satisfies to the condition

$$\int\limits_{-\infty}^{\infty} \frac{f(u)}{1+u^2}\,du < \infty.$$

At the beginning we consider a simple case as

$$s(\cdot) \in \mathscr{L}\,(\varphi;\tau) = \{s: \;\; s = \theta\varphi, \;\; |\theta| \le \tau\}\,,$$

where the function φ is known.

For estimating of unknown θ we can use the observation

$$Y_{\mathrm{T}}\left[\frac{\varphi(t)}{\|\varphi\|^2_T}\right] = \theta + X_{\mathrm{T}}\left[\frac{\varphi(t)}{\|\varphi\|^2_T}\right].$$

It is clear that $X_{\mathrm{T}}\left[\frac{\varphi(t)}{\|\varphi\|^2_T}\right] \in N(0;\sigma^2)$, where

$$\sigma^2 = \sigma^2_f(T;\varphi).$$

So, we have well known problem (I.A.Ibragimov, R.Z.Khasminskii). Let

$$Y = \theta + \sigma X, \quad |\theta| \le \tau, \quad X \sim N(0;1).$$

The minimax linear estimator is

$$\widehat{\theta} = \frac{\tau^2}{\tau^2 + \sigma^2} Y$$

and minimax linear risk $R_L(\tau;\sigma)$ is

$$R_L(\tau;\sigma) = \frac{\tau^2\sigma^2}{\tau^2 + \sigma^2}.$$

Let $R(\tau;\sigma)$ be the minimax risk, then

$$\sup_{\tau,\sigma} \frac{R_L(\tau;\sigma)}{R(\tau;\sigma)} = \mu < \infty.$$

David L. Donoho, Richard C. Liu, Brenda MacGibbon proved that

$$\mu < 1.25.$$

If value σ is unknown it is possible to use the estimator

$$\widetilde{\theta} = \begin{cases} y, & |y| \le \tau \\ 0, & \text{otherwise.} \end{cases}$$

But in general case the observation

$$Y_{\mathrm{T}}\left[\frac{\varphi(\cdot)}{\|\varphi\|_T^2}\right] = \theta + X_{\mathrm{T}}\left[\frac{\varphi(\cdot)}{\|\varphi\|_T^2}\right]$$

is not the sufficient statistics. Possibly (it depends on spectral density f) there exist a function $\psi(\cdot) \in L^2_{loc}$ such that

$$s_{\mathrm{T}}(\psi) = 0, \text{ and } \rho\left(X_{\mathrm{T}}\left[\varphi\right], X_{\mathrm{T}}[\psi]\right) \to 1, \text{ as } T \to \infty.$$

Here we denote by $\rho(\xi, \eta)$ the correlation coefficient between ξ and η). In this case we need use for estimation θ the observation

$$Y^*_{\mathrm{T}} = Y_{\mathrm{T}}\left[\frac{\varphi(\cdot)}{\|\varphi\|_T^2}\right] - \lambda Y_{\mathrm{T}}[\psi],$$

for appropriately chosen λ.

But this λ depends on spectral density f and therefore it is impossible to use the random value Y^*_{T} for estimating when the spectral density is unknown. Denote by $\widetilde{R}(\tau;\sigma)$ the minimax risk of estimating unknown θ based on Y^*_{T}. It is clear that

$$\widetilde{R}(\tau;\sigma) \leq R(\tau;\sigma)$$

and

$$\frac{\widetilde{R}(\tau;\sigma)}{R(\tau;\sigma)} \;\to\; 0, \text{ as } T \to \infty$$

in the case, when

$$\rho\left(X_{\mathrm{T}}\left[\varphi\right], X_{\mathrm{T}}[\psi]\right) \;\to\; 1, \text{ as } T \to \infty.$$

Therefore we have to investigate under what conditions on f uniformly on T

$$\left|\rho\left(X_{\mathrm{T}}\left[\varphi\right], X_{\mathrm{T}}[\psi]\right)\right| \leq \rho < 1.$$

Stepanov space of pseudoperiodic functions

Consider Stepanov space $\mathscr{L}(\Lambda)$ of pseudoperiodic functions

$$s(t) = \sum_{u\in\Lambda} a(u)e^{iut}, \ \ \sum_{u\in\Lambda} |a(u)|^2 < \infty, \tag{5}$$

