

Estimation problems and Hunt-Muckenhoupt-Wheeden condition

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Estimation of function in stationary noise

This talk is connected with nonparametric estimation of the function $s(t)$ as the observation process $Y(t)$ is given by

$$dY(t) = s(t)dt + dX(t), t \in [-T, T].$$

Here s is a locally square-integrable function, $s \in L_{loc}^2$, $X(t)$ is a Gaussian process with stationary increments and spectral density f .

We suppose that unknown function s belongs to a compact subspace \mathcal{L}_* of the Banach space \mathcal{L} with the norm $\|\cdot\|_{\mathcal{L}}$,

$$\|s\|_{\mathcal{L}}^2 = \sup_x \int_x^{x+1} |s(t)|^2 dt < \infty, \quad (1)$$

and assume also that for large T and some positive c, C

$$c \|\varphi\|_T \leq \|\varphi\|_{\mathcal{L}}^2 \leq C \|\varphi\|_T. \quad (2)$$

Here

$$\|\varphi\|_T^2 = \frac{1}{T} \int_{-T}^T |\varphi(t)|^2 dt.$$

For an estimator \hat{s}_T of unknown function s we denote

$$R_T(\hat{s}_T, f) = \sup_{s \in \mathcal{L}_*} \mathbf{E}_{s,f} \|\hat{s}_T - s\|_{\mathcal{L}}^2.$$

Let $R_T(f)$ be the minimax risk,

$$R_T(f) = \inf_{\hat{s}_T} \sup_{s \in \mathcal{L}_*} \mathbf{E}_{s,f} \|\hat{s}_T - s\|_{\mathcal{L}}^2.$$

We plan construct an estimator \hat{s}_T such that for appropriate class \mathcal{K} of spectral densities and sufficiently large T

$$\sup_{f \in \mathcal{K}} \frac{R_T(\hat{s}_T, f)}{R_T(f)} \leq C(\mathcal{K}, \mathcal{L}_*), \quad (3)$$

where the constant $C(\mathcal{K}, \tau, \mathcal{L}_*)$ depends only on class \mathcal{K} .

For a locally square-integrable function h , we denote,

$$Y_T[h] = \frac{1}{T} \int_T^T \overline{h(t)} dY(t) \quad (4)$$

So, our observations coincide with the collection of random variables

$$Y_T[h] = s_T[h] + X_T[h], \quad h \in L_{loc}^2.$$

Here

$$s_T[h] = \frac{1}{T} \int_T^T \overline{h(t)} s(t) dt, \quad X_T[h] = \frac{1}{T} \int_T^T \overline{h(t)} dX(t)$$

For fixed T random process $X_T[h]$ is a Gaussian process with zero mean and the correlation function

$$\mathcal{R}_T(\varphi, \psi) = \frac{1}{T^2} \int_{-\infty}^{\infty} \hat{\psi}_T(u) \overline{\hat{\varphi}_T(u)} f(u) du,$$

where $\widehat{\varphi}_T$ is the Fourier transformation of function $\varphi(t)\mathbf{1}_{[-T, T]}(t)$. So, we assume that the process $X_T(h)$, $h \in L^2_{loc}$, has the spectral density f , which satisfies to the condition

$$\int_{-\infty}^{\infty} \frac{f(u)}{1+u^2} du < \infty.$$

At the beginning we consider a simple case as

$$s(\cdot) \in \mathcal{L}(\varphi; \tau) = \{s : s = \theta\varphi, |\theta| \leq \tau\},$$

where the function φ is known.

