Exponential Semi-martingale models depending on a parameter in Mathematical Finances.

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Modelling

- An investor carry out the trading of risky asset $S(\xi) = \mathcal{E}(X(\xi))$, depending on random parameter ξ, where $X(\xi)$ is a semi-martingale, with maturity time T.
- ξ is random factor for mathematical framework; it can be a random variable or random process
- In mathematical finance ξ often represents the additional information which can have the investor; for example if ξ is default time of some asset, an investor can have the information about it from economical news.
- As we will see this approach can be used also in so called indifference pricing

The aim of insider is utility optimisation

- Let U be utility function satisfying usual properties: concave, strictly increasing, verifying Inada conditions, given on $]\underline{x}, +\infty[$
- $x(\xi)$ is initial capital
- Π class of self-financing admissible strategies

Optimal expected utility of the asset S:

$$V_T(x) = \sup_{\phi \in \Pi} E[U(x(\xi) + \int_0^T \phi_s(\xi)\, dS(\xi)_s)]$$

Level of information

Level of information change the class of self-financing admissible strategies which we use for maximisation. For simplicity suppose that ξ is the final value of some process at time T.

- For non-informed agents, the class self-financing admissible strategies Π related with natural filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ generated by risky asset $S(\xi)$.
- for partially informed agents the class of self-financing admissible strategies will be related with progressively enlarged filtration with the process corresponding to ξ.
- For perfectly informed agents the class of self-financing admissible strategies will be related with initially enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \le t \le T}$

$$\mathcal{G} = \cap_{s>t}(\mathcal{F}_s \otimes \sigma(\xi))$$

We consider the case of perfectly informed agents

Main Assumptions

We denote

- P is the law of $X(\xi)$
- P^u is the regular conditional law of $X(\xi)$ given $\xi = u$
- α is the law of ξ
- $\alpha^t = \mathcal{L}(\xi|\mathcal{F}_t)$

ASSUMPTION 1 For $t \in]0, T]$,

$$\alpha^t \ll \alpha$$

ASSUMPION 2 For all $u \in \Xi$

$$P^u \overset{loc}{\ll} P$$

Reduction to conditional utility maximisation problem

We define $\Pi^u(\mathbf{F})$ as

$$\bigcup_{c>0} \left\{\varphi \in S_u(\mathcal{P}(\mathbf{F})\otimes\sigma(\xi)) \mid \int_0^t \varphi_s(\xi)dS_s(\xi) \geq -c,\ \forall t \in [0,T]\,(\mathbb{P}\text{-a.s.})\right\}$$

We define also conditional maximal utility

$$V^u(x) = \sup_{\varphi\in\Pi^u(\mathbf{F})} E_{P^u}\left[U\left(x(u) + \int_0^T \varphi_s(u)dS_s(u)\right)\right]$$

THEOREM 1 *Let us suppose that Assumptions 1 and 2 holds. Then we can reduce classical utility maximisation problem to the corresponding conditional utility maximisation problem in the sense that*

$$V(x) = \int_{\Xi} V^u(x)d\alpha(u).$$

Dual approach for conditional maximisation problem

Let us denote by f the convex conjugate of U obtained by Frenchel-Legendre transform of U:

$$f(y) = \sup_{x>0} \left(U(x) - yx \right).$$

THEOREM 2 *Suppose that there exists an equivalent f-divergence minimal martingale measure $Q^{u,*}$ for conditional problem and $x(u) > \underline{x}$, then*

$$V^u(x) = E_{P^u} \left[U \left(-f' \left(\lambda(u) \frac{dQ_T^{u,*}}{dP_T} \right) \right) \right]$$

and $\lambda(u)$ is a unique solution of the equation

$$E_{Q^{u,*}} \left[-f' \left(\lambda(u) \frac{dQ_T^{u,*}}{dP_T} \right) \right] = x(u)$$

HARA utilities and information quantities

We introduce three important quantities related with P^u_T and $Q^{u,*}_T$ namely the entropy of P^u with respect to $Q^{u,*}_T$,

$$\mathbf{I}(P^u_T|Q^{u,*}_T) = -E_{P^u}\left[\ln(\frac{dQ^{u,*}_T}{dP_T})\right],$$

the entropy of $Q^{u,*}_T$ with respect to P^u_T,

$$\mathbf{I}(Q^{u,*}_T|P^u_T) = E_{P^u}\left[\frac{dQ^{u,*}_T}{dP_T}\ln(\frac{dQ^{u,*}_T}{dP_T})\right],$$

and Hellinger type integrals

$$\mathbf{H}^{(q),*}_T(u) = E_{P^u}\left[(\frac{dQ^{u,*}_T}{dP_T})^q\right],$$

where $q = \frac{p}{p-1}$ and $p < 1$.

