A GENERALIZATION OF PROJECTIVE ESTIMATES.

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1. Let $(\mathbf{X}, \mathcal{A}, \mu)$ be a measurable space. A function $\theta \in \Theta \subseteq L_2(\mathbf{X})$ is observed in a noise. If the set Θ is infinite-dimensional, the problem arises how to construct statistical estimates for θ. In 1962 N. Chentsov suggested the following method to construct such estimates. Consider N-dimensional subspaces H of $L_2(\mathbf{X})$, take such H which approximates Θ sufficiently well, consider a reasonably good estimate T_N of the projection θ on H and take T_N as an estimate of θ. Thus the problem of estimation of an infinite-dimensional parameter is split into two following problems: 1. the problem of the estimation of an *finite-dimensional* parameter; 2. the problem of approximation of the set Θ by finite-dimensional linear manifolds.

2. In this talk I consider the following generalization of Chentsov's method. Let $H_K \subseteq L_2$ be a subspace of L_2 with a reproducing kernel $K(x, y)$. Our method suggest to project θ on all H_K and estimate these projections (all finite-dimension Hilbert spaces have reproducing kernels). It happens that such generalized projective estimates are in many aspects analogous to the classic Chentsov estimates.

3. In this talk I consider two following estimation problems.

I. Observations in Gaussian noise. The statistician observes for any $\varphi \in L_2(\mathbf{X})$ the random variable

$$X_\varphi = (f, \varphi) + \xi(\varphi),$$

$\xi(\varphi)$ are Gaussian with expectation zero and $\mathbf{E}\xi(\varphi)\xi(\psi) = \int_{\mathbf{X}} \varphi(x)\psi(x)\sigma^2(x)d\mu$. The unknown function f belongs to a known set $F \subseteq L_2$ and the problem is to estimate f. If $L_2(\mathbf{X}) = L_2(R^1)$, the observation can be written as the random process $X(t), dX(t) = f(t)dt + \sigma(t)dw(t)$, $w(t)$ is the standard Wiener process.

II. Poissonian observations. The statistician observes a Poisson random set Π in $\mathbf{X}$ with un unknown intensity density $\lambda(x)$ (with respect to the measure μ), $\lambda \in \Lambda \subseteq L_2(\mathbf{X}, \Lambda$ is a known set. The problem is to estimate λ.

4. Suppose at first that the parametric set belongs some H_K.

Theorem 1. *Let in the problem I the set $F \subseteq H_K$. There exist un estimate $\hat{f}$ such that*

$$\mathbf{E}_f ||\hat{f} - f||^2 \leq \int_{\mathbf{X}} K(x, x)\sigma^2(x)d\mu.$$

Theorem 2. *Let in the problem I $F = H_K$. Then*

$$\inf_{\hat{f}} \sup_{f} \mathbf{E}_f ||\hat{f} - f||^2 = \int_{\mathbf{X}} K(x, x)\sigma^2(x)d\mu.$$

In particular if $\sigma = \varepsilon\sigma_0, \varepsilon$ is a small known parameter,

$$\mathbf{E}_f ||\hat{f} - f||^2 \leq \varepsilon^2 \int_{\mathbf{X}} K(x, x)\sigma_0^2(x)d\mu.$$

Supported by the RFBR (11-01-00577-a), NSh (1216.2012.1).

Thus from the point of view of our estimation problem the set H_K behaves itself as a finite-dimensional set of dimension $\int_{\mathbf{X}} K(x,x)\sigma_0^2(x)d\mu$.

Theorem 3. *Let in the problem II the set $\Lambda \subseteq H_K$. There exists an estimate $\hat{\lambda}$ such that*

$$\mathbf{E}_\lambda ||\hat{\lambda} - \lambda||^2 \leq \int\limits_{\mathbf{X}} K(x,x)\lambda(x)d\mu.$$

Theorem 4. *Let in the problem II $L_2(\mathbf{X})$ is $L_2(\mathbb{R}^d)$. Let G be a set of positive finite Lebesgue measure $m(G)$. Let $H \subset L_2(\mathbb{R}^d)$ consist of all functions $\varphi \in L_2$ whose Fourier transform is zero outside the set G. The space H has the reproducing kernel $K(x,y = (2\pi)^{-d}\int_G e^{-i(t,x-y)}dt$. Suppose that $\lambda(x) = \varepsilon^{-1}\lambda_0(x)$ where ε is known. Suppose that $\lambda_0 \in \Lambda_0$ and that the set Λ_0 contains all positive functions λ_0 from H such that $\int_{\mathbb{R}^d} \lambda_0(x)dx \leq L < \infty$. Then*

$$\lim_{\varepsilon\to 0} \inf_{\hat{\lambda}_0} \sup_{\lambda_0\in\Lambda_0} \varepsilon^{-1}\mathbf{E}_{\lambda_0}||\hat{\lambda}_0 - \lambda_0||^2 = \sup_{\lambda_0} \int\limits_{\mathbf{X}} K(x,x)\lambda_0(x)dx = (2\pi)^{-d}Lm(G).$$

5.Denote $\mathcal{H}(N,\beta) = \mathcal{H}$ the set of all H_K for which $||K(x,x)||_\beta \leq N$ for some$1 \leq \beta \leq \infty$. Define diameters $\delta_N(\Theta,\beta)$ of a set Θ as follows

$$\delta_N(\Theta,\beta) = \inf_{H\in\mathcal{H}} \sup_{\theta\in\Theta} \inf_{h\in H} ||\theta - h||.$$

Below for the sake of simplicity we deal with the case $\beta = \infty$ only and set $\delta_N(\Theta,\infty) = \delta_N(\Theta)$.

Let us show how look generalized projective estimates in the problem II. Consider the spaces $H_K \in \mathcal{H}(N,\infty)$ and define the projective estimates $\hat{T}$ as follows:

$$\hat{T}(y) = \hat{T}(y, H_K) = \sum_{x\in\Pi} K(x,y)$$

. Then

$$\mathbf{E}_\lambda||\hat{T} - \lambda||^2 = \int\limits_{\mathbf{X}} K(x,x)\lambda(x)d\mu + ||\lambda - \lambda_K||^2,$$

λ_K is the projection of λ on H_K.

If for example $\lambda = \varepsilon^{-1}\lambda_0$, ε is a small known parameter, and we would like to estimate $\lambda_0 \in \Lambda_0$ then it is possible to construct a projective estimate $\hat{\lambda}$ such that

$$\mathbf{E}|\lambda_0||\hat{\lambda} - \lambda_0||^2 \leq \sup_{\lambda_0\in\Lambda} \inf_N (2\delta_N^2 + N\varepsilon \int\limits_{\mathbf{X}} \lambda_0(x)d\mu).$$