For estimating of unknown θ we can use the observation

$$Y_T \left[\frac{\varphi(t)}{\|\varphi\|_T^2} \right] = \theta + X_T \left[\frac{\varphi(t)}{\|\varphi\|_T^2} \right].$$

It is clear that $X_T \left[\frac{\varphi(t)}{\|\varphi\|_T^2} \right] \in N(0; \sigma^2)$, where

$$\sigma^2 = \sigma_f^2(T; \varphi).$$

So, we have well known problem (I.A.Ibragimov, R.Z.Khasminskii). Let

$$Y = \theta + \sigma X, \quad |\theta| \leq \tau, \quad X \sim N(0; 1).$$

The minimax linear estimator is

$$\hat{\theta} = \frac{\tau^2}{\tau^2 + \sigma^2} Y$$

and minimax linear risk $R_L(\tau; \sigma)$ is

$$R_L(\tau; \sigma) = \frac{\tau^2 \sigma^2}{\tau^2 + \sigma^2}.$$

Let $R(\tau; \sigma)$ be the minimax risk, then

$$\sup_{\tau, \sigma} \frac{R_L(\tau; \sigma)}{R(\tau; \sigma)} = \mu < \infty.$$

David L. Donoho, Richard C. Liu, Brenda MacGibbon proved that

$$\mu < 1.25.$$

If value σ is unknown it is possible to use the estimator

$$\tilde{\theta} = \begin{cases} y, & |y| \leq \tau \\ 0, & \text{otherwise.} \end{cases}$$

But in general case the observation

$$Y_T \left[\frac{\varphi(\cdot)}{\|\varphi\|_T^2} \right] = \theta + X_T \left[\frac{\varphi(\cdot)}{\|\varphi\|_T^2} \right]$$

is not the sufficient statistics. Possibly (it depends on spectral density f) there exist a function $\psi(\cdot) \in L_{loc}^2$ such that

$$s_T(\psi) = 0, \text{ and } \rho(X_T[\varphi], X_T[\psi]) \rightarrow 1, \text{ as } T \rightarrow \infty.$$

Here we denote by $\rho(\xi, \eta)$ the correlation coefficient between ξ and η). In this case we need use for estimation θ the observation

$$Y_T^* = Y_T \left[\frac{\varphi(\cdot)}{\|\varphi\|_T^2} \right] - \lambda Y_T[\psi],$$

for appropriately chosen λ .

But this λ depends on spectral density f and therefore it is impossible to use the random value Y_T^* for estimating when the spectral density is unknown. Denote by $\tilde{R}(\tau; \sigma)$ the minimax risk of estimating unknown θ based on Y_T^* . It is clear that

$$\tilde{R}(\tau; \sigma) \leq R(\tau; \sigma)$$

and

$$\frac{\tilde{R}(\tau; \sigma)}{R(\tau; \sigma)} \rightarrow 0, \text{ as } T \rightarrow \infty$$

in the case, when

$$\rho(X_T[\varphi], X_T[\psi]) \rightarrow 1, \text{ as } T \rightarrow \infty.$$

Therefore we have to investigate under what conditions on f uniformly on T

$$|\rho(X_T[\varphi], X_T[\psi])| \leq \rho < 1.$$

Stepanov space of pseudoperiodic functions

Consider Stepanov space $\mathcal{L}(\Lambda)$ of pseudoperiodic functions

$$s(t) = \sum_{u \in \Lambda} a(u) e^{iut}, \quad \sum_{u \in \Lambda} |a(u)|^2 < \infty, \quad (5)$$

Here Λ is a countable subset of real line such that

$$\tau = \tau(\Lambda) = \inf_{u, v \in \Lambda, u \neq v} |u - v| > 0, \quad (6)$$

It is well known that $\mathcal{L}(\Lambda) \subset \mathcal{L}$ and Banach norm $\|\cdot\|_{\mathcal{L}}$ is topologically equivalent to Hilbert norm $\|\cdot\|_T$ for sufficiently large T . We denote by \mathcal{L}_* the subset of $\mathcal{L}(\Lambda)$ defined by

$$s(t) = \sum_{u \in \Lambda} a(u) e^{iut}, \quad \sum_{u \in \Lambda} (1 + |u|)^{2\beta} |a(u)|^2 \leq C, \quad (7)$$