Final result for maximisation for HARA utilities

THEOREM 3 *Under the Assumptions 1 and 2 we have the following expressions for $V_T(x)$:*

- *If $U(x) = \ln x$ then*

$$V_T(x) = \int_{\Xi} [\ln(x(u)) + \mathbf{I}(P_T^u | Q_T^{u,*})] d\alpha(u)$$

- *If $U(x) = \frac{x^p}{p}$ with $p < 1, p \neq 0$ then*

$$V_T(x) = \frac{1}{p} \int_{\Xi} (x(u))^p \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u)$$

- *If $U(x) = 1 - e^{-\gamma x}$ with $\gamma > 0$ then*

$$V_T(x) = 1 - \int_{\Xi} \exp\{-[\gamma(x(u)) + \mathbf{I}(Q_T^{u,*} | P_T^u)]\} d\alpha(u)$$

Application to indifference pricing

- An investor carry out the trading of risky asset $S(\xi) = \mathcal{E}(X(\xi))$, depending on random parameter ξ, $X(\xi)$ being a semi-martingale.
- The same investor holds a European type option with pay-off function $G_T = g(\xi)$ which he can not trade because of lack of liquidity or legal restrictions.
- **The question is:** what is indifference price for buyer and seller of the option, i.e. what is the amount of money which buyer would like to pay today (and seller would like to receive today) for the right to receive (to transmit) the option at time T.
- **Interesting fact:** The indifference prices will be the same for non-informed, partially informed and perfectly informed agents.

Indifference price equations

Optimal expected utility with option:

$$V_T(x,g) = \sup_{\phi\in\Pi} E[U(x + \int_0^T \phi_s \, dS_s + g(\xi))]$$

Indifference price for buyer p_T^b is a solution of

$$V_T(x - p_T^b, g) = V_T(x, 0)$$

and the indifference price for seller p_T^s is a solution of

$$V_T(x + p_T^s, -g) = V_T(x, 0)$$

Indifference price for logarithmic utility

PROPOSITION 4 *In the case of logarithmic utility $U(x) = \ln x$, $x > 0$, and under the Assumptions 1,2, the buyer's and seller's indifference price satisfy:*

$$\int_{\Xi} \ln\left[1 - \frac{p_T^b}{x} + \frac{g(u)}{x}\right] d\alpha(u) = 0$$

and

$$\int_{\Xi} \ln\left[1 + \frac{p_T^s}{x} - \frac{g(u)}{x}\right] d\alpha(u) = 0.$$

If $g(\xi) \in]0, x[$ (α-a.s.) and $\ln(g(\xi))$, $\ln(x - g(\xi))$ are integrable functions then the solutions of indifference price equations exist, they are unique and $p_T^b, p_T^s \in [0, x]$.

Interesting fact: Log utility is not sensitive to dependence of ξ and $S(\xi)$!

PROPOSITION 5 *In the case of power utility* $U(x) = \frac{x^p}{p}$, $x > 0$, *with* $p < 1$, $p \neq 0$, *we suppose that the Assumptions 1,2 hold,* $g(\xi) \in]0, x[$ *(*α *-a.s.) and*

$$\int_{\Xi} \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u) < \infty$$

Then, the buyer's and seller's indifference prices are defined respectively from the equations:

$$\int_{\Xi} [(1 - \frac{p_T^b}{x} + \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u) = 0 \qquad (1)$$

and

$$\int_{\Xi} [(1 + \frac{p_T^s}{x} - \frac{g(u)}{x})^p - 1] \left(\mathbf{H}_T^{(q),*}(u) \right)^{1-p} d\alpha(u) = 0 \qquad (2)$$

Moreover, the above the equations have unique solutions.

Indifference price for exponential utility

PROPOSITION 6 *In the case of exponential utility* $U(x) = 1 - e^{-\gamma x}$, $x > 0$, *with* $\gamma > 0$ *and under the Assumptions 1,2 the buyer's and seller's indifference prices verify:*

$$p_T^b = \frac{1}{\gamma} \ln \left[\frac{\int_\Xi \exp\left\{ -\mathbf{I}(Q_T^{u,*}|P_T^u)\right\} d\alpha(u)}{\int_\Xi \exp\left\{ -\gamma g(u) - \mathbf{I}(Q_T^{u,*}|P_T^u)\right\} d\alpha(u)} \right] \qquad (3)$$

and

$$p_T^s = -\frac{1}{\gamma} \ln \left[\frac{\int_\Xi \exp\left\{ -\mathbf{I}(Q_T^{u,*}|P_T^u)\right\} d\alpha(u)}{\int_\Xi \exp\left\{ \gamma g(u) - \mathbf{I}(Q_T^{u,*}|P_T^u)\right\} d\alpha(u)} \right] \qquad (4)$$