Here Λ is a countable subset of real line such that

$$\tau = \tau(\Lambda) = \inf_{u,v\in\Lambda,\, u\neq v} |u - v| > 0, \tag{6}$$

It is well known that $\mathscr{L}(\Lambda) \subset \mathscr{L}$ and Banach norm $\|\cdot\|_{\mathscr{L}}$ is topologically equivalent to Hilbert norm $\|\|_T$ for sufficiently large T. We denote by $\mathscr{L}_*$ the subset of $\mathscr{L}(\Lambda)$ defined by

$$s(t) = \sum_{u\in\Lambda} a(u)e^{iut}, \ \ \sum_{u\in\Lambda} (1 + |u|)^{2\beta}|a(u)|^2 \leq C, \tag{7}$$

We suppose that the unknown function s in the model

$$dY(t) = s(t)dt + dX(t), t \in [-T, T] \ (\text{or in another form } Y_{\mathrm{T}} = s_{\mathrm{T}} + X_{\mathrm{T}}),$$

belongs to $\mathscr{L}_*$. Denote $\varphi_u(t) = e^{iut}\mathbb{1}_{[\text{-T,T}]}$. Let $\mathscr{L}_T(\Lambda)$ be the closure of the linear manifold of the set $\{\varphi_u(t),\ u \in \Lambda\}$ in the Hilbert space L^2_T with the norm $\|\cdot\|_T$, and $H_0(T)$ be the orthogonal compliment of $\mathscr{L}_T(\Lambda)$. It should be recalled that L^2_T is the L^2-space on interval $[-T, T]$ constructed on normalized Lebesgue measure.
Denote by $R^*_T(f)$ the minimax risk in the estimation problem as we observed only the collection

$$Y_T(\varphi_u), u \in \Lambda.$$

That is we reject the observation

$$Y_{\mathrm{T}}(\psi),\ \text{if } s_{\mathrm{T}}(\psi) = 0.$$

It is clear that

$$R^*_T(f) \le R_T(f).$$

We have to answer the question, under what conditions it is true that

$$\inf_T \frac{R_T^*(f)}{R_T(f)} \geq c > 0.$$

Theorem 1. Suppose that

$$\inf_{\varphi \in \mathscr{L}_T(\Lambda),\, \psi \in H_0(T)} \left| \rho\left(X_{\mathrm{T}}(\varphi), X_{\mathrm{T}}(\psi) \right) \right| \geq \rho > 0,$$

where ρ does not depend on T, then

$$\inf_T \frac{R_T^*(f)}{R_T(f)} \geq c > 0.$$

where c depends only on ρ.

Hunt-Muckenhoupt-Wheeden condition

Let f be a nonnegative function, L^2_f be the L^2-space constructed on measure with density f. Denote by $(\cdot,\cdot)_f$ the inner product in the space L^2_f,

$$(h,g)_f = \int\limits_{-\infty}^{\infty} h(u)\overline{g(u)}f(u)\,du.$$

Denote by $H_T(f)$ the closure in L^2_f of linear manifold of the set

$$\left\{\frac{\sin T(u-\cdot)}{u-\cdot},\ u\in R\right\}.$$

Here we assume that

$$\int\limits_{-\infty}^{\infty} \frac{f(u)}{1+u^2}\,du < \infty.$$

In the case as $f \equiv 1$ we shall write H_T instead of $H_T(\mathbb{1})$. It is well known that the space H_T consists of function with Fourier transform, supported on the interval $[-T, T]$. The kernel

$$G_T(u, v) = \frac{\sin T(u - v)}{\pi(u - v)}$$

is the reproducing kernel on H_T in the space L^2, and the integral operator P_T with kernel $G_T(u, v)$ is the orthoprojector on H_T.

We shall say that function f satisfy to the Hunt-Muckenhoupt-Wheeden condition, if

$$\lambda(f) = \sup_A \frac{1}{|A|} \int\limits_A f(u)\, du \, \frac{1}{|A|} \int\limits_A \frac{1}{f(u)}\, du < \infty.$$

The following theorem is well known

Theorem 2. Suppose that $\lambda(f) \leq K$, then the operator P_T is bounded uniformly on T in the space L_f^2 and

$$\|P_T\|_\infty \leq C,$$

where C depends only on K.

From this theorem we deduce the following result.

Let

$$\lambda(f;u) = \sup_{A:\, u\in A} \frac{1}{|A|} \int\limits_A f(u)\, du\, \frac{1}{|A|} \int\limits_A \frac{1}{f(u)}\, du < \infty.$$

Theorem 3. Suppose that $\sup\limits_{u\in\Lambda} \lambda(f;u) \leq K$, and $\tau(\Lambda) > 0$, then

$$\inf_T \frac{R_T^*(f)}{R_T(f)} \geq c > 0.$$

where c depends only on K and $\tau(\Lambda)$.