We suppose that the unknown function s in the model

$$dY(t) = s(t)dt + dX(t), t \in [-T, T] \text{ (or in another form } Y_T = s_T + X_T),$$

belongs to \mathcal{L}_* . Denote $\varphi_u(t) = e^{iut} \mathbf{1}_{[-T, T]}$. Let $\mathcal{L}_T(\Lambda)$ be the closure of the linear manifold of the set $\{\varphi_u(t), u \in \Lambda\}$ in the Hilbert space L_T^2 with the norm $\|\cdot\|_T$, and $H_0(T)$ be the orthogonal complement of $\mathcal{L}_T(\Lambda)$. It should be recalled that L_T^2 is the L^2 -space on interval $[-T, T]$ constructed on normalized Lebesgue measure.

Denote by $R_T^*(f)$ the minimax risk in the estimation problem as we observed only the collection

$$Y_T(\varphi_u), u \in \Lambda.$$

That is we reject the observation

$$Y_T(\psi), \text{ if } s_T(\psi) = 0.$$

It is clear that

$$R_T^*(f) \leq R_T(f).$$

We have to answer the question, under what conditions it is true that

$$\inf_T \frac{R_T^*(f)}{R_T(f)} \geq c > 0.$$

Theorem 1. Suppose that

$$\inf_{\varphi \in \mathcal{L}_T(\Lambda), \psi \in H_0(T)} |\rho(X_T(\varphi), X_T(\psi))| \geq \rho > 0,$$

where ρ does not depend on T , then

$$\inf_T \frac{R_T^*(f)}{R_T(f)} \geq c > 0.$$

where c depends only on ρ .

Hunt-Muckenhoupt-Wheeden condition

Let f be a nonnegative function, L_f^2 be the L^2 -space constructed on measure with density f . Denote by $(\cdot, \cdot)_f$ the inner product in the space L_f^2 ,

$$(h, g)_f = \int_{-\infty}^{\infty} h(u) \overline{g(u)} f(u) du.$$

Denote by $H_T(f)$ the closure in L_f^2 of linear manifold of the set

$$\left\{ \frac{\sin T(u - \cdot)}{u - \cdot}, u \in R \right\}.$$

Here we assume that

$$\int_{-\infty}^{\infty} \frac{f(u)}{1 + u^2} du < \infty.$$

In the case as $f \equiv 1$ we shall write H_T instead of $H_T(\mathbf{I})$. It is well known that the space H_T consists of function with Fourier transform, supported on the interval $[-T, T]$. The kernel

$$G_T(u, v) = \frac{\sin T(u - v)}{\pi(u - v)}$$

is the reproducing kernel on H_T in the space L^2 , and the integral operator P_T with kernel $G_T(u, v)$ is the orthoprojector on H_T .

We shall say that function f satisfy to the Hunt-Muckenhoupt-Wheeden condition, if

$$\lambda(f) = \sup_A \frac{1}{|A|} \int_A f(u) du \frac{1}{|A|} \int_A \frac{1}{f(u)} du < \infty.$$

The following theorem is well known

Theorem 2. Suppose that $\lambda(f) \leq K$, then the operator P_T is bounded uniformly on T in the space L_f^2 and

$$\|P_T\|_\infty \leq C,$$

where C depends only on K .

From this theorem we deduce the following result.

Let

$$\lambda(f; u) = \sup_{A: u \in A} \frac{1}{|A|} \int_A f(u) du \frac{1}{|A|} \int_A \frac{1}{f(u)} du < \infty.$$

Theorem 3. Suppose that $\sup_{u \in \Lambda} \lambda(f; u) \leq K$, and $\tau(\Lambda) > 0$, then

$$\inf_T \frac{R_T^*(f)}{R_T(f)} \geq c > 0.$$

where c depends only on K and $\tau(\Lambda)$.