How it works: BS models

- $(W^{(1)}, W^{(2)})$ bi-dimensional standard Brownian motions with correlation ρ, $|\rho| < 1$ on $[0, T]$.
- $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1 > 0, \sigma_2 > 0$.
- two risky assets

$$S_t^{(1)} = \exp\{(\mu_1 - \frac{\sigma_1^2}{2})t + \sigma_1 W_t^{(1)}\}$$

$$S_t^{(2)} = \exp\{(\mu_2 - \frac{\sigma_2^2}{2})t + \sigma_2 W_t^{(2)}\}$$

- What is $X(\xi)$?

$$X_t(\xi) = \mu_1 t + \sigma_1 W_t^{(1)}$$

- What is ξ? Let $T' > T$. Take

$$\xi = W_{T'}^{(2)}$$

Conditional law of X

- The conditional law of X given $\xi = u$ coincide with the law of

$$X_t(u) = \mu_1 t + \sigma_1 \rho V_t(u) + \sigma_1 \sqrt{1-\rho^2}\gamma_t$$

  where $V(u)$ is a Brownian bridge starting from 0 at $t = 0$ and ending in u at $t = T'$ which is independent from the process γ.

- As known,

$$V_t(u) = \int_0^T \frac{u - V_s(u)}{T' - s} ds + \eta_t$$

  where η is standard Brownian motion independent from γ.

- Finally, since $\hat{\gamma} = \rho\eta + \sqrt{1-\rho^2}\gamma$ is again standard Brownian motion

- We get:

$$X_t(u) = \mu_1 t + \sigma_1 \rho \int_0^t \frac{u - V_s(u)}{T' - s} ds + \sigma_1 \hat{\gamma}_t$$

Assumptions 1 and 2

- Hence, $P_t^u \ll P_t$ for all $u \in \mathbb{R}$ and $t \in]0, T]$.
- The conditional law $\alpha^t = P(\xi \,|\, \mathcal{F}_t)$ given $\mathcal{F}_t = \sigma(W_s^{(1)}, s \leq \; t)$.
- By Markov property we get: for $A \in \mathcal{B}(\mathbb{R})$

$$\alpha^t(A) = P(W_{T'}^{(2)} \in A \,|\, \mathcal{F}_t) = P(W_{T'}^{(2)} \in A \,|\, W_t^{(1)})$$

$$= P(W_{T'}^{(2)} - W_t^{(2)} + W_t^{(2)} \in A \,|\, W_t^{(1)})$$

- Since $W_{T'}^{(2)} - W_t^{(2)}$ are independent from $(W_t^{(1)}, W_t^{(2)})$, the conditional law of ξ given $W_t^{(1)} = x$ is $\mathcal{N}(\rho x, T' - \rho^2 t)$. So, since $T' - \rho^2 t \neq 0$ for $t \in [0, T]$, it is equivalent to the law of $W_{T'}^{(2)}$ being $\mathcal{N}(0, T')$.

PROPOSITION 8 *For mentioned three information quantities we have the following result:*

$$\mathbf{I}(P^u \mid Q^{*,u}) = \frac{\sigma_1^2}{2}\left[\left(\mu_1 - \frac{\sigma_1 \rho u}{T'}\right)^2 T + \frac{\sigma_1^2\rho^2}{T'}\left(T' \ln(\frac{T'}{T'-T}) - T\right)\right],$$

$$\mathbf{I}(Q^{*,u} \mid P^u) = \frac{\sigma_1^2}{2}\left\{\mu_1^2\, T + 2\sigma_1\,\mu_1\,\rho\, u\, \ln(\frac{T'}{T'-T}) + \sigma_1^2\rho^2\, u^2 \frac{T}{T'(T'-T)}\right.$$
$$\left. + \sigma_1^2\rho^2\left[\frac{T}{T'-T} - \ln(\frac{T'}{T'-T})\right]\right\},$$

$$\mathbf{H}_T^{(q)}(u) = \left(\frac{T'}{T'-T+qT}\right)^{1/2} \exp\left\{-\frac{(1-q)}{2}\left[\frac{u^2}{T'} - \frac{(u+cT)^2}{T'-T+qT}\right]\right\}$$

with $q > -(\frac{T'}{T} - 1)$ *and* $c = \frac{\mu_1}{\sigma_1\sqrt{1-\rho^2}}$

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